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Observer Based Nonlinear Control of a Rotating Flexible Beam ⋆

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Abstract: This paper presents an observer based nonlinear control for a flexible beam clamped on a rotating inertia. The considered model is composed by a set of Partial Differential Equations (PDEs) interconnected with an Ordinary Differential Equation (ODE), with control input in the ODE. The control problem consists in orienting the beam at the desired position, maintaining the flexible vibrations as low as possible. To this end, it is presented a nonlinear controller that depends on the beam's state. An Observer is designed to reconstruct the infinite dimensional state, and the estimated state is used in the nonlinear controller instead of the real one. Assuming well-posedness of the closed loop system, it is shown the exponential convergence of the estimated state, and the asymptotic stability of the closed loop system. Numerical simulations are presented to characterize the closed loop behaviour with different choices of observer's parameters.

Keywords: Distributed-parameter system, Nonlinear control, Observers, Asymptotic stability, port-Hamiltonian system.

1. INTRODUCTION

Control of flexible robots has been an highly investigated topic over the last 50 years. The need of precise controllers and stability requirements made necessary to take into consideration distributed flexible phenomena. These processes are modelled using Partial Differential Equations (PDEs), where the state variables are space and time dependent. In the specific case of a rotating flexible beam, the inertia of the hub to which the beam is connected (i.e. the rotor of a motor) cannot be neglected. This scenario brings to a system modelled by an interconnection between a set of PDEs and an ODE, with control input on the ODE. In the literature, the control of a coupled set of PDEs and ODEs is often referred as control of Hybrid systems (Luo et al., 1999). The design of stabilizing controllers for rotating flexible beams can been addressed with the use of a PD controller (Luo and Feng, 1999), but different control methods have been employed to have a faster vibration suppression. In (Morgul, 1991) is shown the asymptotic stability of a rotating Euler-Bernoulli beam with a PD + strain feedback control, while in (Wang et al., 2017) is used a feed-forward control law obtained by model inversion to minimize the flexible vibrations during motion. Another possible strategy is to include in the controller the information about the deformation of the beam. To do so in a passive preserving way, it is necessary to design a nonlinear dynamic controller (Luo and Feng, 1999). This control law have been rewritten as a passively interconnected port-Hamiltonian (PH) system in (Aoues et al., 2019), where Lyapunov stability has been proved. Functional analysis is a powerful tool for studying the asymptotic behaviour of dynamical system described by PDEs or Hybrid systems (Curtain and Zwart, 1995). In the last decades, it has been successfully exploited for the stability study of the class of infinite dimensional PH systems(Jacob and Zwart, 2012; Villegas, 2007; Le Gorrec et al., 2005), that are obtained as an extension of the finite dimensional PH systems (van der Schaft and Maschke, 2002). In a more general fashion, the control problem of nonlinear feedback for a class of infinite dimensional PH systems has been presented in (Ramirez et al., 2017), where conditions for asymptotic and exponential stability are given. In (Mileti et al., 2016), precompactness of trajectories combined with the existence of a limit set is used to prove the asymptotic stability for an Euler-Bernoulli beam subject to a class of nonlinear feedbacks.

In this manuscript we propose a similar control law as proposed in (Luo and Feng, 1999) where, since in practical applications the state of the beam is not directly available, the controller makes use of an observed state instead of the original one. The beam is modelled using the Timoshenko's beam assumptions, and the closed loop system is composed by two linear sets of PDEs interconnected with a nonlinear set of ODEs, with the nonlinearity depending on the infinite dimensional state.

The paper is organized as follows. In Section 2 the PH model of a flexible beam clamped on a rotating inertia and the observer based control design are given. In section 3 is proven the exponential convergence of the observer and the asymptotic stability of the closed-loop system. In Section 4 are shown numerical simulations, while some concluding remarks and comments on future works are given in Section 5.

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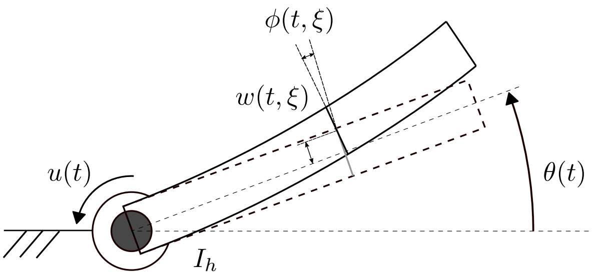

Fig. 1: Rotating flexible Timoshenko's beam.

2. MODELLING AND CONTROL DESIGN

In the following we propose the equations of a rotating flexible beam using Timoshenko's assumptions in the PH framework.

2.1 Modelling

For a sake of clarity, we define the variables and the parameters that are used for the modelling of the system in Figure 1. The rotor angle $\theta(t)$ is a real function of time, while $\xi \in [0, L]$ identify the spatial coordinate of the beam. The deflection of the beam in the rotating frame is defined with $w(t,\xi)$, while $\phi(t,\xi)$ represents the relative rotation of the beam cross section. All the physical parameters are positive real. I_h represents the rotary inertia of the hub to which the beam is connected. E, I are respectively the Young's modulus and the moment of inertia of the beam's cross section. The beam's cross section is assumed to be rectangular, hence its inertia is defined to be $I = \frac{L_w^3 L_t}{12}$, where L_w and L_t are respectively the width and the thickness of the beam. ρ, I_ρ are respectively the density and the mass moment of inertia of the beam's cross section. The mass moment of inertia of the cross section is defined as $I_\rho = I\rho$. K is defined as $K = kGA$, where k is a constant depending on the shape of the cross section ($k = 5/6$ for rectangular cross sections), G is the Shear modulus and A is the cross sectional area.
From now on we will not explicit the dependency from time and space of the variables when it is clear from the context. The kinetic energy H_k and the potential energy H_p, using Timoshenko's assumptions, write

$$H_k = \frac{1}{2} I_h \dot{\theta}^2 + \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t} + \xi\dot{\theta}\right)^2 + I_\rho\left(\frac{\partial \phi}{\partial t} + \dot{\theta}\right)^2\right] d\xi$$
$$H_p = \frac{1}{2}\int_0^L \left[K\left(\frac{\partial w}{\partial \xi} - \phi\right)^2 + EI\left(\frac{\partial \phi}{\partial \xi}\right)^2\right] d\xi.$$

The Hamilton's principle is used to obtain the system's dynamical equations, considering $W_{nc} = u(t)\theta$ the work of non-conservative forces, where $u(t)$ identify the external torque. The derived set of mixed partial and ordinal differential equations write

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\rho\left(\dfrac{\partial w}{\partial t} + \xi\dot{\theta}\right)\right) = \dfrac{\partial}{\partial \xi}\left(K\left(\dfrac{\partial w}{\partial \xi} - \phi\right)\right) \\ \dfrac{\partial}{\partial t}\left(I_\rho\left(\dfrac{\partial \phi}{\partial t} + \dot{\theta}\right)\right) = \dfrac{\partial}{\partial \xi}\left(EI\dfrac{\partial \phi}{\partial \xi}\right) + K\left(\dfrac{\partial w}{\partial \xi} - \phi\right) \\ I_h\ddot{\theta} = +EI\dfrac{\partial \phi(0,t)}{\partial \xi} + u(t). \end{cases} \tag{1}$$

With boundary conditions

$$\begin{aligned} &w(0,t) = 0 && \phi(0,t) = 0 \\ &\frac{\partial w}{\partial \xi}(L,t) - \phi(L,t) = 0 && \frac{\partial \phi}{\partial \xi}(L,t) = 0. \end{aligned} \tag{2}$$

The energy states of the infinite dimensional system are defined by

$$\begin{aligned} \varepsilon_t &= \frac{\partial w}{\partial \xi} - \phi \quad p_t = \rho\left(\frac{\partial w}{\partial t} + \xi\dot{\theta}\right) \\ \varepsilon_r &= \frac{\partial \phi}{\partial \xi} \qquad\quad p_r = I_\rho\left(\frac{\partial \phi}{\partial t} + \dot{\theta}\right). \end{aligned} \tag{3}$$

The equations describing the infinite dimensional system can be written as a PH system

$$\dot{x}_b = \mathcal{J}x_b = P_1\frac{\partial}{\partial \xi}(\mathcal{H}_b x_b) + P_0(\mathcal{H}_b x_b) \tag{4}$$

with $x_b = [p_t\ p_r\ \varepsilon_t\ \varepsilon_r]^T \in X_b \subset L_2([0,L], \mathbb{R}^4)$ representing the system's state. The matrices in equation (4) are defined as

$$P_1 = \begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix}\ P_0 = \begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0 \end{bmatrix}\ \mathcal{H}_b = \begin{bmatrix} \rho^{-1}&0&0&0\\0&I_\rho^{-1}&0&0\\0&0&K&0\\0&0&0&EI \end{bmatrix}.$$

The state space X_b is equipped with the L_2 inner product $\langle x_b, x_b\rangle_{X_b} = \langle x_b, \mathcal{H}_b x_b\rangle_{L_2}$, such to express the energy related to the flexible part of the system as $H_b = \frac{1}{2}\langle x_b, x_b\rangle_{X_b}$. The boundary variables are defined as (Le Gorrec et al., 2005)

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}\begin{bmatrix} (\mathcal{H}_b x_b)(t,0) \\ (\mathcal{H}_b x_b)(t,L) \end{bmatrix}.$$

Then, define the boundary input and output operators as

$$\begin{aligned} u_{b,1} &= \mathcal{B}_1(\mathcal{H}_b x_b) = W_1\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = -I_\rho^{-1}p_r(0,t) \\ u_{b,2} &= \mathcal{B}_2(\mathcal{H}_b x_b) = W_2\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} -\rho^{-1}p_t(0,t) \\ K\varepsilon_t(L,t) \\ EI\varepsilon_r(L,t) \end{bmatrix} \\ y_{b,1} &= \mathcal{C}_1(\mathcal{H}_b x_b) = \tilde{W}_1\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = EI\varepsilon_r(0,t) \\ y_{b,2} &= \mathcal{C}_2(\mathcal{H}_b x_b) = \tilde{W}_2\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} K\varepsilon_t(0,t) \\ \rho^{-1}p_t(L,t) \\ I_\rho^{-1}p_r(L,t) \end{bmatrix} \end{aligned} \tag{5}$$

where $W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ and $\tilde{W} = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix}$ are appropriate matrices, and are such that $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ is non-singular. The total boundary input-output operators are defined as

$$\begin{aligned} \mathcal{B}(\mathcal{H}_b x_b) &= \begin{bmatrix} \mathcal{B}_1(\mathcal{H}_b x_b) \\ \mathcal{B}_2(\mathcal{H}_b x_b) \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ \mathcal{C}(\mathcal{H}_b x_b) &= \begin{bmatrix} \mathcal{C}_1(\mathcal{H}_b x_b) \\ \mathcal{C}_2(\mathcal{H}_b x_b) \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \end{aligned} \tag{6}$$

Denote with u_r the restoring torque $u_r = \frac{\partial \phi(0,t)}{\partial \xi}$ and with y_r the hub's rotating velocity $y_r = \dot{\theta}$. The states related to the finite dimensional part are defined as $p = I_h\dot{\theta}$ and $q = \theta$, and the related equations write

$$\begin{cases} \dot{p} = +u_r(t) + u(t) \\ \dot{q} = I_h^{-1}p \\ y_r(t) = I_h^{-1}p. \end{cases} \tag{7}$$

Using the original boundary conditions (2) together with the state variable definition (3) to derive the interconnection relation between the infinite dimensional and the finite dimensional parts of the system

$$u_{b,1} = -y_r\ \ u_r = +y_{b,1}. \tag{8}$$

while the remaining boundary conditions of (4) are equal to zero, i.e. $u_{b,2} = [0\ 0\ 0]^T$.

2.2 Observer based Control design

The aims of the proposed control law are to orient the beam in the desired configuration and to change the elastic behaviour of the closed loop system using the state of the infinite dimensional part of the system. The nonlinear part of the control law is inspired by (Luo et al., 1999), but here we make use of an observed state $\hat{x}_b \in \hat{X}_b \in L_2([0,L],\mathbb{R}^4)$ instead of the original one. To this end, Define $g : \hat{X}_b \to \mathbb{R}$ and assume that it is linear. The control law writes

$$\begin{cases}\dot{x}_c = -r_c x_c + g(\hat{x}_b)\dot{\theta}(t) \\ u(t) = -k_1(\theta(t)-\theta^*) - g(\hat{x}_b)k_2x_c - k_3\dot{\theta}(t)\end{cases} \tag{9}$$

with $x_c \in \mathbb{R}$ the controller's state. Without loss of generality, we consider the stabilization problem to the origin $\theta^* = 0$. The second term in $u(t)$ is the nonlinear term depending on a function of the observed state $g(\hat{x}_b)$ and on the controller variable x_c. This controller construction makes possible the dependence of the controller dynamics on the observed infinite dimensional state.

For the infinite dimensional state reconstruction we propose a Luenberger observer: starting from the boundary observation of the infinite dimensional system, it reconstructs the original state with an exponential rate. It is assumed that all the physical parameters of the infinite dimensional system are known. The observer equations have the same form of the original system

$$\dot{\hat{x}}_b = P_1\frac{\partial}{\partial\xi}(\mathcal{H}_b\hat{x}_b) + P_0(\mathcal{H}_b\hat{x}_b), \tag{10}$$

with boundary inputs and observations

$$\mathcal{B}(\mathcal{H}_b\hat{x}_b) = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \hat{u}_b(t) = u_b(t) - L(\hat{y}_b(t) - y_b(t))$$
$$\mathcal{C}(\mathcal{H}_b\hat{x}_b) = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \hat{y}_b(t)$$

where, $L = \mathrm{diag}([l_1\ l_2\ l_3\ l_4]) \geq 0$ since $l_1, l_2, l_3, l_4 \geq 0$. For analysis purposes, it is convenient to perform a change of coordinates defining the error state $\tilde{x}_b = \hat{x}_b - x_b$ and its dynamics

$$\begin{aligned}\dot{\tilde{x}}_b &= \dot{\hat{x}}_b - \dot{x}_b \\ &= P_1\frac{\partial}{\partial\xi}(\mathcal{H}_b\hat{x}_b) + P_0(\mathcal{H}_b\hat{x}_b) - P_1\frac{\partial}{\partial\xi}(\mathcal{H}_bx_b) - P_0(\mathcal{H}_bx_b) \\ &= P_1\frac{\partial}{\partial\xi}(\mathcal{H}_b\tilde{x}_b) + P_0(\mathcal{H}_b\tilde{x}_b)\end{aligned} \tag{11}$$

where $\tilde{x}_b \in \tilde{X}_b \subset L_2([0,L],\mathbb{R}^4)$ and the operators' linearity has been used. The Observer's boundary operators are defined as the ones of the original infinite dimensional system (6)

$$\begin{aligned}\mathcal{C}(\mathcal{H}_b\tilde{x}_b) &= \mathcal{C}(\mathcal{H}_b\hat{x}_b) - \mathcal{C}(\mathcal{H}_bx_b) \\ &= \hat{y}_b - y_b = \tilde{y}_b\end{aligned}$$

and,

$$\begin{aligned}\tilde{u}_b = \mathcal{B}(\mathcal{H}_b\tilde{x}_b) &= \mathcal{B}(\mathcal{H}_b\hat{x}_b) - \mathcal{B}(\mathcal{H}_bx_b) \\ &= u_b(t) - L(\hat{y}_b - y(t)) - u_b(t) \\ &= -L\tilde{y}_b.\end{aligned} \tag{12}$$

The controller (9) can be rewritten as a dynamic PH system of the form:

$$\begin{cases}\begin{bmatrix}\dot{q} \\ \dot{x}_c\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & -r_c\end{bmatrix}\begin{bmatrix}k_1q \\ k_2x_c\end{bmatrix} + \begin{bmatrix}1 \\ g(\hat{x}_b)\end{bmatrix}u_c \\ y_c = \begin{bmatrix}1 & g(\hat{x}_b)\end{bmatrix}\begin{bmatrix}k_1q \\ k_2x_c\end{bmatrix} + k_3u_c.\end{cases}$$

To connect the controller to the system we make use of a power preserving interconnection:

$$u_c = y_r,\ u = -y_c. \tag{13}$$

To keep the analysis clear, the x_c dynamics is maintained separated from the rest of the system. Hence, we define the closed loop semilinear equation

$$\begin{aligned}\dot{x} &= \begin{bmatrix} P_1\frac{\partial}{\partial\xi}(\mathcal{H}_bx_b)+P_0(\mathcal{H}_bx_b) \\ P_1\frac{\partial}{\partial\xi}(\mathcal{H}_b\tilde{x}_b)+P_0(\mathcal{H}_b\tilde{x}_b) \\ (J_r-R_r)Q_rx_r+g_r\mathcal{C}_1(\mathcal{H}_bx_b) \\ -r_ck_2x_c\end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -g_rg(\tilde{x}_b+x_b)k_2x_c \\ +g(\tilde{x}_b+x_b)g_r^TQ_rx_r\end{bmatrix} \\ &= \mathcal{A}x + f(x) = \mathcal{A}_{nl}x\end{aligned} \tag{14}$$

where $x = [x_b\ \ \tilde{x}_b\ \ x_r\ \ x_c]^T \in X \subset L_2([0,L],\mathbb{R}^4) \times L_2([0,L],\mathbb{R}^4) \times \mathbb{R}^2 \times \mathbb{R}$, and $x_r = [p\ q]^T$. The new matrices are defined as

$$J_r = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\ R_r = \begin{bmatrix}k_3 & 0 \\ 0 & 0\end{bmatrix}\ Q_r = \begin{bmatrix}\frac{1}{I} & 0 \\ 0 & k_1\end{bmatrix}$$

with $J_r = -J_r^T,\ R_r = R_r^T \geq 0,\ Q_r = Q_r^T > 0$. The linear operator domain is defined as

$$D(\mathcal{A}) = \{x \in X | x_b, \tilde{x}_b \in H^1([0,L],\mathbb{R}^4), W_xx = 0\} \tag{15}$$

where,

$$W_xx = \begin{bmatrix}\mathcal{B}_{b,1}(\mathcal{H}_bx_b) + g_r^TQ_rx_r \\ \mathcal{B}_{b,2}(\mathcal{H}_bx_b) \\ \mathcal{B}_b(\mathcal{H}_b\tilde{x}_b) + \mathcal{C}_b(\mathcal{H}_b\tilde{x}_b)\end{bmatrix}.$$

We equip the state space with the inner product $\langle x,x\rangle_X = \langle x_b, \mathcal{H}_bx_b\rangle_{L_2} + \langle\tilde{x}_b, \mathcal{H}_b\tilde{x}_b\rangle_{L_2} + x_r^TQ_rx_r + k_2x_c^2$, and define the closed loop total energy as $H = \frac{1}{2}\langle x,x\rangle_X$. In the closed-loop energy it appears the square of the controller state x_c, that in turns has its dynamics depending on the function $g(\hat{x})$. Hence, the closed loop energy contains a term depending on the the estimation of the flexible state, and different designs of $g(\hat{x})$ modify the closed loop elastic behaviour in different manners.

3. ASYMPTOTIC STABILITY OF THE CLOSED-LOOP SYSTEM

Since the closed loop system is described by a semilinear equation, it is possible to use perturbation theory to prove its stability (Oostveen, 2000). To this end, we first prove that the linear operator $\mathcal{A}$ of system (14) generates a contraction C_0-semigroup.

Theorem 1. The linear operator $\mathcal{A}$ with domain (15) generates a contraction C_0-semigroup on X. Moreover, $\mathcal{A}$ has a compact resolvent.

Proof. The contraction C_0-semigroup generation is proved applying the Lummer-Phillips theorem. To this end two properties need to be verified: the dissipativity of the operator $\mathcal{A}$, and that $\mathrm{ran}(\lambda I - \mathcal{A}) = X$. Dissipativity consists on showing that $\langle\mathcal{A}x,x\rangle_X \leq 0$, then

$$\begin{aligned}2\langle \mathcal{A}x, x\rangle_X &= \langle \mathcal{A}x, x\rangle_X + \langle x, \mathcal{A}x\rangle_X \\ &= \langle \mathcal{J}x_b, \mathcal{H}_b x_b\rangle_{L_2} + \langle \mathcal{H}_b x_b, \mathcal{J}x_b\rangle_{L_2} \\ &\quad +2(Q_r x_r)^T(J_r - R_r)Q_r x_r \\ &\quad +2\mathcal{C}_{b,1}(\mathcal{H}_b x_b)g_r^T Q_r x_r - 2r_c k_2^2 x_c^2 \\ &\quad +\langle \mathcal{J}\tilde{x}_b, \mathcal{H}_b\tilde{x}_b\rangle_{L_2} + \langle \mathcal{H}_b\tilde{x}_b, \mathcal{J}\tilde{x}_b\rangle_{L_2}\end{aligned}$$

Thanks to the port variable selection is true that

$$\begin{aligned}\langle \mathcal{J}x_b, \mathcal{H}_b x_b\rangle_{L_2} + \langle \mathcal{H}_b x_b, \mathcal{J}x_b\rangle_{L_2} = 2u_{1,b}y_{1,b}, \\ \langle \mathcal{J}\tilde{x}_b, \mathcal{H}_b\tilde{x}_b\rangle_{L_2} + \langle \mathcal{H}_b\tilde{x}_b, \mathcal{J}\tilde{x}_b\rangle_{L_2} = 2\langle \tilde{u}_b, \tilde{y}_b\rangle_{\mathbb{R}^4}\end{aligned} \quad (16)$$

and since $R_c \geq 0,\ r_c > 0,\ k_2 > 0,\ L \geq 0$,

$$\langle \mathcal{A}x, x\rangle_X = -(Q_r x_r)^T R_r Q_r x_r - r_c k_2^2 x_c^2 - \langle L\tilde{y}, \tilde{y}\rangle \leq 0.$$

The range condition consists on finding a $\lambda > 0$ such that for all $f \in X$ there exists $x \in D(\mathcal{A})$ such that $(\lambda I - \mathcal{A})x = f$. The problem is divided in two different parts because the error system is not affected from the rest of the system. The range condition for the part of the system related to the error system follows from Theorem 2.26 of (Villegas, 2007). For the remaining equations, the range condition relies on the existence of the right inverse of the operator $\mathcal{B}_b$ subjected to a perturbation of the form $(\mathcal{B}_b + K\mathcal{C}_b)$, with $\mathcal{B}, \mathcal{C}$ operators defined in equation (6), and K a singular matrix. The existence of this right inverse follows from the non-singularity of $\left[\begin{smallmatrix} W \\ \tilde{W}\end{smallmatrix}\right]$.
To prove the compactness of the resolvent define the sequence

$$\{z_n\} = (\lambda I - \mathcal{A})^{-1}\{x_n\} \quad (17)$$

where, without loss of generality, assume $\{x_n\}$ bounded $\forall n \in \mathbb{N}$. For the compact operator definition, we have to show that $\{z_n\}$ has a converging subsequence on X. Define $\{z_n\} = [\{z_{n,1}\}\ \{z_{n,2}\}\ \{z_{n,3}\}]^T \in H^1([0, L], \mathbb{R}^4) \times H^1([0, L], \mathbb{R}^4) \times \mathbb{R}^3$ and $\{x_n\} = [\{x_{n,1}\}\ \{x_{n,2}\}\ \{x_{n,3}\}]^T \in X$. The operator $\mathcal{A}$ generates a contraction C_0-Semigroup, hence by the Hille-Yoshida theorem (Curtain and Zwart, 1995, Theorem 2.1.12, pag 26) is true that $||(\lambda I - \mathcal{A})^{-1}|| < \frac{1}{\lambda}$. This implies that also $\{z_n\}$ is bounded in X. Since $\{z_{n,3}\}$ belongs to a finite dimensional space, it follows that it has a convergent subsequence. For both $\{z_{n,1}\}$ and $\{z_{n,2}\}$ we have

$$||z_{n,i}||^2_{H^1} = ||\frac{\partial}{\partial z}z_{n,i}||^2_{L_2} + ||z_{n,i}||^2_{L_2} \quad i = \{1, 2\} \quad (18)$$

Using the $\mathcal{J}$ definition and equation (17), it holds

$$\begin{aligned}||\frac{\partial}{\partial z}z_{n,i}||^2_{L_2} &= ||P_1^{-1}\mathcal{J}z_{n,i} + P_1^{-1}P_0 z_{n,i}||^2_{L_2} \\ &\leq ||P_1^{-1}(\lambda z_{n,i} - x_{n,i})||^2_{L_2} + ||P_1^{-1}P_0 z_{n,i}||^2_{L_2} \\ &< \infty\end{aligned} \quad (19)$$

Then, $\{z_{n,i}\}$ is bounded in H^1 and from the Sobolev embedding theorem it implies that $\{z_{n,i}\}$ has a converging subsequence in L_2 for $i = \{1, 2\}$. Therefore, $\mathcal{A}$ has compact resolvent. □

The error system does not receive any input from the other parts of the system, hence its evolution is solely determined by its initial conditions, i.e. the initial error between the observer and the real initial state. Consequently, it is possible to conclude about its stability separately from the rest of the system.

Theorem 2. The error system defined by equation (11) and boundary conditions (12) is exponentially stable if $l_1, l_2 > 0,\ l_3, l_4 \geq 0$ or $l_3, l_4 > 0,\ l_1, l_2 \geq 0$.

Proof. Assume that $l_1, l_2 > 0,\ l_3, l_4 \geq 0$ and define the function $\tilde{E} = \frac{1}{2}\langle \tilde{x}, \mathcal{H}_b\tilde{x}\rangle_{L_2}$. Take its time derivative to obtain

$$\begin{aligned}\frac{1}{2}\dot{E}(x(t, \tilde{x}_0)) &= \frac{1}{2}\langle \mathcal{J}\tilde{x}, \mathcal{H}_b\tilde{x}\rangle_{L_2} + \frac{1}{2}\langle \mathcal{H}_b\tilde{x}, \mathcal{J}\tilde{x}\rangle_{L_2} \\ &= \langle \tilde{u}_b, \tilde{y}_b\rangle_{\mathbb{R}^4} = -\langle L\tilde{y}_b, \tilde{y}_b\rangle_{\mathbb{R}^4}\end{aligned}$$

Where equations (12) and (16) have been used. Then, defining $k = \max\{\frac{1+l_1^2}{l_1^2}, \frac{1+l_2^2}{l_2^2}\}$ and using equation (12),

$$\begin{aligned}||(\mathcal{H}_b\tilde{x})(0)||^2_{\mathbb{R}^4} &= \left(\frac{1}{\rho}\tilde{p}_t(0)\right)^2 + \left(\frac{1}{I_\rho}\tilde{p}_r(0)\right)^2 + (K\tilde{\varepsilon}_t(0))^2 \\ &\quad +(EI\tilde{\varepsilon}_r(0))^2 \\ &= \frac{1+l_1^2}{l_1^2}(l_1 K\tilde{\varepsilon}_t(0))^2 + \frac{1+l_2^2}{l_2^2}(l_2 EI\tilde{\varepsilon}_r(0))^2 \\ &\leq k((l_1 K\tilde{\varepsilon}_t(0))^2 + (l_2 EI\tilde{\varepsilon}_r(0))^2 \\ &\quad + \left(l_3\frac{1}{\rho}\tilde{p}_t(L)\right)^2 + \left(l_4\frac{1}{I_\rho}\tilde{p}_r(L)\right)^2) \\ &= k\langle L\tilde{y}_b, \tilde{y}_b\rangle_{\mathbb{R}^4}\end{aligned}$$

The statement follows from Corollary 5.19 of (Villegas, 2007). Exponential stability assuming $l_3, l_4 > 0,\ l_1, l_2 \geq 0$ follows from very similar arguments computing $||(\mathcal{H}_b\tilde{x})(L)||^2_{\mathbb{R}^4}$ instead of $||(\mathcal{H}_b\tilde{x})(0)||^2_{\mathbb{R}^4}$. □

The previous theorem states that the observer converges exponentially also in case boundary observations are available only at one side of the beam.
To be able to prove the asymptotic stability of the closed loop system it is first necessary to show that p and x_c are square integrable on infinite time.

Lemma 3. The solutions of system (14) are bounded in every interval $[0, t],\ t > 0$, and for all initial condition $x_0 \in D(\mathcal{A})$. Moreover, $p, x_c \in L_2([0, t])\ \forall t > 0$.

Proof. Boundedness of solutions follows from the existence of a Lyapunov function. This means that we have to show that exists a function $V : X \to \mathbb{R}^+$ such that $V(0) = 0$ and with time derivative $\dot{V}(x_0) \leq 0,\ \forall x_0 \in D(\mathcal{A})$. To this end we take $V(x_0) = \frac{1}{2}\langle x_0, x_0\rangle_X$ as candidate Lyapunov function. Its time derivative is defined as

$$\dot{V}(x_0) = \lim_{t\to 0}\frac{V(x(t, x_0)) - V(x_0)}{t}$$

and it can be proven that $\dot{V}(x_0) = dV(x_0)\mathcal{A}_{nl}x$, where $dV(x_0)$ is the Fréchet derivative of the candidate Lyapunov function in x_0. Then,

$$\begin{aligned}dV(x_0)\mathcal{A}_{nl}x &= +\frac{1}{2}\langle \mathcal{J}x_b, \mathcal{H}_b x_b\rangle_{L_2} + \frac{1}{2}\langle \mathcal{H}_b x_b, \mathcal{J}x_b\rangle_{L_2} \\ &\quad +\frac{1}{2}\dot{x}_r^T Q_r x_r + \frac{1}{2}x_r Q_r\dot{x}_r^T + k_c x_c\dot{x}_c \\ &\quad +\frac{1}{2}\langle \mathcal{J}\tilde{x}_b, \mathcal{H}_b\tilde{x}_b\rangle_{L_2} + \frac{1}{2}\langle \mathcal{H}_b\tilde{x}_b, \mathcal{J}\tilde{x}_b\rangle_{L_2}\end{aligned}$$

Similarly to proof of Theorem 1, and substituting x_r and x_c dynamics we obtain

$$\begin{aligned}\dot{V}(x_0) &= u_{b,1}y_{b,1} + \frac{1}{2}\left((J_r - R_r)Q_r x_r + g_r\mathcal{C}_{b,1}(\mathcal{H}_b x_b)\right. \\ &\quad \left.-g_r g(\hat{x})k_2 x_c\right)Q_r x_r + \frac{1}{2}(Q_r x_r)^T((J_r - R_r)Q_r x_r \\ &\quad +g_r\mathcal{C}_{b,1}(\mathcal{H}_b x_b) - g_r g(\hat{x})k_2 x_c) + k_2 x_c(-r_c k_2 x_c \\ &\quad +g(\hat{x})g_r^T Q_r x_r) + \langle \tilde{u}_b, \tilde{y}_b\rangle_{\mathbb{R}^4} \\ &= u_{b,1}y_{b,1} + y_r y_{b,1} - (Q_r x_r)^T R_r Q_r x_r - r_c k_2^2 x_c^2 \\ &\quad +\langle \tilde{u}_b, \tilde{y}_b\rangle_{\mathbb{R}^4}\end{aligned}$$

thanks to the skew symmetry of J_r, and using the boundary input-output definitions (5). Using the interconnection law definition (13), and the error system input definition (12), we obtain

$$\dot{V}(x_0) = -(Q_r x_r)^T R_r (Q_r x_r) - r_c k_2^2 x_c^2 - \langle L\tilde{y}_b, \tilde{y}_b\rangle_{\mathbb{R}^4} \leq 0 \quad (20)$$

From this it follows that $V(x)$ is a Lyapunov function and then that trajectories are bounded. To obtain the second part of the statement we integrate both members of (20) and we note that $(Q_r x_r)^T R_r (Q_r x_r) = \frac{k_3}{I^2} p^2$ to obtain

$$V(x(t, x_0)) = V(x_0) - \int_0^t \frac{k_3}{I^2} p^2 ds - \int_0^t r_c k_2^2 x_c^2 ds - \int_0^t \langle L\tilde{y}_b, \tilde{y}_b\rangle_{\mathbb{R}^4} ds$$

Since the Lyapunov function $V(x)$ is bounded from below, the statement follows. □

We are now in the position to state the result on asymptotic stability of the closed loop system.

Theorem 4. The closed loop system defined by (14) is globally asymptotically stable.

Proof. Notice that we do not have any control on the error system, but with Theorem 2 we have already proven that it is exponentially stable. Hence, it remains to show that the other part of the system described by

$$\dot{z} = \tilde{\mathcal{A}} z + Bu(t) = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathcal{H}_b x_b) + P_0(\mathcal{H}_b x_b) \\ (J_r - R_r) Q_r x_r + g_r \mathcal{C}(\mathcal{H}_b x_b) \\ -r_c k_c x_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g_r & 0 \\ 0 & 1 \end{bmatrix} u(t) \quad (21)$$

with $z = [x_b \;\; x_r \;\; x_c]^T \in Z \subset L^2([0, L], \mathbb{R}^4) \times \mathbb{R}^2 \times \mathbb{R}$, $u(t) = [g(\tilde{x}_b + x_b)k_2 x_c \;\; g(\tilde{x}_b + x_b)Q_r x_r]^T$ and

$$D(\tilde{\mathcal{A}}) = \{z \in Z | x_b \in H^1([0, L], \mathbb{R}^4), \mathcal{B}_1(\mathcal{H}_b x_b) + g_r Q_r x_r = 0, \mathcal{B}_2(\mathcal{H}_b x_b) = 0\},$$

is asymptotically stable. Firstly, define

$$B = \begin{bmatrix} 0 & 0 \\ g_r & 0 \\ 0 & 1 \end{bmatrix}, \; B^* = B^T \begin{bmatrix} \mathcal{H}_b & 0 & 0 \\ 0 & Q_r & 0 \\ 0 & 0 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & g_r^T Q_r & 0 \\ 0 & 0 & k_2 \end{bmatrix}$$
$$K = \begin{bmatrix} k_3 & 0 \\ 0 & r_c \end{bmatrix}. \quad (22)$$

Hence, define the weighted input-output matrices as $\tilde{B} = B\sqrt{K}$ and $\tilde{B}^* = \sqrt{K}B^*$. Then, the system (21) can be rewritten as $\dot{z} = (\tilde{\mathcal{A}}' - \tilde{B}\tilde{B}^*)z + \tilde{B}\tilde{u}(t)$, with $\tilde{u}(t) = \sqrt{K}^{-1} u(t)$. The operator $\tilde{A}'$ is the same as the operator $\tilde{A}$, but without the dissipation terms. With very similar arguments as in the proof of Theorem 1, it is possible to show that the operator $\tilde{\mathcal{A}}'$ generates a contraction C_0-semigroup, and it has compact resolvent. Moreover, the approximate controllability of the couple of operators $(\tilde{A}, B)$ has been proved in (Krabs and Sklyar, 1999), from which it follows the approximate controllability of $(\tilde{A}', \tilde{B})$. From Lemma 2.2.6 of (Oostveen, 2000), we conclude that $\tilde{A} = \tilde{A}' - \tilde{B}\tilde{B}^*$ is strongly stable. Then, using Lemma 2.1.3 of (Oostveen, 2000) it remains to show that the considered nonlinearity $\tilde{u}(t)$ is square integrable in infinite time. By definition of $\tilde{u}(t)$, proving its square integrability is the same as proving the square integrability of $u(t)$, hence

$$\int_0^\infty (g(\hat{x})k_2 x_c)^2 dt = \int_0^\infty (g(x_b + \tilde{x}_b)k_2 x_c)^2 dt \leq M_g^2 k_2^2 \int_0^\infty x_c^2 dt < \infty$$

where for the first inequality it has been used the boundedness of $x_b, \tilde{x}$ and the linearity of $g(\cdot)$, while for the second it has been used the square integrability of x_c shown in Lemma 3. Similarly, the square integrability of $g(\hat{x}_b) g_r^T Q_r x_r$ follows from the square integrability of p.

□

Table 1: Simulation Parameters

Name	Variable	Value
Beam's Length	L	$1\ m$
Beam's Width	L_w	$0.1\ m$
Beam's Thickness	L_t	$0.02\ m$
Density	ρ	$950\ \frac{kg}{m^3}$
Young's modulus	E	$8 \times 10^8\ \frac{N}{m^2}$
Bulk's modulus	K	$1.7 \times 10^9\ \frac{N}{m^2}$
Hub's inertia	I	$1\ kg \cdot m^2$

4. NUMERICAL SIMULATIONS

To perform the numerical simulations, it has been considered a finite dimensional approximation of the system. In particular, it has been used the finite element discretization for infinite dimensional PH systems presented in (Golo et al., 2004). This allows to spatially approximate the resulting linear PDEs with linear PH systems of dimensions depending on the number of discretizing elements (in the shown simulation we used 20 discretizing elements for both the beam's and the observer's PDEs). Simulations were made in the Simulink® environment using the "ode23t" time integration algorithm. The set of parameters used for simulation are listed in Table 1, where a Polyethilene HDPE material has been considered for the beam. For isotropic materials, the Shear modulus is related to the Young's modulus $G = \frac{E}{2(1+\nu)}$, where $\nu = \frac{1}{2} - \frac{E}{6K}$ is the Poisson's ratio. To show the observer action, we initialize the flexible beam to the zero initial state, while we set the observer initial condition different from the origin $\hat{x}_0 = [0.01\chi(z) \;\; 0 \;\; 0.01\chi(z) \;\; 0]^T$, where $\chi(z)$ is the characteristic function on the interval $[0, L]$. As weighting function for the nonlinear controller we select the Beam's tip deformation, that can be reconstructed from the system's state

$$g(\hat{x}_b) = \hat{w}(L, t) = \int_0^L \hat{\varepsilon}_t(z, t) + \left(\int_0^z \hat{\varepsilon}_r(\xi, t) d\xi \right) dz \quad (23)$$

The results are compared with a PD controller defined as $u_{PD}(t) = -k_1(\theta(t) - \theta^*) - k_3\dot{\theta}(t)$. The control parameters of both the PD and the non-linear (9) control law are selected as: $k_1 = e \times 10^2$, $k_2 = 1 \times 10^2$, $k_3 = 5 \times 10^4$ and $r_c = 1 \times 10^{-3}$. The error between the real and observed tip's deformation

$$\tilde{w}_b(L, t) = \hat{w}(L, t) - w(L, t) = \int_0^L \tilde{\varepsilon}_t(z, t) + \left(\int_0^z \tilde{\varepsilon}_r(\xi, t) d\xi \right) dz \quad (24)$$

is shown in Figure 2a. Notice that the Beam's tip deformation error converges to zero, and the convergence rate depends on the value of the diagonal terms of the observer matrix L. The set of values used in the simulation are $l_i = \alpha, \;\; i = \{1, 2, 3, 4\} \;\; \alpha \in \{0.07, 0.2, 0.5\}$. From Figure 3 we note that as far as the observer converges faster to the original state, the control action is more

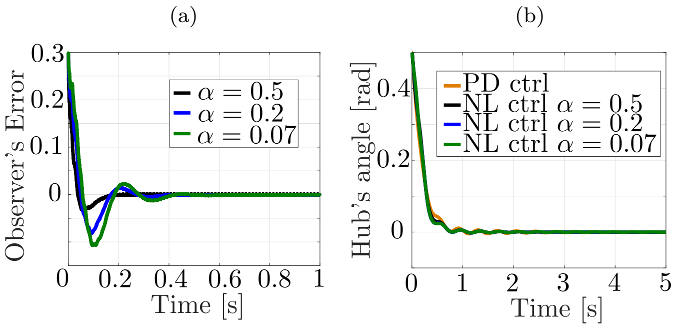

Fig. 2: (a) Beam's tip Observation error $\tilde{w}(L,t)$ (b) Hub's angle $q(t)$.

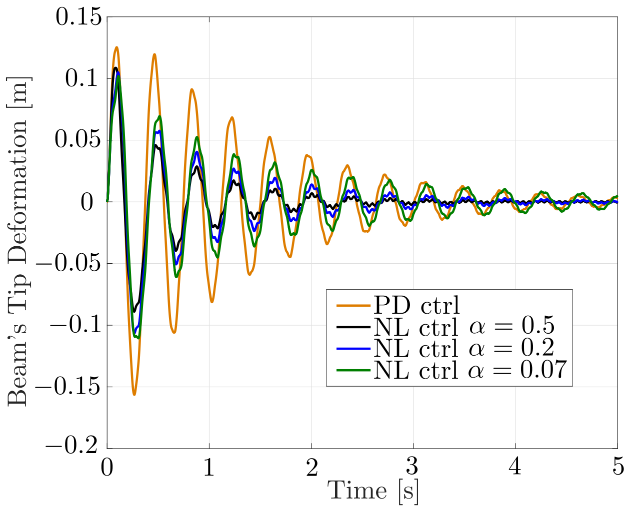

Fig. 3: Tip's deformation $w(L,t)$.

effective in damping the beam's tip vibrations. In case the observer is not converging fast enough, it is shown that the oscillation are kept smaller and the system asymptotically converges to the origin, but with a rate similar to the PD controller. Finally, Figure 2b shows that the hub's angular displacement has a similar rate of convergence in all the different control law applications.

5. CONCLUSIONS

It has been considered a model of the rotating flexible beam composed by a set of PDEs interconnected with an ODE, with actuation in the ODE. Since the control input is not on the PDEs' boundaries, a passive preserving way of using the deformation information in the controller is through the use of a nonlinear dynamic control law. In this paper, the nonlinear controller makes use of an estimated state instead of the original one. Firstly, it has been proven the exponential stability of the observer's state assured that we have at least the complete observation in one side of the beam. Secondly, the nonlinear closed loop system has been analysed using the operator formalism and asymptotic stability has been formally proved. Numerical simulations have been used to show the closed loop behaviour with the use of different observer's parameters. An experimental set-up where it will be possible to test the proposed control law is currently under construction. The future work will deal with the generalization of this type of controller for a class of PDEs-ODE system, that can be frequently encountered in mechanical applications.

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