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On output-based accelerated stabilization of a chain of integrators: Implicit Lyapunov-Krasovskii functional approach ⋆

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Abstract: The problem of output accelerated stabilization of a chain of integrators is considered. Proposed control law nonlinearly depends on the output and its delayed values, and it does not use an observer to estimate the unmeasured components of the state. It is proven that such a nonlinear delayed control law ensures practical output stabilization with rates of convergence faster than exponential. The effective way of computation of feedback gains is given. It is shown that closed-loop system stability does not depend on the value of artificial delay, but the maximum value of delay determines the width of stability zone. The efficiency of the proposed control is demonstrated in simulations.

Keywords: Implicit Lyapunov-Krasovskii Functional (ILKF), nonlinear control, delay-dependent output feedback, Linear Matrix Inequalities (LMIs), practical stability

1. INTRODUCTION

Output feedback stabilization of dynamical systems is one of the central problems in the control theory and application. For linear systems with relative degrees greater than one, output derivatives are necessary to construct a control law. Since sometimes they cannot be measured directly, their estimates should be obtained, and usually for this goal the controller dynamics has to be augmented by an observer Khalil (2002). In some cases, if our control has to be embedded in an actuator with a low computational capacity, any decreasing the control low complexity and dimension is appreciated. To this end one can use finite time approximations of velocities (Selivanov and Fridman (2018)), which can be reconstructed based on the value of position in the current instant of time and a delayed one. It has been shown that delayed-induced feedback preserves the stability for small enough values of artificial delay (Borne et al. (2000); Fridman and Shaikhet (2016)).

It is worth to mention that inducing artificial delays has other benefits also. For instance, in Mazenc and Malisoff (2016); Mazenc et al. (2019) this technique allows one to relax the smoothness requirements imposed on the fictitious controls during backstepping design.

In addition, in many applications convergence rate and the settling time are the main optimization criteria. In such cases the problem of non-asymptotic (finite-time and fixed-time) stable system design arises. For example, a convergence rate of homogeneous systems could be changed significantly by modifying only their degree of homogeneity. Another conventional solution for acceleration consists in feedback gains increasing. Nevertheless, for the time-delayed systems such a method has a limited use: for any given delay h sufficiently large gains make the closed-loop system unstable.

Non-asymptotic stability analysis could be performed by using different methods. For instance, homogeneous systems described by ordinary differential equation can be investigated in a simple way. The main feature of such systems is that it is necessary and sufficient to study their behavior on a sphere in state space. However, time-delayed systems in general do not inherit such properties (Efimov et al. (2014)).

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One of the main theoretical tools for stability analysis and control synthesis of nonlinear systems is the Lyapunov function method (Khalil (2002)). In Polyakov et al. (2013) the Implicit Lyapunov Function (ILF) method has been extended to non-asymptotic analysis. Theorems on Implicit Lyapunov-Krasovskii Functional (ILKF) are presented in Polyakov et al. (2015). Both implicit methods allow to check all stability conditions by analyzing the algebraic equations. Therefore, a simple and effective procedure for parameter tuning based on Linear Matrix Inequalities (LMIs) can be obtained (see, for example, Lopez-Ramirez et al. (2018)).

Nevertheless, stability analysis of time-delay systems can be done not only in the time domain. For example, in Kharitonov et al. (2005) necessary conditions for the existence of multiple delay controllers are presented in terms of Hurwitz stability of some polynomials. However, feedback gains and admissible delays could be found only by solving transcendental equations.

The present work studies the problem of nonlinear output delay-dependent feedback design providing a practical accelerated (faster than exponential) stabilization of a chain of integrators. Although the chain of integrators is a quite simple model, it is also a very useful benchworking tool since all linear controllable systems and many nonlinear ones can be transformed into this particular form.

The goal of this work is to extend the results obtained in Efimov et al. (2018) for the case of a second order system to the chain of $n \geq 2$ integrators. However, in this paper the closed-loop system is not homogeneous, therefore, for stability analysis Implicit Lyapunov-Krasovskii functional is introduced. It is proven that the proposed control law ensures practical output stabilization with the rates of convergence faster than exponential. In contrast to Efimov et al. (2018) the effective way of computation of feedback gains is given. Moreover, it is shown that closed-loop system stability does not depend on the value of artificial delay. Nevertheless, maximum value of delay determines the width of stability zone.

The outline of this work is as follows. The notations and auxiliary lemmas are given in Section 2. The Implicit Lyapunov-Krasovskii approach for stability analysis of time-delay systems is introduced in Section 3. The finite-differences approximation of derivatives is touched upon in Section 4. The problem statement and the control design with stability analysis are considered in section 5. An example is presented in Section 6.

2. PRELIMINARIES

2.1 Notations

Through the paper the following notations will be used:

- $\mathbb{N}$ is the field of natural numbers;
- a series of integers $1, 2, \ldots, n$ is denoted by $\overline{1,n}$;
- $\mathbb{R}$ is the field of real numbers, $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_{\geq 0} := \mathbb{R}_{>0} \cup 0$;
- C^i is a class of i times continuously differentiable functions $\mathbb{R}_{>0} \to \mathbb{R}$;
- $\lceil \cdot \rfloor^{\alpha} := \mathrm{sign}(\cdot)|\cdot|^{\alpha}$;
- $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$;
- $\mathbb{C}_h^1$ is the space of continuously differentiable functions $[-(n-1)h, 0] \to \mathbb{R}^n$ with the norm $\|\cdot\|_h$ defined as follows $\|\Phi\|_h := \max\limits_{\tau \in [-(n-1)h, 0]} \|\Phi(\tau)\|$ for $\Phi \in \mathbb{C}_h^1$;
- $\mathbb{C}_h^{1,0} = \{\Phi \in \mathbb{C}_h^1 : \Phi(0) = 0\}$ is a subspace of $\mathbb{C}_h^1$;
- $\mathrm{diag}\{\lambda_i\}_{j=1}^n$ is the diagonal matrix with the elements $\lambda_j \in \mathbb{R}, j = \overline{1,n}$ on the main diagonal;
- if $P \in \mathbb{R}^{n\times n}$ is symmetric, then the inequalities $P > 0$ ($P < 0$) and $P \geq 0$ ($P \leq 0$) mean that P is positive (negative) definite and semidefinite, respectively;
- $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of a symmetric matrix $P \in \mathbb{R}^{n\times n}$, respectively;
- $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix and $O_n \in \mathbb{R}^{n\times 1}$ is the zero column;
- $\binom{j}{i} := \frac{j!}{i!(j-i)!}$ is the binomial coefficient;
- $|\epsilon|_{\mathcal{A}} := \inf_{\eta\in\mathcal{A}} \|\epsilon - \eta\|$ is the distance between $\epsilon \in \mathbb{R}^n$ and $\mathcal{A} \subseteq \mathbb{R}^n$.

2.2 Auxiliary lemmas

Lemma 1. (Solomon and Fridman (2013)). Let $f, \kappa : [a, b] \to [0, \infty)$ and $\phi : [a, b] \to \mathbb{R}$ be such that integration concerned is well defined. Then:

$$\Big[\int_a^b \kappa(s)\phi(s)ds\Big]^2 \leq \int_a^b f^{-1}(s)\kappa(s)ds \int_a^b f(s)\kappa(s)\phi^2(s)ds.$$

Lemma 2. (Lopez-Ramirez et al. (2018)). For $\forall s \in [0, 1]$, $\beta \in \mathbb{R}_{>0} \setminus \{1\}$ the function $g_\beta(s) := |s^\beta - s|$ admits the following estimate

$$\max_{s\in[0,1]} g_\beta(s) \leq g_\beta(\beta^{1/(1-\beta)}).$$

3. IMPLICIT LYAPUNOV-KRASOVKSII FUNCTIONAL APPROACH

Consider the system of the form

$$\dot{x}(t) = f(x_t),\ t \in \mathbb{R}_{>0}, \quad x_0 = \Phi \in \mathcal{D} \subseteq \mathbb{C}_h^1, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$, $x_t \in \mathbb{C}_h^1$ is the state function defined by $x_t(\tau) := x(t+\tau)$ with $\tau \in [-(n-1)h, 0]$ with $h > 0$ (time delay) and $f : \mathbb{C}_h^1 \to \mathbb{R}^n$ is a continuous operator. Assume that the origin is an equilibrium point of the system (1), i.e. $f(0) = 0$. A solution of the system (1) with the initial function $\Phi \in \mathcal{D}$ is denoted by $x(t, \Phi)$.

In applications frequently it is only required to stabilize the origin of the system in some zone, the width of which is determined by technical requirements. The next definition present an asymptotically stability with respect to the set.

Definition 3. The system (1) is said to be *asymptotically stable with respect to the set* $\mathcal{A}$, if it is:

a) *Lyapunov stable with respect to the set* $\mathcal{A}$:
for any $\varepsilon > 0$ there exists $\Delta(\varepsilon) > 0$ such that $|x(t, \Phi)|_{\mathcal{A}} \leq \varepsilon$ for all $t \geq 0$ and $\Phi \in \mathcal{D} : \|\Phi\|_h \leq \Delta$;
b) *attractive with respect to the set* $\mathcal{A}$:
$|x(t, \Phi)|_{\mathcal{A}} \to 0$ as $t \to \infty$ for any $\Phi \in \mathcal{D}$.

If the set could be chosen as $\mathcal{A} = 0$, then the *origin* of the system (1) is *asymptotically stable*. The set $\mathcal{D}$ is called the *domain of attraction* of the system (1). If $\mathcal{D} = \mathbb{C}_h^1$, then the corresponding stability becomes *global*. ■

In this paper only asymptotic stability is considered. Therefore, hereafter *stability* relates to asymptotic one.

Before formulating the theorem concerning stability analysis of time-delay systems using ILKF, a special class of comparison functions introduced by the following definition.

Definition 4. (Polyakov et al. (2015)). The function $q : \mathbb{R}^2_{>0} \to \mathbb{R}, (\rho, s) \to q(\rho, s)$ is said to be of the class $\mathcal{IK}_\infty$ if and only if:

1) q is continuous on $\mathbb{R}^2_{>0}$;
2) for any $s \in \mathbb{R}_{>0}$ there exists $\rho \in \mathbb{R}_{>0}$ such that $q(\rho, s) = 0$;
3) for any fixed $s \in \mathbb{R}_{>0}$ the function $q(\cdot, s)$ is strictly decreasing on $\mathbb{R}_{>0}$;
4) for any fixed $\rho \in \mathbb{R}_{>0}$ the function $q(\rho, \cdot)$ is strictly increasing on $\mathbb{R}_{>0}$;
5) $$\lim_{\substack{s\to 0^+\\(\rho,s)\in\Gamma}} \rho = \lim_{\substack{\rho\to 0^+\\(\rho,s)\in\Gamma}} s = 0, \quad \lim_{\substack{s\to\infty\\(\rho,s)\in\Gamma}} \rho = \infty,$$
where $\Gamma = \{(\rho, s) \in \mathbb{R}^2_{>0} : q(\rho, s) = 0\}$. ■

The next theorem on Implicit Lyapunov-Krasovskii functional presents conditions that guarantee stability of the nonlinear time-delayed system.

Denote partial derivatives of functional $Q(V, \chi)$ at time instant t as follows: $Q'_t(V, x_t) := \left.\frac{\partial Q(V,\chi)}{\partial \chi}\right|_{\chi=x_t} f(x_t)$ and $Q'_V(V, x_t) := \left.\frac{\partial Q(V,\chi)}{\partial V}\right|_{\chi=x_t}$.

Theorem 5. If there exists a continuous functional $Q : \mathbb{R}_{>0} \times \mathbb{C}^1_h \to \mathbb{R}$ such that:

C1) for all $\chi \in \mathbb{C}^1_h$ the function $(V, \tau) \to Q(V, \chi(\tau))$ is continuously differentiable;
C2) for any $\chi \in \mathbb{C}^1_h$ there exists $V \in \mathbb{R}_{>0}$ such that $Q(V, \chi) = 0$;
C3) there exist $q_i \in \mathcal{IK}_\infty, i = 1, 2$ such that for all $V \in \mathbb{R}_{>0}$
$$q_1(V, \|\chi(0)\|) \le Q(V, \chi), \quad \forall\chi \in \mathbb{C}^1_h \setminus \mathbb{C}^{1,0}_h,$$
$$Q(V, \chi) \le q_2(V, \|\chi\|_h), \quad \forall\chi \in \mathbb{C}^1_h \setminus \{0\};$$
C4) $Q'_V(V, \chi) < 0$ for all $V \in \mathbb{R}_{>0}$ and $\chi \in \mathbb{C}^1_h$ such that $Q(V, \chi) = 0$;
C5) for all $(V, x_t) \in \Omega = \{(V, x_t) \in (V_{\min}, V_{\max}) \times \mathbb{C}^1_h : Q(V, x_t) = 0\}$ such that x_t satisfies (1), we have
$$Q'_t(V, x_t) \le \theta V^{1+\mu} Q'_V(V, x_t), \quad \forall t \in \mathbb{R}_{>0},$$
where $\mu \in (-1, 0) \cup \mathbb{R}_{>0}$, $\theta > 0$, $V_{\min}, V_{\max} \in \mathbb{R}_{\ge 0}$.

Then the system is stable with the domain of attraction
$$\mathcal{D} := \{\Psi \in \mathbb{C}^1_h : Q(V_{\max}, \Psi) \le 0\} \tag{2}$$
(*globally stable* if $V_{\max} = +\infty$) with respect to *the set*
$$\mathcal{A} := \{\Psi \in \mathbb{C}^1_h : Q(V_{\min}, \Psi) \le 0\} \tag{3}$$
(at *the origin* if $V_{\min} = 0$).

The transition between sets $\mathcal{S}_1 := \{\Psi \in \mathbb{C}^1_h : Q(V_1, \Psi) = 0\}$ and $\mathcal{B}_2 := \{\Psi \in \mathbb{C}^1_h : Q(V_2, \Psi) \le 0\}$ is bounded as:
$$T(\mu, \mathcal{S}_1, \mathcal{B}_2) \le (V_2^{-\mu} - V_1^{-\mu})/(\theta\mu), \tag{4}$$
where $V_{\min} \le V_2 < V_1 \le V_{\max}$. ■

Proof. This Theorem can be proven in the same way as Theorems 2 and 3 in Polyakov et al. (2015) and for brevity the proof is omitted. Nevertheless, it is worth to note that condition C5) provides $\dot V \le -\theta V^{1+\mu}$, since $\dot Q := Q'_t(V, x_t) + Q'_V(V, \chi)\dot V = 0$ for all $(V, x_t) \in \Omega$. Hence, Theorem 5 implies the stability and the transition time estimate (4). □

4. FINITE-DIFFERENCES APPROXIMATION OF DERIVATIVES

In this section the exact form of finite-differences approximation error $\delta_i(t) := \tilde y_i(t) - y^{(i)}(t)$ is presented.

Proposition 6. (Selivanov and Fridman (2018)). If $y \in C^i$ and $y^{(i)}$ is absolutely continuous with $i \in \mathbb{N}$, then the following relation holds:
$$\begin{aligned}\delta_i(t) &= \frac{\tilde y_{i-1}(t) - \tilde y_{i-1}(t-h)}{h} - y^{(i)}(t)\\ &= -\int_{-ih}^{0} \varphi_i(-\tau) y^{(i+1)}(t+\tau)d\tau,\end{aligned} \tag{5}$$
where $\tilde y_0(t) := y(t)$ and
$$\varphi_1(\xi) := 1 - \xi/h, \qquad \xi \in [0, h]$$
$$\varphi_{i+1}(\xi) := \begin{cases} \int_0^\xi \frac{\varphi_i(\lambda)}{h} d\lambda + \frac{h-\xi}{h}, & \xi \in [0, h],\\ \int_{\xi-h}^{\xi} \frac{\varphi_i(\lambda)}{h} d\lambda, & \xi \in (h, ih),\\ \int_{\xi-h}^{ih} \frac{\varphi_i(\lambda)}{h} d\lambda, & \xi \in [ih, ih+h]. \blacksquare \end{cases} \tag{6}$$

For further stability analysis some characteristics of the functions φ_i are introduced.

Proposition 7. The functions $\varphi_i(\xi)$ defined in (6) and $\psi_i(\xi) := \int_\xi^{ih} \varphi_i(\lambda)d\lambda$ possess the following properties:

P1) $\varphi'_i \le 0$;
P2) $\varphi_i(\xi) + \varphi_i(ih - \xi) = 1$;
P3) $\psi_i(0) = ih/2$ and $\psi_i(ih) = 0$;
P4) $\psi'_i(\xi) = -\varphi_i(\xi)$;
P5) $\varphi_i(\xi)$ is concave on $\xi \in [0, ih/2]$ and convex on $\xi \in [ih/2, ih]$ for $i \ge 2$;
P6) $\psi_i(\xi) \le (ih/2)\varphi_i(\xi)$;
P7) $\zeta_i := \int_0^{ih} \frac{\varphi_i^2(\xi)}{\psi_i(\xi)} d\xi$ does not depend on h;
P8) for all $x_{i+1}(t) = y^{(i)}(t)$ it holds:
$$\int_{-ih}^{0} \varphi_i(-\tau)\dot x^2_{i+1}(t+\tau)d\tau \ge \frac{2}{ih}\delta_i^2(t), \tag{7a}$$
$$\int_{-ih}^{0} \psi_i(-\tau)\dot x^2_{i+1}(t+\tau)d\tau \ge \frac{1}{\zeta_i}\delta_i^2(t). \quad \blacksquare \tag{7b}$$

Proof of the properties P1)–P4) can be found in Selivanov and Fridman (2018). Other properties can be proven using P1)–P4) and Lemma 1, but due to space limitation are not presented in this paper.

It is worth to note that ζ_i is well-defined. Indeed, function $\tilde\varphi_i(\xi) := \varphi_i^2(\xi)/\psi_i(\xi)$ is continuous on $\xi \in [0, ih)$ and $\tilde\varphi_i(ih-) = 0$. Therefore, function $\tilde\varphi_i(\xi)$ is integrable.

5. NONLINEAR DELAY-DEPENDENT CONTROL

5.1 Problem statement

Consider a chain of $n \ge 2$ integrators:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{8}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the output available for measurements and

$$A = \begin{bmatrix} O_{n-1} & I_{n-1} \\ 0 & O_{n-1}^\top \end{bmatrix}, \; B = [0 \, \dots \, 0 \; 1]^\top, \; C = [1 \; 0 \, \dots \, 0].$$

The goal is to design an output-feedback control practically stabilizing the system (8) with the rate of convergence faster than exponential.

5.2 Main result

Define nonlinear delay-dependent control algorithm as follows:

$$u = K\lceil \tilde{x} \rfloor^{\alpha} \tag{9}$$

where

$$K = [K_1 \, \dots \, K_n], \quad \lceil \tilde{x} \rfloor^{\alpha} = [\lceil \tilde{x}_1 \rfloor^{\alpha_1} \, \dots \, \lceil \tilde{x}_n \rfloor^{\alpha_n}]^\top,$$

$$\tilde{x}_1(t) := x_1(t), \; \tilde{x}_{i+1}(t) := \frac{\tilde{x}_i(t) - \tilde{x}_i(t-h)}{h}, \; i = \overline{1, n-1},$$

$$\alpha_j := 1/r_j, \; r_j(\mu) := 1 - (n+1-j)\mu, \; j = \overline{1,n}. \tag{10}$$

Let us rewrite control law (9) in a form suitable for practical implementation:

$$u(t) = \sum_{j=1}^{n} K_j \Big\lceil \sum_{i=0}^{j-1} \frac{(-1)^i}{h^{j-1}} \binom{j-1}{i} z_i(t) \Big\rfloor^{\alpha_j}, \tag{11}$$

where

$$z_i(t) := \begin{cases} y(t - ih), & t > ih, \\ y(0), & t \leq ih. \end{cases}$$

Therefore, control (11) depends on only $n+2$ parameters: $h > 0$ is the artificial delay, $\mu \in (-1, 0) \cup (0, 1/n)$ is the degree of nonlinearity and $K_j < 0, j = \overline{1, n}$ are feedback gains. The restrictions on effective selection of these parameters and the conditions to check are given in the following theorem.

Before formulating the main result, let us introduce an Implicit Lyapunov-Krasovskii functional (ILKF) by the equality:

$$\begin{aligned} Q(V, \chi) &:= -1 + \chi^\top(0)\Lambda_V^{-r} P \Lambda_V^{-r} \chi(0) \\ &+ V^{\mu} \sum_{i=1}^{n-1} \int_{-ih}^{0} \psi_i(-\tau) \frac{2\zeta}{ihS} \Big[V^{-r_{i+2}} \dot{\chi}_{i+1}(\tau)\Big]^2 d\tau, \end{aligned} \tag{12}$$

where $P = P^\top > 0$ and $\Lambda_V^{-r} := \mathrm{diag}\{V^{-r_j}\}_{j=1}^n$.

Theorem 8. Let for some $\mu \in (-1, 0) \cup (0, 1/n)$ the system of LMIs:

$$0 < XH_r + H_r X \leq 2\omega(\mu)X \tag{13a}$$

$$\begin{bmatrix} X & Y^\top \\ Y & 1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} X & I_n \\ I_n & SI_n \end{bmatrix} \geq 0, \quad \begin{bmatrix} aI_n & X \\ X & SI_n \end{bmatrix} \geq 0,$$
$$\begin{bmatrix} b & S \\ S & 1 \end{bmatrix} \geq 0, \quad X \leq \frac{1}{2} I_n, \tag{13b}$$

$$\begin{bmatrix} \Phi_{11} & b\|\sigma\|B & B & Y^\top \\ * & -S & 0 & b\|\sigma\| \\ * & * & -4\zeta(1-a) & 1 \\ * & * & * & -S/\zeta \end{bmatrix} \leq 0, \tag{13c}$$

where

$$H_r := \mathrm{diag}\{r_j\}_{j=1}^n, \; \omega(\mu) := \begin{cases} 1 - (n+1)\mu, & \mu < 0, \\ 1, & \mu > 0, \end{cases}$$

$$\Phi_{11} := XA^\top + Y^\top B^\top + AX + BY + (\zeta + 2)aI_n,$$

$$\sigma_j := g_{\alpha_j}(\alpha_j^{1/(1-\alpha_j)}), \quad \zeta := \max_{i=\overline{1,n-1}} \int_0^i \frac{\varphi_i^2(\bar{\xi})}{\int_{\bar{\xi}}^i \varphi_i(\bar{\lambda}) d\bar{\lambda}} d\bar{\xi}$$

be feasible for some $S, a, b > 0$, $X \in \mathbb{R}^{n \times n}$, $X = X^\top > 0$, $Y \in \mathbb{R}^{1 \times n}$, with $j = \overline{1,n}$.

Then for any $h > 0$ the system (8), (11) with $K := YX^{-1}$:

a) is globally stable with respect to the set $\mathcal{A}$ (3), where $V_{\min} := ((n-1)hS)^{-1/\mu}$, if $\mu < 0$;
b) is stable at the origin with the domain of attraction $\mathcal{D}$ (2), where $V_{\max} := ((n-1)hS)^{-1/\mu}$, if $\mu > 0$;
c) converges from the set $\mathcal{S}_1$ to the set $\mathcal{B}_2$ with time estimate (4), where $1/\theta := 2\omega(\mu)S^2$. ■

The proof of Theorem 8 is given in Appendix A.

Corollary 9. If LMIs (13) from Theorem 8 are feasible for some $\mu_{\max}$, then they also hold for all $\mu = \gamma\mu_{\max}$, $0 \leq \gamma \leq 1$. ■

Proof. Indeed, $r_j(\mu)$ could be rewritten as:
$r_j(\mu) = (1-\gamma) + \gamma - (n+1-j)\gamma\mu_{\max} = (1-\gamma) + \gamma r_j(\mu_{\max})$.
Therefore, condition (13a) can be expressed as follows

$$0 < 2(1-\gamma)X + \gamma(XH_{r_{\max}} + H_{r_{\max}}X) \leq 2\omega(\mu)X,$$

where $r_{\max} = r(\mu_{\max})$.

Clearly, this condition holds for all $\mu = \gamma\mu_{\max}$, $0 \leq \gamma \leq 1$, since $0 < XH_{r_{\max}} + H_{r_{\max}}X \leq 2\omega(\mu_{\max})X$ and $\omega(\mu) = (1-\gamma) + \gamma\omega(\mu_{\max})$.

And taking into account that:

$$\mu \frac{\partial \sigma_j}{\partial \mu} = \mu r_j^{\frac{r_j}{1-r_j}} \frac{n+1-j}{1-r_j} \ln(r_j) \,\mathrm{sign}(g_{\alpha_j}(\alpha_j^{1/(1-\alpha_j)})) > 0,$$

one can deduce that $\max_{\mu} \|\sigma\| = \|\sigma(\mu_{\max})\|$. Hence, inequality (13c) also holds for all $\mu = \gamma\mu_{\max}$. □

Therefore, Corollary 9 postulates that sufficiently small μ, for which LMIs (13) are feasible, always could be found.

The following proposition claims that for any feedback gains K, obtained as the solution of LMIs (13) from Theorem 8, and fixed delay h, the nonlinear closed-loop system (8), (11) with $\mu \neq 0$ is always converging faster than its linear analog with $\mu = 0$ between properly selected sets $\mathcal{S}_1$ and $\mathcal{B}_2$, which depend on μ and h.

Proposition 10. Let conditions of Theorem 8 hold for some $\mu \in (-1, 0) \cup (0, 1/n)$. Then there exist two sets $\mathcal{S}_1 := \{V_1 \geq 1, \Psi \in \mathbb{C}_h^1 : Q(V_1, \Psi) = 0\}$ and $\mathcal{B}_2 := \{V_2 \leq 1, \Psi \in \mathbb{C}_h^1 : Q(V_2, \Psi) \leq 0\}$ such that in the system (8) and the control (11):

$$T(\mu, \mathcal{S}_1, \mathcal{B}_2) < T(0, \mathcal{S}_1, \mathcal{B}_2) \tag{14}$$

for fixed K and $h > 0$ satisfying

$$h \leq \begin{cases} V_2^{-\mu}/((n-1)S) & \text{for } \mu < 0, \\ V_1^{-\mu}/((n-1)S) & \text{for } \mu > 0. \end{cases}$$ ■

A procedure for selection of sets $\mathcal{S}_1$ and $\mathcal{B}_2$ is given in the proof below.

Proof. For linear system $\mu = 0$ the time of convergence between sets $\mathcal{S}_1$ and $\mathcal{B}_2$ is lower-bounded:

$$T(0, \mathcal{S}_1, \mathcal{B}_2) \geq T_0(h)(\ln V_1 - \ln V_2),$$

where $T_0(h)$ is the fastest time of transition between levels $V_1' = e$ and $V_2' = 1$ for $\mu = 0$, given delay h and fixed K.

From (4) it follows that condition (14) is verified if:

$$(V_2^{-\mu} - V_1^{-\mu})/(\theta\mu) \le T_0(h)(\ln V_1 - \ln V_2)$$

or taking into account property c) from Theorem 8

$$\frac{V_1^{-\mu} - V_2^{-\mu}}{\ln V_1^{-\mu} - \ln V_2^{-\mu}} < \frac{T_0(h)}{2(1-(n+1)\mu)S^2} \quad \text{for } \mu < 0,$$

$$\frac{V_2^{-\mu} - V_1^{-\mu}}{\ln V_2^{-\mu} - \ln V_1^{-\mu}} < \frac{T_0(h)}{2S^2} \quad \text{for } \mu > 0,$$

which for any $\mu \in (-1,0)$ ($\mu \in (0,1/n)$), $h > 0$ and $V_2 \in [V_{\min}, 1]$ ($V_1 \in [1, V_{\max}]$) could be guaranteed by decreasing V_1 (increasing V_2).

Conditions on h guarantee that $\mathcal{A} \subset \mathcal{B}_2$ for $\mu < 0$ and $\mathcal{S}_1 \subset \mathcal{D}$ for $\mu > 0$. □

Remark 11. Since the system (8), (11) keeps a constant value of μ for all $t \ge 0$, then it is clear that the set $\mathcal{S}_1$ represents the set of initial conditions.

Remark 12. Taking into account that $V_2^{-\mu} \le 1$ for $\mu < 0$ and $V_1^{-\mu} \le 1$ for $\mu > 0$, it follows that the maximum value of artificial delay is bounded as:

$$h_{\max} := 1/((n-1)S). \tag{15}$$

On the one hand, the greater values of h leads to the bigger stable set for $\mu < 0$ and smaller domain of attraction for $\mu > 0$. On the other hand, the smaller values of h could make the control system (11) unimplementable. Therefore, there is the minimum value of artificial delay $h_{\min}$ depending on technical realization.

6. EXAMPLE

Consider a chain of three integrators to show the main advantages of the proposed control law. Selecting $\mu = -0.01$ and $\mu = 0.01$ one can obtain, respectively, the following feedback gains $K^- = [-0.9754, \; -2.4398, \; -1.9303]$, $K^+ = [-0.9667, \; -2.4473, \; -1.9459]$ and $S^- = 11.4824$, $S^+ = 11.5149$.

According to (15), the maximum value of artificial delay is $h_{\max} = 0.0434$ s. Therefore, let us choose the nominal value as $h = 0.025$ s to enlarge the stability zone and to keep the control system implementable.

The norm of the state $x(t)$ obtained in simulation of the system (8) governed by the proposed nonlinear output delay-induced feedback (11) (blue solid line) in comparison with linear one (red solid line), as well as nonlinear (blue dashed line) and linear (red dashed line) state feedback, are depicted in figures 1 and 2 in logarithmic scale. For the sake of clarity, the feedback gains K are chosen the same for all control schemes as well as parameter μ for both nonlinear feedback and artificial delay for delay-dependent control laws.

Obviously, in the nonlinear case the rate of convergence is faster than in the linear one close to the origin for $\mu < 0$ and far outside for $\mu > 0$. Moreover, proposed control law and nonlinear state feedback show the same performance.

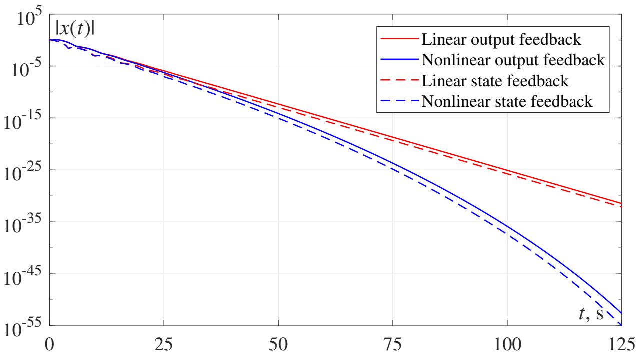

Fig. 1. Trajectories of stabilized system with $\mu = -0.01$

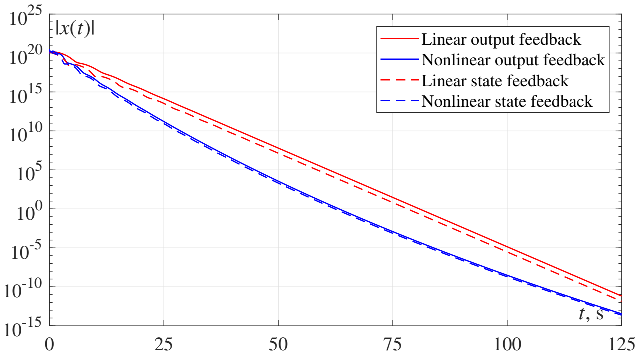

Fig. 2. Trajectories of stabilized system with $\mu = 0.01$

7. CONCLUSION

The paper addresses the problem of output feedback stabilization of a chain of integrators using a nonlinear delay-dependent control law that achieves the rates of convergence faster than exponential. The effective way of computation of feedback gains is presented. It is shown that the value of artificial delay does not affect closed-loop system stability, but determines the width of stability zone. Simulation results show feasibility of the proposed control scheme. The efficiency of the proposed approach is demonstrated in simulations, and a comparison with a linear analogue as well as nonlinear and linear state feedback is carried out.

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Appendix A. PROOF OF THEOREM 8

Let us show that ILKF (12) satisfies all conditions from Theorem 5. Since proof of conditions C1)–C3) can be done in a straightforward way (see, for example, Polyakov et al. (2015)), we just focus on conditions C4) and C5).

A. Proof of condition C4) of Theorem 5

The derivative of $Q(V,\chi)$ with respect to V is:

$$VQ'_V(V,\chi) = -\chi(0)^\top\Lambda_V^{-r}[H_rP+PH_r]\Lambda_V^{-r}\chi(0)$$
$$-V^\mu\sum_{i=1}^{n-1}(2r_{i+2}-\mu)\int_{-ih}^{0}\psi_i(-\tau)\frac{2\zeta}{ihS}\Big[V^{-r_{i+2}}\dot\chi_{i+1}(\tau)\Big]^2d\tau.$$

Taking into account (13a), one can see that:

$$0 < H_rP+PH_r \le 2\omega(\mu)P,\ \max_{i=\overline{1,n-1}} 2r_{i+2}-\mu < 2\omega(\mu).$$

So we finally conclude that

$$-\omega(\mu) \le VQ'_V(V,\chi) < 0 \tag{A.1}$$

for all $V\in\mathbb{R}_{>0}$ and $\chi\in\mathbb{C}_h^1$ such that $Q(V,\chi)=0$. Therefore, the condition C4) of Theorem 5 hold. □

B. Proof of condition C5) of Theorem 5

If $x(t)$ is the solution of the system (8), (9), then using properties P3) and P4), we obtain

$$Q'_t(V,x_t) = 2x^\top\Lambda_V^{-r}P\Lambda_V^{-r}\Big[Ax+BK\lceil\tilde x\rfloor^\alpha\Big]$$
$$-V^\mu\sum_{i=1}^{n-1}\int_{-ih}^{0}\varphi_i(-\tau)\frac{2\zeta}{ihS}\Big[V^{-r_{i+2}}\dot x_{i+1}(t+\tau)\Big]^2d\tau \tag{A.2}$$
$$+V^\mu\sum_{i=1}^{n-1}\frac{\zeta}{S}\Big[V^{-r_{i+2}}\dot x_{i+1}(t)\Big]^2 = \Sigma_1+\Sigma_2+\Sigma_3.$$

Taking into account that $\Lambda_V^{-r}A = V^\mu A\Lambda_V^{-r}$ and $\Lambda_V^{-r}BK = V^{-r_n}BK = V^{-1+\mu}BK$, the first term in (A.2) could be rewritten as follows:

$$\Sigma_1 = 2V^\mu x^\top\Lambda_V^{-r}P(A+BK)\Lambda_V^{-r}x$$
$$+2V^\mu x^\top\Lambda_V^{-r}PBKd_1 + 2V^\mu x^\top\Lambda_V^{-r}PBKd_2, \tag{A.3}$$

where

$$d_1 := V^{-1}\lceil\tilde x\rfloor^\alpha - \Lambda_V^{-r}\tilde x = \lceil V^{-r}\tilde x\rfloor^\alpha - \Lambda_V^{-r}\tilde x,$$
$$d_2 := \Lambda_V^{-r}\tilde x - \Lambda_V^{-r}x = \Lambda_V^{-r}[0,\delta_1,\ldots,\delta_{n-1}]^\top.$$

Obviously, disturbance terms d_1 and d_2 represent nonlinear deviation of feedback and finite-differences approximation error respectively.

The second term in (A.2) either can be upper-bounded by using property P6) and supposing that $d_1^\top d_1 \le \|\sigma\|^2$:

$$\Sigma_2 \le -V^\mu S\Big(\frac{V^{-\mu}}{(n-1)hS}\Big)\Big[1+\frac{d_1^\top d_1}{\|\sigma\|^2} - 2x^\top\Lambda_V^{-r}P\Lambda_V^{-r}x\Big]$$

or by using (7a):

$$\Sigma_2 \le -4\zeta V^\mu\Big(\frac{V^{-\mu}}{(n-1)hS}\Big)^2 Sd_2^\top d_2.$$

It is easy to see that the term Σ_2 could compensate cross-terms in (A.3) for $V\ge V_{\min}$ if $\mu<0$ or $V\le V_{\max}$ if $\mu>0$. Therefore, only global stability with respect to the set $\mathcal{A}$ (3) and stability at the origin with the domain of attraction $\mathcal{D}$ (2) could be obtained for $\mu<0$ and $\mu>0$ respectively. Hence, we deduce:

$$\Sigma_2 \le -S^{-2}V^\mu(1+\|\sigma\|^{-2}d_1^\top d_1 - 2x^\top\Lambda_V^{-r}P\Lambda_V^{-r}x)$$
$$-4\zeta(1-S^{-3})V^\mu Sd_2^\top d_2.$$

Taking into account that from (13b) it follows that:

$$K^\top K \le P \le SI_n \le aS^2P^2 \Rightarrow 1/S^3 \le a,\quad S^2\le b,$$

the term Σ_2 could be finally estimated as follows:

$$\Sigma_2 \le -S^{-2}V^\mu - S(b\|\sigma\|)^{-2}V^\mu d_1^\top K^\top Kd_1$$
$$+2aV^\mu x^\top\Lambda_V^{-r}P^2\Lambda_V^{-r}x - 4\zeta(1-a)V^\mu d_2^\top K^\top Kd_2. \tag{A.4}$$

Since $S^{-1}I_n \le aP^2$, the third term in (A.2) is bounded:

$$\Sigma_3 \le \zeta V^\mu(ax^\top\Lambda_V^{-r}P^2\Lambda_V^{-r}x + S^{-1}[V^{-1}\dot x_n]^2), \tag{A.5}$$

where

$$V^{-1}\dot x_n = K\Lambda_V^{-r}x + Kd_1 + Kd_2.$$

Combining (A.3)–(A.5), applying the Schur complement with respect to the vector $[x^\top\Lambda_V^{-r},(b\|\sigma\|)^{-1}Kd_1,Kd_2]^\top$ and using (13c), we deduce that $Q'_t(V,x_t)\le -S^{-2}V^\mu$. Taking into account (A.1), we conclude the proof. □

C. Proof of the estimate $d_1^\top d_1 \le \|\sigma\|^2$

The nonlinear disturbance term $d_1^\top d_1$ can be rewritten as:

$$d_1^\top d_1 = \sum_{j=1}^{n}\Big[|V^{-r_j}\tilde x_j|^{\alpha_j} - |V^{-r_j}\tilde x_j|\Big]^2. \tag{A.6}$$

Applying (7b) to (12) for all $(V,x_t)\in\Omega$, we deduce that:

$$1 \ge x^\top\Lambda_V^{-r}P\Lambda_V^{-r}x + \sum_{i=1}^{n-1}\frac{2\zeta}{\zeta_i}\Big(\frac{V^{-\mu}}{ihS}\Big)\Big[V^{-r_{i+1}}\delta_i\Big]^2.$$

Since $\zeta\ge\zeta_i$ for any $h>0$ due to P7) and $V^{-\mu}\ge(n-1)hS$, the following inequality holds for $P\ge 2I_n$:

$$1 \ge 2\sum_{j=1}^{n}\Big[V^{-r_j}x_j\Big]^2 + 2\sum_{i=1}^{n-1}\Big[V^{-r_{i+1}}\delta_i\Big]^2 \ge \sum_{j=1}^{n}\Big[V^{-r_j}\tilde x_j\Big]^2.$$

Hence, it follows that $|V^{-r_j}\tilde x_j|\le 1$ and, applying Lemma 2 to (A.6), one can finally conclude that $d_1^\top d_1 \le \|\sigma\|^2$. □