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Exploiting Wireless Interference For Distributively Solving Linear Equations

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Abstract: Interference in wireless communication is generally considered as an undesired phenomenon which needs to be combatted. Unlike traditional approaches, this paper investigates the possibility of exploiting interference for the cooperative solution of a linear algebraic equation, of which each agent knows only a portion. The presented communication system, together with the designed iterative algorithm, guarantees that agents converge exponentially to a global solution (unique or non-unique) of the algebraic linear equation, notwithstanding the presence of the unknown fading wireless channel. Guaranteeing privacy is one of the benefits of this approach: the unknown fading channel prohibits the access to neighboring agents' local equations. Such a feature is extremely useful in case local equations contain sensitive information. Randomized simulations reinforce our theoretical results.

Keywords: Multi-agent systems; Distributed control and estimation; Cooperative systems.

1. INTRODUCTION

Starting from Mou and Morse (2013), there has been considerable interest in networks of cooperative autonomous agents solving linear algebraic equations, i.e.,

$$A\mathbf{x} = b$$

in a distributed way, see, e.g., Mou et al. (2015), Shi et al. (2016), and Mou et al. (2016). Each agent can access only a distinct partition of the equation (i.e., a subset of rows of A and b) and aims at reaching the global solution by cooperating with the rest of the network. Mou and Morse (2013) consider a simple setting in which A is a nonsingular square matrix and the underlying network topology is fixed; their approach guarantees exponentially fast convergence to the global solution. Later on, many contributions extended this first result, e.g., Liu et al. (2017) considers non-square A, nonunique solutions, jointly-connected time-varying network topologies, and asynchronous communication. As shown in Mou et al. (2016), this result involves many possible applications, e.g., solving least square or *network localization* problems. Also, by Anderson et al. (2016), the cooperative solution of linear equations finds an important application in clustered computation involving sensitive data, such as business financial records, personally identifiable health information, etc.

In most applications, agents communicate over wireless networks, in which signal interference is an inherent phenomenon, see, e.g., Utschick (2016). It has been traditionally combatted by letting agents access the channel separately. In fact, all standard wireless communication protocols, e.g., TDMA (Time Division Multiple Access), are based on this principle. According to Goldenbaum et al. (2013), though standard, combatting interference leads to a waste of wireless resources. Following this idea, recently, an increasing number of contributions have considered cooperative multiagent systems exploiting interference, see, e.g., Goldenbaum and Stanczak (2013) and Liu and Zang (2019). Concerning consensus problems (see Ren et al. (2007) for an introduction to the topic), Molinari et al. (2018a) analyze the impact that the exploitation of channel interference has on convergence. Results show a multitude of benefits using this approach. For instance, Molinari et al. (2018b) prove that exploiting interference for a max-consensus problem saves wireless resources and leads to faster convergence. Similar results apply also for consensus-based formation control problems exploiting interference, see, e.g., Molinari and Raisch (2019).

For distributively solving linear algebraic equations, most literature has been focusing on improving the convergence rate and on relaxing conditions on neighbor graphs, see (Liu et al., 2017, Table I). To the best of our knowledge, the impact of the communication medium has not been taken into account. For this reason, the focus of this work is on the exploitation of interference for distributively solving linear algebraic equations in cooperative networks. We argue that this has a collection of additional benefits:

a) **Privacy**. Sets of neighbors and arc weights are unknown to agents, see, e.g., Molinari et al. (2018a). Therefore, it is impossible to use the received signals to have access to neighboring agents' local equations. This is a useful feature when different agents are not in the same domain of trust and each local equation may contain sensitive information, see, e.g., Anderson et al. (2016) and Wang et al. (2012).
b) **Saving resources**. Exploiting interference instead of getting rid of it allows for saving wireless resources,

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see, Goldenbaum et al. (2013) for a theoretical explanation. Section 6.2 will show similar results also for the case at hand.

The paper is organized as follows: Section 2 introduces the problem and its conditions. Sections 3 and 4 present, respectively, the communication system and the iterative algorithm. A proof of convergence to the global solution is derived in Section 5, both for cases of unique and nonunique solutions. Numerical simulations are shown in Section 6, and final remarks are stated in Section 7.

1.1 Notation

Throughout this paper, $\mathbb{N}_0$, respectively $\mathbb{N}$, denotes the set of nonnegative, respectively positive, integers. The sets of real numbers, nonnegative real numbers, and positive real numbers are, respectively, denoted $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$. Given a matrix A of dimension $n \times m$, the entry in position (i, j) is $[A]_{ij}$. Given a matrix A, its kernel is $\ker(A)$, its image is $\text{image}(A)$, its rank is denoted by $\text{rank}(A)$, and the nullity of A (i.e., the dimension of its kernel) is $\text{nullity}(A)$. A^{-1} is the inverse of A (if it exists) and A' its transpose. The Kronecker product of matrices A and B is denoted $A \otimes B$. By $\text{blockdiag}\,(A_1, \ldots, A_n)$ we represent a block-diagonal matrix with blocks $A_1, \ldots, A_n$ on the diagonal. The n-dimensional column of zeros is $\mathbf{0}_n$.

A directed graph $\mathcal{G}$ is a pair $(\mathcal{N}, \mathcal{A})$, where $\mathcal{N} = \{1 \ldots n\}$ is the set of nodes and $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of arcs. $(i, j) \in \mathcal{A}$ if and only if an arc goes from node $i \in \mathcal{N}$ to node $j \in \mathcal{N}$. Given a sequence of graphs constructed on the same set of nodes, i.e. $\{\mathcal{G}(k)\}_{k \in \mathbb{N}_0}$, the set

$$\mathcal{N}_i(k) := \{j \in \mathcal{N} \mid (j, i) \in \mathcal{A}(k)\}$$

contains the (in-)neighbors of agent i in the graph $\mathcal{G}(k)$. A path from node i to node j in graph $\mathcal{G}(k)$ is a sequence of arcs

$$(l_0, l_1), (l_1, l_2), \ldots, (l_{p-1}, l_p)$$

with $p \geq 1$, $l_0 = i$, $l_p = j$, and $(l_i, l_{i+1}) \in \mathcal{A}(k)$, $\forall i = 0 \ldots p - 1$. Graph $\mathcal{G}(k)$ is strongly connected if, $\forall i, j \in \mathcal{N}$, there exists a path from node i to node j. Graph $\mathcal{G}(k)$ is fully connected if, $\forall i, j \in \mathcal{N}$, there is an arc from i to j, $i \neq j$. A sequence of directed graphs is denoted by

$$\mathfrak{G} := (\mathcal{N}, \mathcal{A}(k))_{k \in \mathbb{N}_0}.$$

2. PROBLEM DESCRIPTION

A group of $\nu \in \mathbb{N}$ autonomous agents, grouped in the set $\mathcal{N} := \{1 \ldots \nu\}$, need to cooperate to solve the linear algebraic equation

$$A\mathbf{x} = b, \tag{1}$$

where $A \in \mathbb{R}^{n \times m}$ has full row-rank, $b \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^m$, and $\nu \leq n \leq m$. Each agent $i \in \mathcal{N}$ has access only to a distinct subset of $n_i \in \mathbb{N}$ rows of (1), i.e., it can locally solve

$$A_i \boldsymbol{x}_i = b_i, \tag{2}$$

where $A_i \in \mathbb{R}^{n_i \times m}$, $b_i \in \mathbb{R}^{n_i}$, and $\boldsymbol{x}_i \in \mathbb{R}^m$. Clearly, $\sum_{i \in \mathcal{N}} n_i = n$. Agents communicate over the wireless channel and exploit its interference property in a way already proposed in literature, see, e.g., Kortke et al. (2014); Molinari et al. (2018a); Liu and Zang (2019).

In what follows, we present both a communication protocol and an iterative algorithm allowing agents to distributively solve (1). The infinite sequence of graphs modeling the network topology, namely $\mathfrak{G}$, is assumed to be a sequence of fully-connected graphs.

3. COMMUNICATION SYSTEM

The wireless channel is a shared broadcast medium; when multiple users simultaneously access the same channel frequency spectrum, there is interference, see (Utschick, 2016, pg. 100). Physically, this means that the electromagnetic waves broadcast by a set of transmitters in the same frequency band superimpose at the receiver.

3.1 Wireless Multiple Access Channel

A set of transmitting agents, say $\mathcal{M} = \{1, \ldots, \mu\}$, broadcast real-valued signals $\omega_i \in \mathbb{R}$, $i \in \mathcal{M}$. Then, a simple model known as *Wireless Multiple Access Channel (WMAC)*, see e.g. (Utschick, 2016, Definition 5.2.1), allows to model the value at a receiver.

Definition 1. (WMAC). The WMAC between transmitters in $\mathcal{M}$ and a receiver is a map $\mathcal{W} : \mathbb{R}^{|\mathcal{M}|} \mapsto \mathbb{R}$ such that

$$y = \mathcal{W}(\omega_1, \ldots, \omega_\mu) := \sum_{j \in \mathcal{M}} \xi_j \omega_j + \eta, \tag{3}$$

where, $\forall j \in \mathcal{M}$, $\xi_j \in \mathbb{R}$ is the (unknown) *channel fading coefficient* between a transmitter j and the destination, and η is the receiver noise, see Utschick (2016).

In most literature, see, e.g., Goldenbaum et al. (2013) and Utschick (2016), an *ideal WMAC*, i.e.,

$$\forall j \in \mathcal{M},\ \xi_j = 1 \text{ and } \eta = 0,$$

has been considered. Molinari et al. (2018a) consider a noiseless WMAC with *power modulation*, i.e.,

$$\forall j \in \mathcal{M},\ \xi_j \in \mathbb{R}_{>0} \text{ and } \eta = 0, \tag{4}$$

which will be assumed also throughout this paper.

3.2 Communication System Design

Based on the WMAC model (3)-(4), the communication protocol is suggested. At every iteration $k \in \mathbb{N}_0$, each agent $j \in \mathcal{N}$ simultaneously broadcasts two pieces of information

$$\boldsymbol{\tau}_j^a(k) := \boldsymbol{x}_i(k), \tag{5a}$$

$$\tau_j^b(k) := 1. \tag{5b}$$

(5a) and (5b) are broadcast orthogonally, i.e., independent from each other (e.g., on a different frequency). For the synchronous broadcast of these two signals, at every algorithm iteration, agents needs $m + 1$ orthogonal transmissions (namely, $m + 1$ wireless resources are used per every iteration), since $\boldsymbol{\tau}_j^a(k) \in \mathbb{R}^m$ and $\tau_j^b(k) \in \mathbb{R}$. By (3)-(4), each agent $i \in \mathcal{N}$ receives

$$\boldsymbol{\vartheta}_i^a(k) = \sum_{j=1}^{\nu} \xi_{ij}(k) \boldsymbol{\tau}_j^a(k) = \sum_{j=1}^{\nu} \xi_{ij}(k) \boldsymbol{x}_j(k), \tag{6a}$$

$$\vartheta_i^b(k) = \sum_{j=1}^{\nu} \xi_{ij}(k) \tau_j^b(k) = \sum_{j=1}^{\nu} \xi_{ij}(k), \tag{6b}$$

where $\xi_{ij}(k) \in \mathbb{R}_{>0}$ is the (unknown) channel fading coefficient between transmitter j and receiver i at iteration $k \in \mathbb{N}_0$. Fading coefficients are unknown to agents. This explains the *privacy* benefit introduced in Section 2.

4. ALGORITHM

Each agent $i \in \mathcal{N}$, at every iteration $k \in \mathbb{N}_0$, obtains $\boldsymbol{\vartheta}_i^a(k)$ and $\vartheta_i^b(k)$, which may be used for updating $\boldsymbol{x}_i(k)$. $\boldsymbol{\vartheta}_i^a(k)$ represents a linear combination of the local estimates $\boldsymbol{x}_j(k)$ of all agents that can transmit information to agent i. $\vartheta_i^b(k)$ is the sum of the corresponding channel coefficients. The proposed algorithm is executed $\forall i \in \mathcal{N}$ and $\forall k \in \mathbb{N}_0$ and is as follows:

$$\boldsymbol{x}_i(k+1) = \boldsymbol{x}_i(k) - \frac{1}{\vartheta_i^b(k)} P_i \left(\vartheta_i^b(k)\boldsymbol{x}_i(k) - \boldsymbol{\vartheta}_i^a(k) \right), \quad (7)$$

where $P_i \in \mathbb{R}^{m \times m}$ is the orthogonal projection matrix onto the kernel of A_i. $\forall i \in \mathcal{N}$, the orthogonal projection matrix P_i is computed as follows, see (Spence et al., 2000, Ch. 7.3):

$$P_i := \kappa_i(\kappa_i'\kappa_i)^{-1}\kappa_i',$$

where $\kappa_i \in \mathbb{R}^{m \times \rho}$, $\rho = \text{nullity}(A_i)$, and its columns form a basis for the kernel of A_i.

Remark 1. $\forall \boldsymbol{v} \in \mathbb{R}^m$, $P_i\boldsymbol{v} \in \ker(A)_i$.

Remark 2. $\forall \boldsymbol{v} \in \mathbb{R}^m$, $A_iP_i\boldsymbol{v} = \mathbf{0}_{n_i}$.

Proposition 1. If $\boldsymbol{x}_i(k)$ solves (2) and $\boldsymbol{x}_i(k)$ is updated according to (7), then $\boldsymbol{x}_i(k+1)$ solves (2).

Proof. $\boldsymbol{x}_i(k+1)$ solves (2) iff $A_i\boldsymbol{x}_i(k+1) = b_i$, i.e., by (7),

$$A_i \left(\boldsymbol{x}_i(k) - \frac{1}{\vartheta_i^b(k)} P_i \left(\vartheta_i^b(k)\boldsymbol{x}_i(k) - \boldsymbol{\vartheta}_i^a(k) \right) \right) = b_i.$$

By Remark 2, the latter becomes

$$A_i\boldsymbol{x}_i(k) - \frac{1}{\vartheta_i^b(k)} A_iP_i \left(\vartheta_i^b(k)\boldsymbol{x}_i(k) - \boldsymbol{\vartheta}_i^a(k) \right) = A_i\boldsymbol{x}_i(k),$$

which is, by hypothesis of the proposition, equal to b_i. This concludes the proof.

Convergence to a global solution can be formalized as follows.

Theorem 1. A set $\mathcal{N}$ of communicating agents update their estimates according to (7). If $\mathfrak{G}$ is a sequence of fully connected graphs, then all $\boldsymbol{x}_i(k)$ converge to the same global solution of (1) notwithstanding the unknown fading channel.

Remark 3. The same result can be proven for $\mathfrak{G}$ being a sequence of repeatedly jointly $\mathcal{D}$-connected graphs, whose meaning is thoroughly analyzed in (Liu et al., 2017, Sec. 3). In this work, we assume a sequence of fully-connected network topologies, since this is a valid assumption in the context of analog transmissions.

5. PROOF OF THEOREM 1

The proof of Theorem 1 is inspired by Liu et al. (2017) and extends the proofs of Mou and Morse (2013) to the case of communication interference. We first deal with the case of (1) having a unique solution (i.e. $m = n$); after that, we extend the proof to the more general case (i.e. $m \geq n$).

5.1 Unique solution

Let $x^* \in \mathbb{R}^{m \times 1}$ be the unique solution of (1). Let's define error variables, i.e.,

$$\forall i \in \mathcal{N},\ \forall k \in \mathbb{N}_0, \quad \boldsymbol{e}_i(k) := \boldsymbol{x}_i(k) - x^*. \quad (8)$$

Clearly, all agents have the global solution if and only if, $\forall i \in \mathcal{N}$, $\boldsymbol{e}_i(k) = \mathbf{0}_m$. As in Molinari and Raisch (2019), we define normalized channel coefficients $h_{ij}(k)$, i.e., $\forall k \in \mathbb{N}_0,\ \forall i, j \in \mathcal{N}$,

$$h_{ij}(k) := \frac{\xi_{ij}(k)}{\sum_{j=1}^{\nu} \xi_{ij}(k)} \quad (9)$$

Remark 4. Normalized channel coefficients sum up to 1, i.e., $\forall i \in \mathcal{N}$, $\forall k \in \mathbb{N}_0$,

$$\sum_{j=1}^{\nu} h_{ij}(k) = 1. \quad (10)$$

Inserting (9) into (7) results in, $\forall k \in \mathbb{N}_0,\ \forall i \in \mathcal{N}$,

$$\boldsymbol{x}_i(k+1) = \boldsymbol{x}_i(k) - P_i \left(\boldsymbol{x}_i(k) - \sum_{j=1}^{\nu} h_{ij}(k)\boldsymbol{x}_j(k) \right). \quad (11)$$

Note that the individual coefficients $h_{ij}(k)$ in (11) are unknown. The only available information is that they sum up to 1. In order to bring (11) to a compact form, let $\boldsymbol{x}(k) \in \mathbb{R}^{\nu m}$ be the column vector stacking local solutions, i.e.,

$$\forall k \in \mathbb{N}_0, \quad \boldsymbol{x}(k) := [\boldsymbol{x}_1(k)', \ldots, \boldsymbol{x}_\nu(k)']'.$$

Similarly, let $\boldsymbol{x}^* \in \mathbb{R}^{\nu m \times 1}$ be the column stacking ν vectors x^*, i.e. $\boldsymbol{x}^* = [x^{*'}, \ldots, x^{*'}]'$. Moreover, let $P \in \mathbb{R}^{\nu m \times \nu m}$ be the blockdiagonal matrix composed of the ν orthogonal projection matrices P_i, i.e.,

$$P := \text{blockdiag}\left(P_1, \ldots, P_\nu\right).$$

Define the matrix $H(k) \in \mathbb{R}^{\nu \times \nu}$,

$$\forall k \in \mathbb{N}_0,\ \forall i \in \mathcal{N},\ \forall j \in \mathcal{N}, \quad [H(k)]_{ij} = h_{ij}(k).$$

Clearly, because of (10), $H(k)$ is row-stochastic, $\forall k \in \mathbb{N}_0$. Iteration (11) can now be rewritten as, $\forall k \in \mathbb{N}_0$,

$$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) - P\left(\boldsymbol{x}(k) - (H(k) \otimes \mathbf{I}_m)\boldsymbol{x}(k)\right), \quad (12)$$

where $H(k) \otimes \mathbf{I}_m$ denotes the Kronecker product of $H(k)$ and $\mathbf{I}_m$. Let, $\forall k \in \mathbb{N}_0$, $\boldsymbol{e}(k) \in \mathbb{R}^{\nu m}$ stack all $\boldsymbol{e}_i(k)$, i.e.

$$\boldsymbol{e}(k) := [\boldsymbol{e}_1(k)', \ldots, \boldsymbol{e}_\nu(k)']'. \quad (13)$$

Lemma 1. $\forall k \in \mathbb{N}_0$, $(H(k) \otimes \mathbf{I}_m)\boldsymbol{x}^* = \boldsymbol{x}^*$.

Proof.

$$(H(k) \otimes \mathbf{I}_m)\boldsymbol{x}^* = \begin{pmatrix} \sum_{j=1}^{\nu} h_{1j}(k)x^* \\ \vdots \\ \sum_{j=1}^{\nu} h_{\nu j}(k)x^* \end{pmatrix} = \boldsymbol{x}^*,$$

where the second equality is a consequence of (10).

Lemma 2. $\forall k \in \mathbb{N}_0$, $P\boldsymbol{e}(k) = \boldsymbol{e}(k)$.

Proof. Since $\boldsymbol{x}_i(k)$ is a local solution of (1), $A_i\boldsymbol{e}_i(k) = A_i(\boldsymbol{x}_i(k) - x^*) = b_i - b_i = 0$. Therefore, $\boldsymbol{e}_i(k) \in \ker(A_i)$. $\forall i \in \mathcal{N}$, P_i is the orthogonal projection onto the kernel of A_i, therefore, since $\boldsymbol{e}_i(k)$ is already in $\ker(A_i)$,

$$P_i\boldsymbol{e}_i(k) = \boldsymbol{e}_i(k).$$

This is valid $\forall i \in \mathcal{N}$, thus the proof is concluded.

By Lemma 1 and by subtracting $\boldsymbol{x}^*$ from both sides of (12), one gets

$$\boldsymbol{e}(k+1) = \boldsymbol{e}(k) - P\left(\boldsymbol{e}(k) - (H(k) \otimes \mathbf{I}_m)\boldsymbol{e}(k)\right).$$

By Lemma 2, the latter becomes, $\forall k \in \mathbb{N}_0$,

$$\boldsymbol{e}(k+1) = Q(k)\boldsymbol{e}(k), \quad (14)$$

where $Q(k) := P(H(k) \otimes \mathbf{I}_m)$. If the matrix sequence $\{Q(k)\}_{k\in\mathbb{N}_0}$ converges exponentially, Theorem 1 is proven. Similarly to Liu et al. (2017), we are going to employ a *mixed norm* for proving this.

Next, we discuss the mixed l_2/l_∞ vector norm and its induced matrix norm. In literature, the concept of mixed matrix norm is closely related to the subject of *norm compression of block-partitioned matrices*, see, e.g., Audenaert (2007), and it has been employed in the context of *compressed sensing*, see Eldar et al. (2010).

Assume $\boldsymbol{v} \in \mathbb{R}^{dm}$, $d, m \in \mathbb{N}$, Partition $\boldsymbol{v}$ as $\boldsymbol{v} = [\tilde{\boldsymbol{v}}_1', \dots, \tilde{\boldsymbol{v}}_m']'$, where $[\tilde{\boldsymbol{v}}_i]_j := [\boldsymbol{v}]_{(i-1)d+j}$, $i = 1, \dots, m$, $j = 1, \dots, d$. Define $\boldsymbol{w} \in \mathbb{R}^m_{\geq 0}$ by $[\boldsymbol{w}]_i = \|\tilde{\boldsymbol{v}}_i\|_2$, $i = 1, \dots, m$. Then the mixed l_2/l_∞ norm of $\boldsymbol{v}$ corresponding to the integer d is defined as

$$\|\boldsymbol{v}\|_{2,\infty} := \|\boldsymbol{w}\|_\infty . \tag{15}$$

Assume $A \in \mathbb{R}^{dm\times dm}$, $d, m \in \mathbb{N}$. Partition A as

$$A = \begin{bmatrix} \tilde{A}_{11} & \dots & \tilde{A}_{1m} \\ \dots & \dots & \dots \\ \tilde{A}_{m1} & \dots & \tilde{A}_{mm} \end{bmatrix},$$

where $\tilde{A}_{ij} \in \mathbb{R}^{d\times d}$ is given by $[\tilde{A}_{ij}]_{kl} := [A]_{(i-1)d+k,(j-1)d+l}$, $i, j = 1, \dots, m$ and $k, l = 1, \dots, d$. Define $B \in \mathbb{R}^{m\times m}_{\geq 0}$ by $[B]_{ij} = \left\|\tilde{A}_{ij}\right\|_2$ where the latter is the matrix norm induced by the l_2 vector norm. Then the mixed l_2/l_∞ norm of matrix A corresponding to the integer d is defined as

$$|||A|||_{2,\infty} := \|B\|_\infty ,$$

where $\|B\|_\infty$ is the norm of matrix B induced by the l_∞ vector norm. It is straightforward to show that $\|\cdot\|_{2,\infty}$ and $|||\cdot|||_{2,\infty}$ indeed satisfy all norm axioms. In Appendix A, we show that $|||\cdot|||_{2,\infty}$ is the matrix norm induced by the vector norm $\|\cdot\|_{2,\infty}$, i.e.

$$|||A|||_{2,\infty} = \sup_{\boldsymbol{v}\neq 0} \frac{\|A\boldsymbol{v}\|_{2,\infty}}{\|\boldsymbol{v}\|_{2,\infty}}.$$

Furthermore, (Liu et al., 2017, Lemma 3) have shown that $|||\cdot|||_{2,\infty}$ is submultiplicative, i.e., $|||A_1A_2|||_{2,\infty} \leq |||A_1|||_{2,\infty} |||A_2|||_{2,\infty}$.

Lemma 3. ((Mou et al., 2015, Lemma 2)). For any nonempty set of $m \times m$ real orthogonal projection matrices $\{T_1, \dots, T_\ell\}$,

$$\|T_\ell T_{\ell-1} \cdot \dots \cdot T_1\|_2 \leq 1. \tag{16}$$

In particular,

$$\|T_\ell T_{\ell-1} \cdot \dots \cdot T_1\|_2 < 1 \tag{17}$$

if and only if

$$\dim\left(\bigcap_{i=1}^{\ell} \mathrm{image}(T_i)\right) = 0. \tag{18}$$

Proof. See Mou et al. (2015).

Corollary 1. Given the projection matrices employed in (7),

$$\|P_{\ell_1} \cdot \dots \cdot P_{\ell_\nu}\|_2 < 1, \tag{19}$$

for $\{\ell_1 \dots \ell_\nu\} = \{1 \dots \nu\}$.

Proof. In case (1) has a unique solution, $\mathrm{rank}(A) = n = m$, thus $\dim(\ker(A)) = 0$. The latter implies that $\dim\left(\bigcap_{i=1}^{\nu} \ker(A_i)\right) = 0$. Since, $\forall i \in \mathcal{N}$, P_i is the orthogonal projection matrix onto the kernel of A_i, $\ker(A_i) = \mathrm{image}(P_i)$, then

$$\dim\left(\bigcap_{i=1}^{\nu} \mathrm{image}(P_i)\right) = 0,$$

which, by Lemma 3, implies (19). This concludes the proof.

Lemma 4. $\exists\gamma \in (0, 1)$:

$$\forall k \in \mathbb{N}_0, \quad \|\boldsymbol{e}(k+\nu)\|_{2,\infty} \leq \gamma \|\boldsymbol{e}(k)\|_{2,\infty} . \tag{20}$$

Proof. By (14),

$$\begin{aligned} \boldsymbol{e}(k+\nu) &= Q(k+\nu-1) \cdot \dots \cdot Q(k)\boldsymbol{e}(k), \\ &= \underbrace{P(H(k+\nu-1) \otimes \mathbf{I}_m) \cdot \dots \cdot P(H(k) \otimes \mathbf{I}_m)}_{:=Q_k^\nu} \boldsymbol{e}(k). \end{aligned}$$

Matrix $Q_k^\nu \in \mathbb{R}^{\nu m\times \nu m}$ can be seen as a block-matrix composed of $m \times m$ blocks of dimension $\nu \times \nu$. Each block i, j $(i, j \in \{1 \dots \nu\})$ is denoted by $Q_k^\nu[i, j]$ and is explicitly written as

$$\begin{aligned} &Q_k^\nu[i, j] = \\ &\sum_{\ell_1=1}^{m} \cdots \sum_{\ell_{\nu-1}=1}^{m} P_i h_{i\ell_1}(k+\nu-1) P_{\ell_1} h_{\ell_1\ell_2}(k+\nu-2) \cdot \ \dots \\ &\qquad \dots \ \cdot P_{\ell_{\nu-2}} h_{\ell_{\nu-2}\ell_{\nu}-1}(k+1) P_{\ell_{\nu-1}} h_{\ell_{\nu-1}j}(k). \end{aligned}$$

By the triangle inequality for norms,

$$\begin{aligned} \|Q_k^\nu[i, j]\|_2 \leq \sum_{\ell_1=1}^{m} \cdots \sum_{\ell_{\nu-1}=1}^{m} \left\|P_i \cdot \dots \cdot P_{\ell_{\nu-1}}\right\|_2 \cdot \dots \\ \dots \ \cdot h_{i\ell_1}(k+\nu-1) \cdot \dots \cdot h_{\ell_{\nu-1}j}(k). \end{aligned} \tag{21}$$

In general, by Lemma 3,

$$\forall\{i, \ell_1, \dots, \ell_{\nu-1}\}, \ \left\|P_i P_{\ell_1} \cdot \dots \cdot P_{\ell_{\nu-1}}\right\|_2 \leq 1. \tag{22}$$

Among all considered ν-dimensional sets $\{i, \ell_1, \dots, \ell_{\nu-1}\}$ in (22), there is also set $\{1 \dots \nu\}$. For this set, by Corollary 1,

$$\left\|P_i P_{\ell_1} \cdot \dots \cdot P_{\ell_{\nu-1}}\right\|_2 < 1. \tag{23}$$

The corresponding product of channel coefficients is always positive, i.e.,

$$h_{i\ell_1}(k+\nu-1) h_{\ell_1\ell_2}(k+\nu-2) \cdot \dots \cdot h_{\ell_{\nu-1}j}(k) > 0. \tag{24}$$

Inserting (23) and (24) into (21) yields

$$\begin{aligned} &\|Q_k^\nu[i, j]\|_2 < \\ &\quad \sum_{\ell_1=1}^{m} \cdots \sum_{\ell_{\nu-1}=1}^{m} h_{i\ell_1}(k+\nu-1) \cdot \dots \cdot h_{\ell_{\nu-1}j}(k). \end{aligned} \tag{25}$$

To prove the Lemma, we need to show that

$$\forall k \in \mathbb{N}_0, \ |||Q_k^\nu|||_{2,\infty} \leq \gamma < 1.$$

By definition of l_2/l_∞ mixed matrix norm,

$$|||Q_k^\nu|||_{2,\infty} = \max_{i=1\dots m} \left(\sum_{j=1}^{m} \|Q_k^\nu[i, j]\|_2\right). \tag{26}$$

By Remark 4, the rows of $H(k)$ sum up to 1, $\forall k \in \mathbb{N}_0$, therefore, $\forall k \in \mathbb{N}_0$, $\forall i \in \mathcal{N}$,

$$\sum_{j=1}^{m} \sum_{\ell_1=1}^{m} \cdots \sum_{\ell_{\nu-1}=1}^{m} h_{i\ell_1}(k+\nu-1) \cdot \dots \cdot h_{\ell_{\nu-1}j}(k) =$$

$$= \left(\sum_{\ell_1=1}^{m} h_{i\ell_1}(k+\nu-1) \cdot \dots \cdot \left(\sum_{j=1}^{m} h_{\ell_{\nu-1}j}(k)\right) \cdots \right) = 1.$$

Hence, using (25),

$$\forall k \in \mathbb{N}_0,\ \forall i \in \mathcal{N}, \quad \sum_{j=1}^{m} \|Q_k^\nu[i,j]\|_2 < 1,$$

thus, by (26),

$$\forall k \in \mathbb{N}_0, \quad |||Q_k^\nu|||_{2,\infty} < 1. \tag{27}$$

Note that set $\{Q_k^\nu\}_{k\in\mathbb{N}_0}$ is finite. This is a realistic assumption if we consider the presence of a quantization effect on the fading channel, which implies that set $\{H(k)\}_{k\in\mathbb{N}_0}$ is finite. By definition of Q_k^ν, if $\{H(k)\}_{k\in\mathbb{N}_0}$ is finite, then also $\{Q_k^\nu\}_{k\in\mathbb{N}_0}$ is finite. By this consideration and (27),

$$\gamma := \sup_{k\in\mathbb{N}_0} |||Q_k^\nu|||_{2,\infty} < 1. \tag{28}$$

By definition of the induced vector norm,

$$\forall k \in \mathbb{N}_0, \quad \|\boldsymbol{e}(k+\nu)\|_{2,\infty} \leq |||Q_k^\nu|||_{2,\infty} \|\boldsymbol{e}(k)\|_{2,\infty},$$

which, by (28), yields (20) thus concluding the proof of Lemma 4.

The proof for Theorem 1 follows right from Lemma 4. In fact, by (20),

$$\lim_{k\to\infty} \|\boldsymbol{e}(k\nu)\|_{2,\infty} \leq \gamma^k \|\boldsymbol{e}(0)\|_{2,\infty}. \tag{29}$$

Since $\|\cdot\|_{2,\infty}$ is a norm (29) yields that the error approaches 0 exponentially, therefore the local estimates $\boldsymbol{x}_i(k)$ approach x^* exponentially (independently of the unknown and time-varying normalized channel coefficients $h_{ij}(k)$).

5.2 Multiple solutions

For proving convergence in case of multiple solutions, we make use of the same tools employed by (Liu et al., 2017, Sec. 4.2). This is possible because, in the previous section, we have reduced (7) to a form similar to what presented in Liu et al. (2017). For this scenario, in fact, Corollary 1 cannot be used since having more than one solution implies that $\dim\left(\bigcap_{i=1}^{\nu} \ker(A_i)\right) \neq 0$.

Define the subspace $\mathcal{P}$ as

$$\mathcal{P} := \bigcap_{i=1}^{\nu} \mathrm{image}(P_i) = \bigcap_{i=1}^{\nu} \ker(A_i),$$

and $\tilde{m} = m - \dim(\mathcal{P})$.

Definition 2. Let the columns of the $m \times \tilde{m}$ matrix L' be an orthonormal basis for the orthogonal complement of $\mathcal{P}$.

We define the following $\tilde{m} \times \tilde{m}$ matrix, $\forall i \in \mathcal{N}$,

$$\bar{P}_i := L P_i L'.$$

Lemma 5. ((Mou et al., 2015, Lemma 1)). The following statements are true

(1) $\forall i \in \mathcal{N}$, $\bar{P}_i$ is an orthogonal projection matrix;
(2) $\forall i \in \mathcal{N}$, $LP_i = \bar{P}_i L$
(3) $\forall i \in \mathcal{N}$, $P_i L' = L' \bar{P}_i$;
(4) $\bigcap_{i=1}^{\nu} \mathrm{image}(\bar{P}_i) = 0$.

Proof. See (Mou et al., 2015, Lemma 1).

Corollary 2. $\forall i,j \in \mathcal{N}$, $LP_iP_j = \bar{P}_i\bar{P}_j L$.

Proof. The proof follows directly from point (2) of Lemma 5.

In what follows, we consider two different sets of transformed error variables, i.e.,

$$\forall i \in \mathcal{N}, \qquad \bar{\boldsymbol{e}}_i(k) := L\boldsymbol{e}_i(k) \in \mathbb{R}^{\tilde{m}} \tag{30}$$

and

$$\forall i \in \mathcal{N}, \qquad \tilde{\boldsymbol{e}}_i(k) := \boldsymbol{e}_i(k) - L'\bar{\boldsymbol{e}}_i(k) \in \mathbb{R}^m. \tag{31}$$

Lemma 6. $\forall i \in \mathcal{N}$, $\bar{P}_i\bar{\boldsymbol{e}}_i(k) = \bar{\boldsymbol{e}}_i(k)$.

Proof. By (30), $\bar{P}_i\bar{\boldsymbol{e}}_i(k) = \bar{P}_iL\boldsymbol{e}_i(k)$. By property (2) of Lemma 5, $\bar{P}_iL\boldsymbol{e}_i(k) = LP_i\boldsymbol{e}_i(k)$, which, by Lemma 2, yields

$$\bar{P}_i\bar{\boldsymbol{e}}_i(k) = LP_i\boldsymbol{e}_i(k) = L\boldsymbol{e}_i(k) = \bar{\boldsymbol{e}}_i(k),$$

thus concluding the proof.

Lemma 7. $\forall i,j \in \mathcal{N}$, $P_j\tilde{\boldsymbol{e}}_i(k) = \tilde{\boldsymbol{e}}_i(k)$.

Proof. Note that

$$L\tilde{\boldsymbol{e}}_i(k) = L\boldsymbol{e}_i(k) - LL'\bar{\boldsymbol{e}}_i(k) = L\boldsymbol{e}_i(k) - \bar{\boldsymbol{e}}_i(k) = 0,$$

since the columns of L are orthonormal. This implies $\tilde{\boldsymbol{e}}_i(k) \in \ker(L)$, which yields $\tilde{\boldsymbol{e}}_i(k) \in \bigcap_{j=1}^{\nu} \mathrm{image}(P_j)$, then, $\forall i,j \in \mathcal{N}$, $P_j\tilde{\boldsymbol{e}}_i(k) = \tilde{\boldsymbol{e}}_i(k)$. This concludes the proof.

Lemma 8. If, $\forall i \in \mathcal{N}$,

$$\lim_{k\to\infty} \bar{\boldsymbol{e}}_i(k) = \mathbf{0}_{\tilde{m}}$$

and

$$\lim_{k\to\infty} \tilde{\boldsymbol{e}}_i(k) = \boldsymbol{\epsilon}^*,$$

with $\boldsymbol{\epsilon}^* \in \mathbb{R}^m$, then, $\forall i \in \mathcal{N}$,

$$\lim_{k\to\infty} \boldsymbol{x}_i(k) = \boldsymbol{\epsilon}^* + x^*.$$

Proof. By merging (8) and (31), one obtains

$$\boldsymbol{x}_i(k) = \tilde{\boldsymbol{e}}_i(k) + L'\bar{\boldsymbol{e}}_i(k) + x^*.$$

The hypotheses of the Lemma yield

$$\lim_{k\to\infty} \boldsymbol{x}_i(k) = \lim_{k\to\infty} \tilde{\boldsymbol{e}}_i(k) + L'\bar{\boldsymbol{e}}_i(k) + x^* = \boldsymbol{\epsilon}^* + x^*,$$

which concludes the proof.

If the hypotheses of Lemma 8 are proven to hold for the problem at hand, then the local solution of each agent will converge to the same value. This can be done as follows.

By (11), (30), and Corollary 2, $\forall k \in \mathbb{N}_0$,

$$\bar{\boldsymbol{e}}_i(k+1) = \bar{P}_i \sum_{j=1}^{m} h_{ij}(k)\bar{\boldsymbol{e}}_j(k), \tag{32}$$

which in compact form becomes, $\forall k \in \mathbb{N}_0$,

$$\bar{\boldsymbol{e}}(k+1) = \bar{Q}(k)\bar{\boldsymbol{e}}(k), \tag{33}$$

where $\bar{Q}(k) := \mathrm{blockdiag}\left(\bar{P}_1, \ldots, \bar{P}_\nu\right)(H(k) \otimes \mathbf{I}_m)$. Equations (33) and (14) are the same, apart from having $\bar{P}_i$ instead of P_i. By Lemma 5, $\bar{P}_i$ is also an orthogonal projection matrix, $i = 1 \ldots \nu$, and $\bigcap_{i=1}^{\nu} \mathrm{image}(\bar{P}_i) = 0$. We can therefore repeat the argument from Section 5.1 to show that the mixed l_2/l_∞ norm of the product $\bar{Q}(k+\nu-1)\ldots\bar{Q}(k)$ is strictly smaller than 1. Therefore, $\bar{\boldsymbol{e}}(k)$ converges exponentially to $\mathbf{0}_{\nu\tilde{m}}$, i.e., $\forall i \in \mathcal{N}$,

$$\lim_{k\to\infty} \bar{\boldsymbol{e}}_i(k) = \mathbf{0}_{\tilde{m}}. \tag{34}$$

Concerning $\tilde{\boldsymbol{e}}_i(k)$, by its definition,

$$\begin{aligned}\tilde{\boldsymbol{e}}_i(k+1) &= \boldsymbol{e}_i(k+1) - L'\bar{\boldsymbol{e}}_i(k+1)\\ &= P_i \sum_{j=1}^{m} h_{ij}(k)\boldsymbol{e}_j(k) - L'\bar{P}_i \sum_{j=1}^{m} h_{ij}(k)\bar{\boldsymbol{e}}_j(k),\end{aligned}$$

which, by property (3) in Lemma 5, becomes

$$= P_i \sum_{j=1}^{m} h_{ij}(k) \boldsymbol{e}_j(k) - P_i L' \sum_{j=1}^{m} h_{ij}(k) \bar{\boldsymbol{e}}_j(k),$$
$$= P_i \sum_{j=1}^{m} h_{ij}(k) \tilde{\boldsymbol{e}}_j(k).$$

The latter, by Lemma 7, yields, $\forall i \in \mathcal{N}$, $\forall k \in \mathbb{N}_0$,

$$\tilde{\boldsymbol{e}}_i(k+1) = \sum_{j=1}^{m} h_{ij}(k) \tilde{\boldsymbol{e}}_j(k). \tag{35}$$

By (Molinari et al., 2018a, Sec. 3.2), being the underlying network topology fully connected, the system in (35) achieves consensus, namely, $\forall i \in \mathcal{N}$,

$$\lim_{k \to \infty} \tilde{\boldsymbol{e}}_i(k) = \boldsymbol{\epsilon}^*, \tag{36}$$

with $\boldsymbol{\epsilon}^*$ unique and bounded. Under the hypotheses of the Theorem, equations (34) and (36) hold. These are the hypotheses of Lemma 8, which, in turn, implies that all $\boldsymbol{x}_i(k)$ converge to the same solution. This shows that Theorem 1 is proven also in case of (1) having multiple solutions.

6. SIMULATIONS

In what follows, we simulate a set of agents trying to solve (1) by running algorithm (7) together with the communication protocol (6), under different conditions. Also channel coefficients $\xi_{ij}(k)$ of each fully connected graph $\mathcal{G}(k)$ are drawn out of an uniform distribution [1], i.e., $\forall i, j \in \mathcal{N}$, $\forall k \in \mathbb{N}_0$, $\xi_{ij}(k) \sim \mathcal{U}(0, \mu]$, where $\mu \in \mathbb{R}_{>0}$. In all simulations, $\nu = m$, thus, $\forall i \in \mathcal{N}$, $n_i = 1$. The matrices of equation (1) are in Appendix B.

6.1 Unique Solution

We first analyze the case of (1) having a unique solution, namely $n = m$ and A full-row rank. Fig. 1 represents the evolution of $\boldsymbol{x}_i(k)$, $i = 1 \ldots \nu$, through iterations. The evolution of $\|\boldsymbol{e}(k)\|_{2,\infty}$ through iterations can be seen in Fig. 2. In the same figure, we have plotted also $\log(\|\boldsymbol{e}(k)\|_{2,\infty})$, thus showing the exponential convergence to a solution.

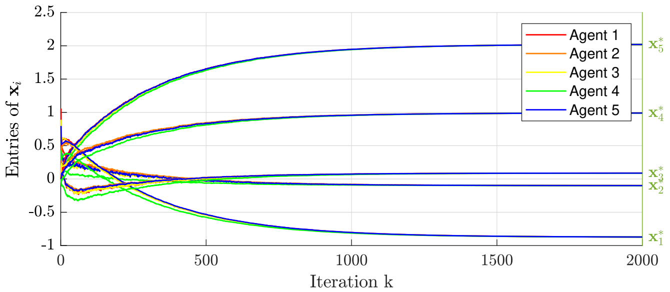

Fig. 1. Convergence to the solution for (1) having an unique solution.

[1] They are independent and identically distributed.

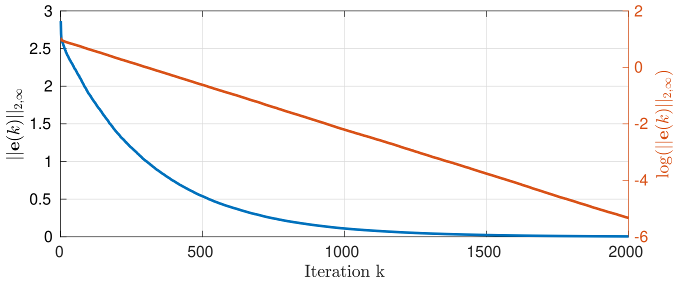

Fig. 2. $\|\boldsymbol{e}(k)\|_{2,\infty}$ and $\log(\|\boldsymbol{e}(k)\|_{2,\infty})$ through iterations.

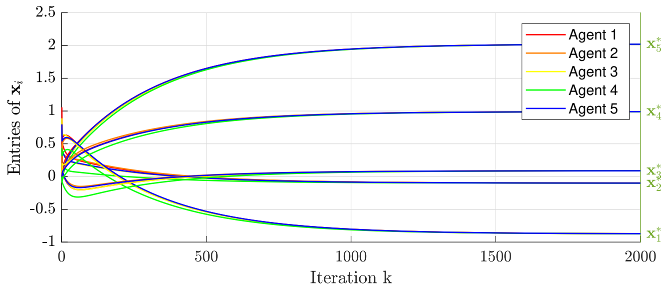

Fig. 3. $\boldsymbol{x}_i(k)$ through iterations, with an ideal channel.

6.2 Ideal Channel

The impact of the unknown fading channel can be addressed by looking at Fig. 3, where uncertainty and fading are removed, namely, $\forall i, j \in \mathcal{N}$, $\forall k \in \mathbb{N}_0$, $\xi_{ij}(k) = 1$. Note that this can only be done using *standard* orthogonal channel access methods, i.e., at every iteration step, all agents would have to transmit the current estimations independently. Hence, if this is done, e.g., by TDMA (Time-division multiple access), we would need νm orthogonal transmissions per iteration. The comparison of Fig. 1 and Fig. 3 illustrates that the presence of an unknown and fading channel does not have a noticeable impact.

Comparing Fig. 1 with Fig. 3 helps quantifying the savings of wireless resources in case interference is exploited. In fact, although Fig. 1 and Fig. 3 exhibit the same convergence rate, the traditional communication approach (in which agents access the channel separately) requires at least $m\nu$ independent channel accesses per every iteration. In fact, each agent $(1 \ldots \nu)$ separately communicates to neighbors each entry $(1 \ldots m)$ of its information state. By Section 3.2, our designed communication system requires only $m + 1$, but guarantees the same convergence rate in terms of iterations as the traditional approach. Exploiting interference, therefore, significantly reduces the consumption of wireless resources.

6.3 Multiple Solution

In case (1) has multiple solutions, e.g., $n = m - 1$ (see Appendix B for the matrices), the convergence of $\boldsymbol{x}_i(k)$, $i = 1 \ldots \nu$, to the solution is depicted by Fig. 4.

6.4 Noisy Fading Channel

We try now to relax assumption (4) and consider a noisy fading channel, in which $\eta \neq 0$. This case is only considered in simulation here. A theoretical investigation is the topic

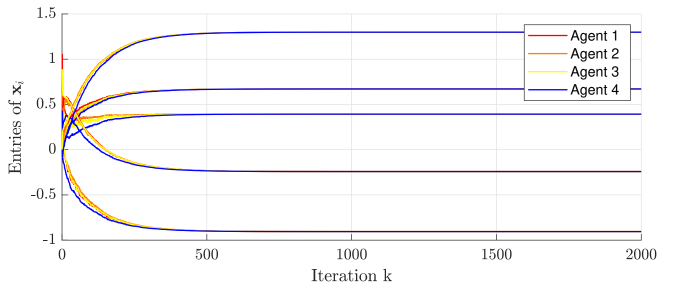

Fig. 4. $\boldsymbol{x}_i(k)$ through iterations for the case of multiple solutions.

of ongoing work. Let additive noise be white Gaussian and affect both $\boldsymbol{\vartheta}_i^a(k)$ and $\vartheta_i^b(k)$, such that SNR (signal to noise ratio) equals $1e-3$. The evolution of $\boldsymbol{x}_i(k)$ in this case is depicted in Fig. 5. The analysis of $\log(\|\boldsymbol{e}(k)\|_{2,\infty})$ in Fig. 6 shows that the error converges until up to a point from where the presence of noise is prohibiting the exponential convergence.

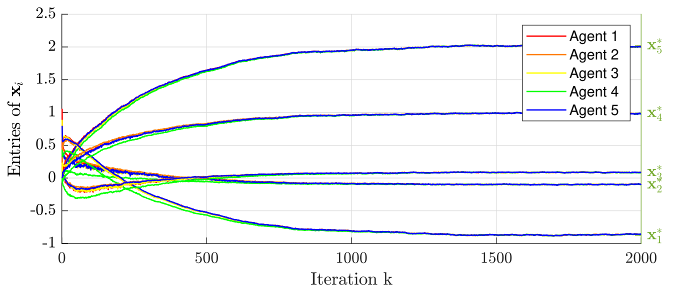

Fig. 5. $\boldsymbol{x}_i(k)$ through iterations in presence of additive white Gaussian noise.

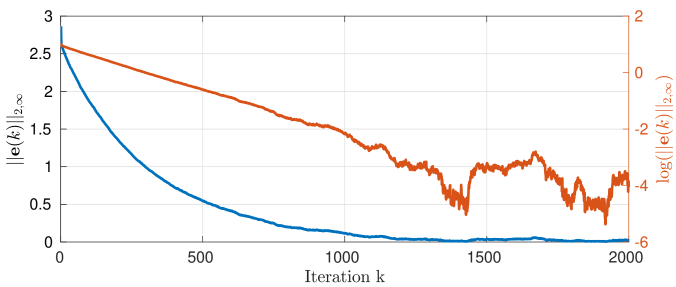

Fig. 6. $\|\boldsymbol{e}(k)\|_{2,\infty}$ and $\log(\|\boldsymbol{e}(k)\|_{2,\infty})$ through iterations, with a noisy fading channel.

7. CONCLUSION

This paper has investigated a way to exploit wireless interference for the distributed solution of linear algebraic equations. Combining the designed communication system together with the designed iterative algorithm yields a method which guarantees exponential convergence to a solution notwithstanding the presence of unknown fading channel coefficients. Simulations have confirmed the theoretical analysis.

In ongoing work, we aim at extending the strategy to the case of a noisy fading channel. Moreover, we aim at developing a real-life demonstrator to prove all benefits of employing interference for the distributed solution of linear algebraic equations.

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Appendix A. MIXED NORMS

A.1 Induced mixed norms

The two defined matrix and vector norms are *compatible* if

$$\|A\boldsymbol{v}\|_{2,\infty} \leq |||A|||_{2,\infty} \, \|\boldsymbol{v}\|_{2,\infty}\,, \tag{A.1}$$

for any matrix $A \in \mathbb{R}^{dm\times dm}$ and any vector $\boldsymbol{v} \in \mathbb{R}^{dm}$. In fact, by the definition of $\|\cdot\|_{2,\infty}$ and by the triangular inequality for $\|\cdot\|_2$,

$$\begin{aligned}
\|A\boldsymbol{v}\|_{2,\infty} &= \max_{i=1\dots m}\left(\left\|\sum_{\ell=1}^{m}\tilde{A}_{i\ell}\tilde{\boldsymbol{v}}_\ell\right\|_2\right)\\
&\leq \max_{i=1\dots m}\left(\sum_{\ell=1}^{m}\left\|\tilde{A}_{i\ell}\right\|_2\|\tilde{\boldsymbol{v}}_\ell\|_2\right)\\
&\leq \max_{i=1\dots m}\left(\sum_{\ell=1}^{m}\left\|\tilde{A}_{i\ell}\right\|_2\right)\max_{\ell=1\dots m}\left(\|\tilde{\boldsymbol{v}}_\ell\|_2\right)\\
&= |||A|||_{2,\infty}\,\|\boldsymbol{v}\|_{2,\infty}\,.
\end{aligned}$$

This proves that the presented mixed-matrix and mixed-vector norms are compatible. Moreover, we aim at showing that $|||\cdot|||_{2,\infty}$ is *induced* by $\|\cdot\|_{2,\infty}$. By (Horn and Johnson, 2012, Def. 5.6.1), this is the case if

$$|||A|||_{2,\infty} = \sup_{\|\boldsymbol{v}\|_{2,\infty}\neq 0}\frac{\|A\boldsymbol{v}\|_{2,\infty}}{\|\boldsymbol{v}\|_{2,\infty}}. \tag{A.2}$$

Since (A.1) is true, condition (A.2) is proven if $\exists \boldsymbol{v}$ for which we get

$$|||A|||_{2,\infty} = \frac{\|A\boldsymbol{v}\|_{2,\infty}}{\|\boldsymbol{v}\|_{2,\infty}}. \tag{A.3}$$

To verify this, let's expand the mixed-matrix norm, so that

$$|||A|||_{2,\infty} = \max_{i=1\dots m}\left(\sum_{\ell=1}^{m}\left\|\tilde{A}_{i\ell}\right\|_2\right). \tag{A.4}$$

Since the spectral norm for matrix is induced by the l_2 vector norm,

$$\max_{i=1\dots m}\left(\sum_{\ell=1}^{m}\left\|\tilde{A}_{i\ell}\right\|_2\right) = \max_{i=1\dots m}\left(\sum_{\ell=1}^{m}\left\|\tilde{A}_{i\ell}\tilde{\boldsymbol{w}}_\ell\right\|_2\right), \tag{A.5}$$

for some $\boldsymbol{w}_\ell$, $\ell = 1\dots m$, such that $\|\tilde{\boldsymbol{w}}_\ell\|_2 = 1$, $\forall \ell = 1\dots m$. By the triangular inequality applied to (A.5),

$$\max_{i=1\dots m}\left(\sum_{\ell=1}^{m}\left\|\tilde{A}_{i\ell}\tilde{\boldsymbol{w}}_\ell\right\|_2\right) \geq \max_{i=1\dots m}\left(\left\|\sum_{\ell=1}^{m}\tilde{A}_{i\ell}\tilde{\boldsymbol{w}}_\ell\right\|_2\right), \tag{A.6}$$

where the right-hand side equals $\|A\boldsymbol{w}\|_{2,\infty}$. Also, by definition of l_2/l_∞ vector norm,

$$\|\boldsymbol{w}\|_{2,\infty} = \max_{\ell=1\dots m}\left(\|\tilde{\boldsymbol{w}}_\ell\|_2\right) = 1,$$

since $\|\tilde{\boldsymbol{w}}_\ell\|_2 = 1$, $\forall \ell = 1\dots m$.

By merging all these considerations, one obtains

$$|||A|||_{2,\infty} \geq \frac{\|A\boldsymbol{w}\|_{2,\infty}}{\|\boldsymbol{w}\|_{2,\infty}}. \tag{A.7}$$

By the latter and (A.1), equation (A.3) is verified.

Appendix B. SIMULATION DATA

B.1 Unique Solution

$$A = \begin{pmatrix}
\frac{22}{25} & \frac{61}{100} & \frac{31}{50} & \frac{1}{10} & \frac{8}{25}\\
\frac{27}{50} & \frac{3}{100} & \frac{19}{100} & \frac{49}{100} & \frac{23}{50}\\
\frac{1}{100} & \frac{8}{25} & \frac{81}{100} & \frac{8}{25} & \frac{19}{25}\\
\frac{87}{100} & \frac{8}{25} & \frac{22}{25} & \frac{18}{25} & \frac{53}{100}\\
\frac{13}{100} & \frac{41}{100} & \frac{59}{100} & \frac{57}{100} & \frac{3}{25}
\end{pmatrix}, \qquad B = \begin{pmatrix}
\frac{93}{100}\\ \frac{17}{50}\\ \frac{18}{25}\\ \frac{19}{50}\\ \frac{47}{100}
\end{pmatrix}$$

B.2 Multiple Solutions

$$A = \begin{pmatrix}
\frac{22}{25} & \frac{61}{100} & \frac{31}{50} & \frac{1}{10} & \frac{8}{25}\\
\frac{27}{50} & \frac{3}{100} & \frac{19}{100} & \frac{49}{100} & \frac{23}{50}\\
\frac{1}{100} & \frac{8}{25} & \frac{81}{100} & \frac{8}{25} & \frac{19}{25}\\
\frac{87}{100} & \frac{8}{25} & \frac{22}{25} & \frac{18}{25} & \frac{53}{100}
\end{pmatrix}, \qquad B = \begin{pmatrix}
\frac{93}{100}\\ \frac{17}{50}\\ \frac{18}{25}\\ \frac{19}{50}
\end{pmatrix}$$