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On the LQ Based Stabilization for a Class of Switched Dynamic Systems

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Abstract: This paper deals with the stabilization of a class of time-dependent linear autonomous systems with a switched structure. For this aim, the switched dynamic system is modeled by means of an implicit representation combined with a Linear-Quadratic (LQ) type control design. The proposed control design stabilizes the resulting system for all of the possible realizations of its locations. In order to solve the Algebraic Riccati Equation (ARE) associated with the LQ control strategy one only needs the knowledge of the algebraic structure related to the switched system. We finally prove that the proposed optimal LQ type state feedback stabilizes the closed-loop switched system no matter which location is active. The proposed theoretical approaches are illustrated by a numerical example.

Keywords: switched dynamic systems, implicit control systems, linear quadratic regulator (LQR), Riccati equation, Lyapunov stability.

1. INTRODUCTION

In (Bonilla *et al*, 2015a) is shown that a wide class of time-dependent autonomous systems with a switched structure (Liberzon, 2003) can be adequately modeled by the state space representation:

$$\dot{\overline{x}} = A_q\overline{x} + Bu\,, \quad y = C_q\overline{x}\,. \qquad (1.1)$$

$B \in R^{\overline{n}\times m}$ is an injective matrix and the A_q and C_q have the following structure (see *e.g.*, (Narendra *et al*, 1994)):

$$A_q = \overline{A}_0 + \overline{A}_1\overline{D}(q) \ \text{ and } \ C_q = \overline{C}_0 + \overline{C}_1\overline{D}(q), \qquad (1.2)$$

The system remains in a specific location,

$$q \in \mathcal{Q} = \Big\{(q_1, ..., q_\eta) \mid q_i \in \mathbb{R}^n, \quad i \in \{1, ..., \eta\}\Big\}, \qquad (1.3)$$

for all time instants $t \in [T_{i-1}, T_i)$, where $T_i \in \mathbb{R}^+$, $T_0 = 0$, $T_{i-1} < T_i$, for all $i \in \mathbb{N}$, $\lim_{i\to\infty} T_i = \infty$, and $\mathfrak{s} : \{[T_{i-1}, T_i) \subset \mathbb{R}^+,\ i \in \mathbb{N}\} \to \mathcal{Q}$, $\mathfrak{s}([T_{i-1},\ T_i)) = q$. Moreover, $\overline{A}_0 \in \mathbb{R}^{\overline{n}\times\overline{n}}$ and $\overline{C}_0 \in \mathbb{R}^{p\times\overline{n}}$, $\overline{A}_1 \in \mathbb{R}^{\overline{n}\times\hat{n}}$ is an injective matrix and $\overline{C}_1 \in \mathbb{R}^{p\times\hat{n}}$ and $\overline{D}(q) \in \mathbb{R}^{\hat{n}\times\overline{n}}$ are surjective matrices with the property $\overline{D}(0) = 0$.

In (Bonilla *et al*, 2015b) authors additionally propose a specific variable structure decoupling control strategy based on the ideal proportional and derivative (PD) feedback control law. Finally a proper practical approximation of the above ideal PD feedback is developed. Such feedback control strategies reject the initially given "variable structure" and make it possible to establish the required stability property of both control strategies.

In this paper we consider the stabilizing problem for a class of time-dependent switched dynamic systems equipped with a relative simple static state feedback. The paper is organized as follows: in Section 2 we formally introduce a class of switched systems and represent the given dynamics using the global implicit representation technique from (Bonilla *et al*, 2019). Section 3 contains a proper self-closed solution procedure for the main LQR design problem under specific assumption of unknown location. In Section 4 we show that the developed LQ type optimal feedback control additionally stabilizes the switched dynamic system. In Section 5 we discuss an illustrative numerical example and Section 6 summarizes our paper.

2. NON-STATIONARY SWITCHED SYSTEM WITH TIME-DRIVEN SWITCHING STRUCTURE

Let us consider the global implicit representation:

$$\mathbb{E}\dot{x} = \mathbb{A}x + \mathbb{B}u\,, \quad y = Cx. \qquad (2.1)$$

Here

$$\mathbb{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \ \ \mathbb{A} = \begin{bmatrix} A \\ D_q \end{bmatrix}, \ \ \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad (2.2)$$

$$E = [I \ \ 0]\,, \ \ A = \big[\overline{A}_0 \ \ -\overline{A}_1\big]\,, \ \ C = \big[\overline{C}_0 \ \ -\overline{C}_1\big] \ \text{ and } \ D_q = \big[\overline{D}(q) \ \ I\big]\,. \qquad (2.3)$$

We now assume that the locations set has a specific structure described by the following Hypothesis.

H1. (Bonilla *et al*, 2015b) Given $\overline{q}_0,\ \overline{q}_1,\ \ldots,\ \overline{q}_\ell \in \mathcal{Q}$, $g = [g_1 \ldots\ g_\ell]^T$, $g_1,\ \ldots,\ g_\ell \in \mathbb{R}^+$, the locations $q_i \in \mathcal{Q}$ belong to the convex set described as follows

$$\overline{\mathcal{Q}}_{\overline{q}_0}(g) = \left\{ q_i \in \mathcal{Q} \,\middle|\, q_i = \overline{q}_0 + \sum_{j=1}^{\ell} \gamma(i,j) g_j \overline{q}_j \,,\, i \in \{1, , \ldots, \ell\} \right\}, \tag{2.4}$$

and moreover, for each $[T_{i-1}, T_i) \in \mathfrak{s}^{-1}(q)$, $\gamma(i,j)$ takes constant values in the closed subset of $\mathbb{R}$: $[0,1]$.

H2. (Bonilla *et al*, 2015b) There exist $\overline{\Delta}_0$, $\overline{\Delta}_1$, $\ldots$, $\overline{\Delta}_\ell$ such that

$$\overline{D}(q_i) = \overline{\Delta}_0 - \sum_{j=1}^{\ell} \gamma(i,j) g_j \overline{\Delta}_j, \tag{2.5}$$

where $\gamma(i,j)$ and g_j are determined by *H1*.

3. THE MAIN LQ-TYPE OPTIMAL CONTROL PROBLEM

Let us now formulate the following natural problem associated with a dynamics.

Problem 1. Given a switched system represented by (1.1) and (1.2) (*viz* (2.1)–(2.3) with a nonzero initial condition $\overline{x}_0$, and one unknown $q \in \mathcal{Q}$, determine a control design such that the descriptor variable $\overline{x}$ tends to zero and moreover, the following objective

$$J = \int_0^\infty \left(\overline{x}^T Q(q) \overline{x} + u^T R u \right) dt \tag{3.1}$$

attains its minimal value, where $Q(q) = Q(q)^T \geq 0$ and $R = R^T > 0$.

For the concrete treatment of the above Problem 1 we next follow Kailath (1980). Let us also refer to Lewis *et al* (2012) for more technical details. We now define the conventionally augmented objective associated with the originally given costs functional:

$$J_a(x, u, \lambda) = \int_0^\infty \left(H(x, u, \lambda, t) - \lambda^T \mathbb{E} \dot{x} \right) dt, \tag{3.2}$$

where the system's Hamiltonian H is defined as follows

$$H(x, u, \lambda, t) := L(x, u, t) + \lambda^T (\mathbb{A} x + \mathbb{B} u), \tag{3.3}$$

$$L(x, u, t) := \frac{1}{2} \left(x^T Q(q) x + u^T R u \right). \tag{3.4}$$

The above formalism implies the associated Euler-Lagrange equation (see Azhmyakov (2019) for details)

$$\frac{\partial f}{\partial v} - \frac{d}{dt} \left(\frac{\partial f}{\partial (\dot{v})} \right) = 0,$$

where: $f = H - \lambda^T \mathbb{E} \dot{x}$, and $v \in \{x, \, u, \, \lambda\}$ have the following constructive definition (Luenberger, 1969):

$$v = x : \qquad Q(q)x + \mathbb{A}^T \lambda = -\mathbb{E}^T \dot{\lambda}, \tag{3.5}$$

$$v = \lambda : \qquad \mathbb{E} \dot{x} = \mathbb{A} x + \mathbb{B} u, \tag{3.6}$$

$$v = u : \qquad u = -R^{-1} \mathbb{B}^T \lambda. \tag{3.7}$$

From (3.5)–(3.7) we next deduce the Hamiltonian equation

$$\begin{bmatrix} \mathbb{E} \dot{x} \\ \mathbb{E}^T \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbb{A} & -\mathbb{B} R^{-1} \mathbb{B}^T \\ -Q(q) & -\mathbb{A}^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}. \tag{3.8}$$

and from (2.2)–(3.8) we additionally obtain the following useful expressions

$$\begin{bmatrix} \begin{bmatrix} E \\ 0 \end{bmatrix} \dot{x} \\ \begin{bmatrix} E^T & 0 \end{bmatrix} \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A \\ D_q \end{bmatrix} & -\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B^T & 0 \end{bmatrix} \\ -Q(q) & -\begin{bmatrix} A^T & D_q^T \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}.$$

Taking into consideration the above relation (2.3), we have the final expressions:

$$\begin{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{x} \\ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 \\ \overline{D}(q) & I \end{bmatrix} & -\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B^T & 0 \end{bmatrix} \\ -Q(q) & -\begin{bmatrix} \overline{A}_0^T & \overline{D}^T(q) \\ -\overline{A}_1^T & I \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}.$$

Let us now define $x = \begin{bmatrix} x_c \\ x_\ell \end{bmatrix}$ and $\lambda = \begin{bmatrix} \lambda_c \\ \lambda_\ell \end{bmatrix}$. Additionally let us denote

$$Q(q) = \begin{bmatrix} \overline{Q}_q & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{Q}_q = \overline{Q}_q^T \geq 0. \tag{3.9}$$

Taking into account the above formalism, we obtain the formal consequences:

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_c \\ \dot{x}_\ell \\ \dot{\lambda}_c \\ \dot{\lambda}_\ell \end{bmatrix} = \begin{bmatrix} \overline{A}_0 & -\overline{A}_1 & -BR^{-1}B^T & 0 \\ \overline{D}(q) & I & 0 & 0 \\ -\overline{Q}_q & 0 & -A_0^T & -\overline{D}^T(q) \\ 0 & 0 & \overline{A}_1^T & -I \end{bmatrix} \begin{bmatrix} x_c \\ x_\ell \\ \lambda_c \\ \lambda_\ell \end{bmatrix} \tag{3.10}$$

We next formally describe the constructive "separation idea" and split (3.10) into two subsystems. The obtained "dynamic part" has the following form

$$\begin{bmatrix} \dot{x}_c \\ \dot{\lambda}_c \end{bmatrix} = \begin{bmatrix} \overline{A}_0 & -BR^{-1}B^T \\ -\overline{Q}_q & -\overline{A}_0^T \end{bmatrix} \begin{bmatrix} x_c \\ \lambda_c \end{bmatrix} + \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & -\overline{D}^T(q) \end{bmatrix} \begin{bmatrix} x_\ell \\ \lambda_\ell \end{bmatrix}. \tag{3.11}$$

The resulting algebraic part of the proposed "separation" has the corresponding representation:

$$\begin{bmatrix} x_\ell \\ \lambda_\ell \end{bmatrix} = \begin{bmatrix} -\overline{D}(q) & 0 \\ 0 & \overline{A}_1^T \end{bmatrix} \begin{bmatrix} x_c \\ \lambda_c \end{bmatrix} \tag{3.12}$$

Now from (3.12) and (3.11) we can easily deduce the next relations

$$\begin{aligned} \begin{bmatrix} \dot{x}_c \\ \dot{\lambda}_c \end{bmatrix} &= \begin{bmatrix} \overline{A}_0 & -BR^{-1}B^T \\ -\overline{Q}_q & -\overline{A}_0^T \end{bmatrix} \begin{bmatrix} x_c \\ \lambda_c \end{bmatrix} \\ &+ \begin{bmatrix} -\overline{A}_1 & 0 \\ 0 & -\overline{D}^T(q) \end{bmatrix} \begin{bmatrix} -\overline{D}(q) & 0 \\ 0 & \overline{A}_1^T \end{bmatrix} \begin{bmatrix} x_c \\ \lambda_c \end{bmatrix}, \end{aligned} \tag{3.13}$$

which are equivalent to the condition

$$\begin{bmatrix} \dot{x}_c \\ \dot{\lambda}_c \end{bmatrix} = \begin{bmatrix} (\overline{A}_0 + \overline{A}_1 \overline{D}(q)) & -BR^{-1}B^T \\ -\overline{Q}_q & -(\overline{A}_0^T + \overline{D}^T(q) \overline{A}_1^T) \end{bmatrix} \begin{bmatrix} x_c \\ \lambda_c \end{bmatrix}. \tag{3.14}$$

Moreover, we also deduce (*cf.* (1.2)) the following:

$$\begin{bmatrix} \dot{x}_c \\ \dot{\lambda}_c \end{bmatrix} = \begin{bmatrix} A_q & -BR^{-1}B^T \\ -\overline{Q}_q & -A_q^T \end{bmatrix} \begin{bmatrix} x_c \\ \lambda_c \end{bmatrix}. \tag{3.15}$$

The obtained relation (3.15) provides a basis for the celebrated Algebraic Riccati Equation (ARE)

$$A_q^T P + P A_q - P B R^{-1} B^T P + \overline{Q}_q = 0, \tag{3.16}$$

where q is one unknown element of the given locations set $\{q_1, \ldots, q_\ell\}$.

Finally let us recall the common knowledge of the feedback-type optimal control design for the generic LQ problem (see for example Kailath (1980) and Lewis *et al* (2012))

$$F_* = R^{-1}B^T P, \quad u = -F_*\overline{x}. \tag{3.17}$$

We next use the well-known facts from this section as a theoretical basis for the stabilization approach we propose.

3.1 On the ARE Involved Solution Approach

Which location q is currently active being unknown implies the conceptual difficulties in solving the ARE (3.16). Note that the solvability property of the "switched" ARE under consideration is assumed for every admissible location. For the concrete treatment of (3.16) we next assume that $\overline{Q}_q$ has the same structure as in *H1*. That means

$$\overline{Q}_q = \overline{Q}_0 + \sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{Q}_j, \tag{3.18}$$

where $\overline{Q}_0 = \overline{Q}_0^T > 0$ and (see (2.5)):

$$\overline{Q}_j = (\overline{A}_1\overline{\Delta}_j)^T P_0 + P_0(\overline{A}_1\overline{\Delta}_j). \tag{3.19}$$

P_0 is a positive definite matrix and a solution of the ARE:

$$A_q^T P_0 + P_0 A_q - P_0 B R^{-1} B^T P_0 = -\overline{Q}_0. \tag{3.20}$$

From (1.2) and (3.16) we next conclude that

$$(\overline{A}_0 + \overline{A}_1\overline{D}(q_i))^T P + P(\overline{A}_0 + \overline{A}_1\overline{D}(q_i)) - PBR^{-1}B^T P + \overline{Q}_{q_i} = 0 \tag{3.21}$$

and taking into account the basic relation (2.5) and (3.18) we finally get the useful relation

$$\begin{aligned}&\left(\overline{A}_0 + \overline{A}_1\left(\overline{\Delta}_0 - \sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{\Delta}_j\right)\right)^T P+\\ &P\left(\overline{A}_0 + \overline{A}_1\left(\overline{\Delta}_0 - \sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{\Delta}_j\right)\right) -\\ &PBR^{-1}B^T P + \left(\overline{Q}_0 + \sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{Q}_j\right) = 0\end{aligned} \tag{3.22}$$

viz:

$$(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0)^T P + P(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0) - PBR^{-1}B^T P + \overline{Q}_0 = \sum_{j=1}^{\ell} g_j\gamma(i,j)\left[\left(\overline{A}_1\overline{\Delta}_j\right)^T P + P\left(\overline{A}_1\overline{\Delta}_j\right)\right] - \sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{Q}_j. \tag{3.23}$$

Relations (3.19) and (3.23) imply the next formal consequence

$$\begin{aligned}&(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0)^T P + P(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0) - PBR^{-1}B^T P + \overline{Q}_0\\ &= \sum_{j=1}^{\ell} g_j\gamma(i,j)\Big((\overline{A}_i\overline{\Delta}_j)^T P + P(\overline{A}_i\overline{\Delta}_j) - (\overline{A}_i\overline{\Delta}_j)^T P_0\\ &\qquad - P_0(\overline{A}_i\overline{\Delta}_j)\Big).\end{aligned} \tag{3.24}$$

which finally involve (under assumption $P = P_0$) the resulting ARE of the following type:

$$(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0)^T P_0 + P_0(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0) - P_0BR^{-1}B^T P_0 + \overline{Q}_0 = 0 \tag{3.25}$$

Let us note that the obtained ARE (3.25) does not include any (unknown) active location. It depends only on the given structure of (2.1). The essential parameters, namely, $(\overline{A}_0, \overline{A}_1, \overline{\Delta}_0)$ determine the above equation structure. Let us summarize the obtained result in the form of a theorem.

Theorem 1. Assume that all the technical conditions of this Section are satisfied. Then the optimal feedback solution to the main Problem 1, where $Q(q)$ is given by (3.9), (3.18) and (3.19), has the following form

$$F_{*0} = R^{-1}B^T P_0, \quad u = -F_{*0}\overline{x}. \tag{3.26}$$

Here P_0 is solution of the ARE (3.25).

The presented analytic result constitutes a theoretical basis for the stabilization problem studied in the next section.

4. THE LQ BASED STABILIZATION OF SWITCHED SYSTEMS

As a well-known stabilizability property of the classic optimal LQ control design (see for example Lewis *et al* (2012); Kailath (1980)), we next show that the generic optimal control from Theorem 1 also stabilizes system (1.1)–(1.3) even in the case of unknown locations $q \in \mathcal{Q}$.

Applying the optimal control feedback (3.26) to the switched system representation (1.1), we obtain the following closed loop state space form

$$\begin{aligned}\dot{\overline{x}} &= Aq_i\overline{x} - BF_{*0}\overline{x} = (Aq_i - BF_{*0})\overline{x}\\ &= (Aq_i - BR^{-1}B^T P_0)\overline{x}\end{aligned} \tag{4.1}$$

Taking into consideration the previously derived formulae (1.2) and (2.5) in (4.1), we also get:

$$\begin{aligned}\dot{\overline{x}} &= (\overline{A}_0 + \overline{A}_1(\overline{\Delta}_0 - \sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{\Delta}_j) - BR^{-1}B^T P_0)\overline{x}\\ &= F(\overline{\Delta}_j, P_0)\,\overline{x},\end{aligned} \tag{4.2}$$

where: $F(\overline{\Delta}_j, P_0) \triangleq \overline{A}_0 + \overline{A}_1\overline{\Delta}_0 - \sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{A}_1\overline{\Delta}_j - BR^{-1}B^T P_0$. Let us define the Lyapunov function:

$$V(x) = \overline{x}(t)^T P_0\overline{x}(t). \tag{4.3}$$

The usual Lie derivative of (4.3) (along the trajectories of system (4.2)) next implies:

$$\begin{aligned}\dot{V}(t) &= \dot{\overline{x}}^T(t)P_0\overline{x} + \overline{x}^T(t)P_0\dot{\overline{x}}\\ &= \overline{x}^T F^T(\overline{\Delta}_j, P_0)P_0\overline{x}(t) + \overline{x}^T(t)P_0 F(\overline{\Delta}_j, P_0)\overline{x}\\ &= \overline{x}^T\Bigg(\left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0\right)^T P_0 - \sum_{j=1}^{\ell} g_j\gamma(i,j)\left(\overline{A}_1\overline{\Delta}_j\right)^T P_0\\ &\quad - \left(BR^{-1}B^T P_0\right)^T P_0\Bigg)\overline{x} + \overline{x}^T\Bigg(P_0\left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0\right)\\ &\quad - \sum_{j=1}^{\ell} g_j\gamma(i,j)P_0\left(\overline{A}_1\overline{\Delta}_j\right) - P_0\left(BR^{-1}B^T P_0\right)\Bigg)\overline{x}\end{aligned}$$

$$\dot{V}(t) = \overline{x}^T\Big(\left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0\right)^T P_0 + P_0\left(\overline{A}_0 + \overline{A}_1\overline{\Delta}_0\right) - P_0BR^{-1}B^TP_0 - \sum_{j=1}^{\ell} g_j\gamma(i,j)\Big(\left(\overline{A}_1\overline{\Delta}_j\right)^T P_0 + P_0\left(\overline{A}_1\overline{\Delta}_j\right)\Big) - P_0BR^{-1}B^TP_0\Big)\overline{x}\,.$$

From (3.25) and (3.26) we deduce

$$\dot{V}(\overline{x}) = -\overline{x}^T\Big(\overline{Q}_0 + \sum_{j=1}^{\ell} g_j\gamma(i,j)\Big(\left(\overline{A}_1\overline{\Delta}_j\right)^T P_0 + P_0\left(\overline{A}_1\overline{\Delta}_j\right)\Big) + F_{*0}^T R F_{*0}\Big)\overline{x}\,,$$

$$\dot{V}(\overline{x}) = -\overline{x}^T\Bigg(\begin{bmatrix}\sqrt{\overline{Q}_0} & F_{*0}\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & R\end{bmatrix}\begin{bmatrix}\sqrt{\overline{Q}_0} \\ F_{*0}^T\end{bmatrix} + \sum_{j=1}^{\ell} g_j\gamma(i,j)\Big(\left(\overline{A}_1\overline{\Delta}_j\right)^T P_0 + P_0\left(\overline{A}_1\overline{\Delta}_j\right)\Big)\Bigg)\overline{x}\,,$$

where $\overline{Q}_0 = \left(\sqrt{\overline{Q}_0}\right)^T\left(\sqrt{\overline{Q}_0}\right)$. Defining: $Q_0 = \begin{bmatrix}\sqrt{\overline{Q}_0} & F_{*0}\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & R\end{bmatrix}\begin{bmatrix}\sqrt{\overline{Q}_0} \\ F_{*0}^T\end{bmatrix}$, we finally get:

$$\dot{V}(\overline{x}) = -\overline{x}^T\Bigg(Q_0 + \sum_{j=1}^{\ell} g_j\gamma(i,j)\left[\left(\overline{A}_1\overline{\Delta}_j\right)^T P_0 + P_0\left(\overline{A}_1\overline{\Delta}_j\right)\right]\Bigg)\overline{x}. \tag{4.4}$$

The analytic relations obtained above constitute in fact a formal proof of our next stability result [1]

Theorem 2. Assume that all the technical assumptions of this section are fulfilled. Then the system (4.1) is stable in the sense of Lyapunov if one of the two following conditions is satisfied:

$$\lambda_{min}\left(Q_0\right) + \lambda_{min}\left(\sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{Q}_j\right) > 0\,, \tag{4.5}$$

$$Q_0 + \sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{Q}_j > 0\,, \tag{4.6}$$

or if the pair $\left(\sqrt{\overline{Q}_0},\, (\overline{A}_0 + \overline{A}_1\overline{\Delta}_0)\right)$ is observable and one of the following conditions holds true:

$$\lambda_{min}\left(Q_0\right) + \lambda_{min}\left(\sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{Q}_j\right) \geq 0\,, \tag{4.7}$$

$$\lambda_{min}\left(Q_0\right) + \lambda_{min}\left(\sum_{j=1}^{\ell} g_j\gamma(i,j)\overline{Q}_j\right) \geq 0\,. \tag{4.8}$$

[1] Let us also recall Theorem 5.10 of Chapter 6, Section 5 of Stewart (1973), and Corollary 2.6-2 of Kailath (1980).

The obtained result provides a stability criterion for the switched systems under consideration in absence of the exact *a priori* information about a concrete switching mechanism.

Starting from a model in the form (1.1), the procedure to design the feedback is summarized as follows.

(1) Identify the parameters of the implicit representation (2.1)–(2.5), in particular the matrices $\overline{A}_0$, $\overline{A}_1$, B, and $\overline{\Delta}_0$.
(2) Choose matrices R and $Q(q)$ in the form defined by (3.9) and (3.18), satisfying one of the four conditions of Theorem 2, namely, (4.5), (4.6), (4.7) or (4.8).
(3) Solve the Riccati equation (3.25), and define the feedback by (3.26).

5. NUMERICAL ASPECTS

In this section we apply the theoretical results (stability results) developed in the previous parts of the manuscript and study an illustrative example taken from Azhmyakov (2019). Let us also refer to Bonilla *et al* (2019) for some further examples and theoretical details. Consider now the state space representation (1.1) determined by the following matrices

$$A_q = \begin{bmatrix} \alpha & \beta+1 \\ \alpha+1 & \beta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_q = \begin{bmatrix} -\alpha & -\beta \end{bmatrix}. \tag{5.1}$$

and with the locations (*cf.* (1.3)):

$$q \in \mathcal{Q} = \Big\{(\alpha,\,\beta) \mid q_1 = (-1.5,\,-0.8),\, q_2 = (-1,\,-2),\, q_3 = (-1,\,0),\, q_4 = (-1,\,-3),\, q_5 = (-1,\,-1)\Big\}. \tag{5.2}$$

Comparing with (1.2), we can observe that

$$\overline{A}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{D}_q = [\alpha \;\; \beta], \quad \overline{C}_0 = [0\;\;0],\; \overline{C}_1 = 1, \tag{5.3}$$

The conventional systems transfer functions of (1.1) and (5.1) for each pair $(\alpha,\,\beta)$ are:

$$F_q(s) = \frac{(\beta+2)s - \alpha}{(s+1)(s-(\alpha+\beta+1))}\,. \tag{5.4}$$

5.1 Global Implicit Representation

The global implicit representation associated with (1.1) and (5.1) can be formalized as

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]\begin{bmatrix} \dot{\overline{x}} \\ - \\ \dot{\hat{x}} \end{bmatrix} = \left[\begin{array}{cc|c} \alpha & (1+\beta) & -1 \\ (1+\alpha) & \beta & -1 \\ \hline 0 & 0 & 1 \end{array}\right]\begin{bmatrix} \overline{x} \\ - \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ - \\ 0 \end{bmatrix} u$$
$$y = [-\alpha \;\, -\beta \mid -1]\begin{bmatrix} \overline{x} \\ - \\ \hat{x} \end{bmatrix}, \tag{5.5}$$

viz (*cf.* $\sum_0^{gir}(\mathbb{E}, \mathbb{A}_i, \mathbb{B}, C)$. Moreover, we have

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]\dot{x} = \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & -1 \\ \hline \alpha & \beta & 1 \end{array}\right] x + \begin{bmatrix} 0 \\ 1 \\ - \\ 0 \end{bmatrix}$$
$$y = [0 \;\; 0 \mid 1]\, x \tag{5.6}$$

i	q_i	(α, β)	$\gamma(i,1)$	$\gamma(i,2)$
1	q_1	$(-1.5, -0.8)$	1	4/15
2	q_2	$(-1, -2)$	0	2/3
3	q_3	$(-1, 0)$	0	0
4	q_4	$(-1, -3)$	0	1
5	q_5	$(-1, -1)$	0	1/3

Table 1. α, β values and the values of $\gamma(i, j)$

We now consider the systems controllability requirement and examine the characteristic determinant of the controllability matrix for the pair (A_q, B) which is:

$$\det [B \quad A_{qi}] = \begin{vmatrix} 0 & \beta+1 \\ 1 & \beta \end{vmatrix} = -\beta - 1. \tag{5.7}$$

The characteristic polynomial of A_q is:

$$\begin{aligned} |sI_2 - A_{q_i}| &= \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} \alpha & \beta+1 \\ \alpha+1 & \beta \end{bmatrix} \right| \\ &= s^2 - (\alpha+\beta)s - (\alpha+\beta+1) \\ &= (s+1)(s-(1+\alpha+\beta)). \end{aligned} \tag{5.8}$$

Hence, we are ready to calculate the eigenvalues of A_q

$$\begin{aligned} s_1 &= -1 \\ s_2 &= 1+\alpha+\beta \end{aligned} \tag{5.9}$$

From (5.7), we obtain the controllability region associated with the pair (A_q, B):

$$CR_{(A_{qi},B)} = \{\beta \in \mathbb{R} : \ \beta \neq -1\}. \tag{5.10}$$

From (5.8), we have the Hurwitz region of A_q:

$$HR_{A_q} = \{(\alpha, \beta) \in \mathbb{R}^2 : 1+\alpha+\beta < 0\}. \tag{5.11}$$

Let us note that the locations set (5.2) has the same structure as *H1*, which implies the next constructive relation:

$$\begin{aligned} q \in \overline{\mathcal{Q}}_{(-\underline{\alpha},-\underline{\beta})} = \Big\{ (\alpha,\beta) \in \mathcal{Q} \ \Big| \\ (\alpha,\beta) = (-\underline{\alpha},-\underline{\beta}) + \gamma(i,1)(\overline{\alpha}-\underline{\alpha})\overline{q}_1 + \gamma(i,2)(\overline{\beta}-\underline{\beta})\overline{q}_2, \\ \gamma(i,1), \gamma(i,2) \in [0,1] \Big\} \end{aligned} \tag{5.12}$$

where

$$\overline{\alpha} = 1.5, \quad \overline{\beta} = 3, \quad \underline{\alpha} = 1, \quad \underline{\beta} = 0. \tag{5.13}$$

Moreover, we also have $\overline{q}_0 = (-\underline{\alpha}, -\underline{\beta}) = (-1, 0)$, $\overline{q}_1 = (-1, 0)$ and $\overline{q}_2 = (0, -1)$. In Table 1 we present the values of $\gamma(i, j)$ and we additionally deduce:

$$\begin{aligned} \Delta_0 = (-\underline{\alpha}, -\underline{\beta}) = (-1, 0), \\ \Delta_1 = -\overline{q}_1 = (1, 0), \quad \Delta_2 = -\overline{q}_2 = (0, 1). \end{aligned} \tag{5.14}$$

5.2 LQ Feedback Stabilization

We now solve the ARE (3.25) with

$$\overline{Q}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad R = 1. \tag{5.15}$$

Note that a concrete numerical solution procedure for the resulting ARE can be easily found in MATLAB® and Python numerical packages. From (5.3), (5.14) and (5.15) we deduce that this solution of (3.25) has the following form

$$P_0 = \begin{bmatrix} 0.4765 & 0.2168 \\ 0.2168 & 1.1974 \end{bmatrix}, \tag{5.16}$$

Hence, the feedback (3.26) is:

$$F_{*0} = R^{-1} B^T P_0 = [0.2168 \quad 1.1974], \quad u = -F_{*0}\overline{x}. \tag{5.17}$$

As proposed in Section 4, the above optimal LQ - type control feedback is finally used for the system stability design. From (5.3), (5.14) and (5.16) we can deduce the concrete stability relations and the corresponding numerical parameters

$$\begin{aligned} \overline{Q}_1 &= (\overline{A}_1 \overline{\Delta}_1)^T P_0 + P_0 (\overline{A}_1 \overline{\Delta}_1) = \begin{bmatrix} 1.3867 & 1.4142 \\ 1.4142 & 0 \end{bmatrix}, \\ \overline{Q}_2 &= (\overline{A}_1 \overline{\Delta}_2)^T P_0 + P_0 (\overline{A}_1 \overline{\Delta}_2) = \begin{bmatrix} 0 & 0.6933 \\ 0.6933 & 2.8284 \end{bmatrix}. \end{aligned} \tag{5.18}$$

The spectra σ_j of the $\overline{Q}_j$, $j \in \{1, 2\}$ also have the concrete numerical expressions:

$$\sigma_1 = \{-0.8817, 2.2684\}, \quad \text{and} \quad \sigma_2 = \{-0.1608, 2.9892\}.$$

Hence

$$\begin{aligned} \lambda_{min}(\overline{Q}_0) = 1, \\ \lambda_{min}(\overline{Q}_1) = -0.8817, \quad \lambda_{min}(\overline{Q}_2) = -0.1608. \end{aligned} \tag{5.19}$$

Finally, from (5.19) and (5.13), we get (*cf.* (4.5) with $g_1 = (\overline{\alpha} - \underline{\alpha})$ and $g_2 = (\overline{\beta} - \underline{\beta})$):

$$\begin{aligned} \lambda_{min}(\overline{Q}_0) + (\overline{\alpha} - \underline{\alpha})\lambda_{min}(\overline{Q}_1) + (\overline{\beta} - \underline{\beta})\lambda_{min}(\overline{Q}_2) \\ = 1 + (0.5)(-0.8817) + (3)(-0.1608) \\ = 0.0767 > 0. \end{aligned} \tag{5.20}$$

The above inequality implies the conventional Lyapunov stability conditions (4.5).

Figure 1 depicts all the feasible points that satisfy the sufficient condition. Analysing this Figure, we can conclude that all the locations are inside the sufficiently big stability region, where the formal analytic stability conditions are expressed by (4.5).

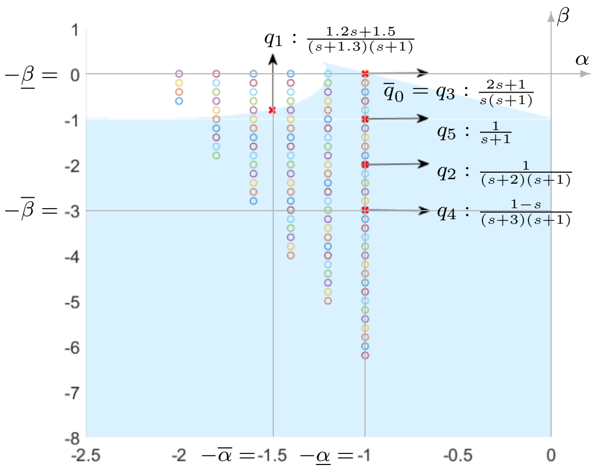

Fig. 1. The region in blue is the open loop necessary stability region, namely $\det(\lambda \mathbb{E} - \mathbb{A}_i)$ is a Hurwitz polynomial. The region depicted by the little circles is the closed loop guaranteed Lyapunov stability region (4.5).

5.3 Simulation Results

In this last part of the numerical section we deal with the concrete simulation results when applying a random switching signal $\mathfrak{i}(t)$ with

$$Aq_1 = \begin{bmatrix} -1.5 & 0.2 \\ -0.5 & -0.8 \end{bmatrix},\ Aq_2 = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix},$$
$$Aq_3 = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix},\ Aq_4 = \begin{bmatrix} -1 & -2 \\ 0 & -3 \end{bmatrix},\ Aq_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The generated signal $\mathfrak{i}(t)$ involves a random switching mechanism and hence the sub-systems and the realizations of the switching times are unknown. Figure 2 shows the time response under the initial condition $\overline{x} = [8 \;\; 3]^T$ and the switching signal $\mathfrak{i}(t)$.

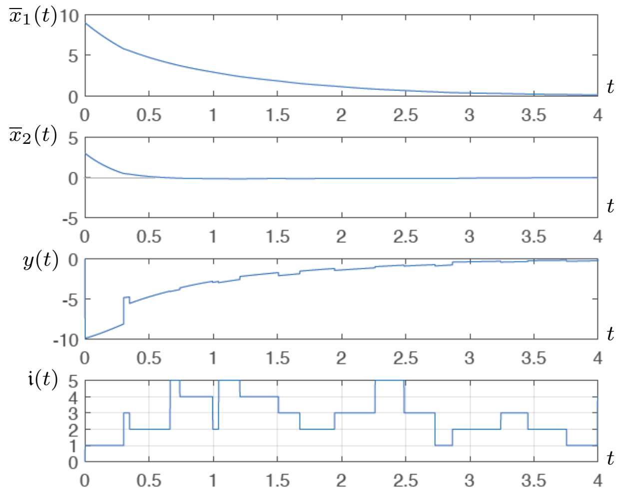

Fig. 2. Time response of $\overline{x}_1$ and $\overline{x}_2$ under the initial condition $\overline{x} = [8 \;\; 3]^T$ for the switching signal $\mathfrak{i}(t)$.

Clearly, the effect caused by a concrete realization of the switching mechanism implies changes on the states at every time a location switching occurs. On the other side, the system possesses the evident convergence properties with respect to the equilibrium point.

Considering the numerical results for that concrete illustrative example we can conclude that the resulting switched system (1.1), (5.1), (5.2), and (5.17) closed by the LQ-type optimal control is stable for every admissible realisation of systems locations.

6. CONCLUDING REMARKS

In this paper, we have proposed an optimal stabilizing state feedback for a wide class of switched dynamic models. The obtained matrices of the state space representation (1.1) associated to the class of switched system possess a specific structure similar to (Narendra *et al*, 1994). Moreover, the admissible switching mechanisms have a generic nature studied in Azhmyakov (2019). These switched dynamic models make it possible to consider the useful state space representation (1.1)–(1.2) and the corresponding global implicit representation (2.1)–(2.3). Note that the combinatorial structure of the locations set in switched systems we examined is in fact represented by the matrix $\overline{D}(q_i)$ (*cf.* (2.5)).

We firstly solve the conventional LQ optimization problem with the standard objective (3.1). The obtained optimal control feedback (3.17) naturally involves the necessary solution of the related ARE of the type (3.16) which in fact is formally undefined and depends on the given unknown active location q. In order to solve this generic undefined ARE we determine a specific structure of the weight matrix $\overline{Q}_q$, namely, we define the same structures as *H1* and (2.5). This fact makes it possible to obtain the ARE (3.25) which depends only on the known switching structure of (2.1): in that specific case on the following triplet: $(\overline{A}_0, \overline{A}_1, \overline{\Delta}_0)$. Finally, we are able to synthesize the optimal stabilizing state feedback (3.26) as shown in the main Theorem 1.

In fact, we have proven a kind of a "robustness" result with respect to a possible (admissible) switching mechanism: the LQ-type optimal control design from the main Theorem 1 stabilizes system (1.1)–(1.3) under the assumption of an unknown dynamic location $q \in \mathcal{Q}$. Note that the formal proof of Theorem 2 involves some recent results from (Bonilla *et al*, 2015b).

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