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On the Problems of Optimal and Minmax Type Control Under Vector Criteria ⋆

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Abstract: The topic of this paper is to solve two types of problems in controlled system dynamics formulated in terms of vector-valued criteria whose application depends on the type of ordering for the scalar participants of each such criterion. The first problem is that of optimal control under vector criterion with ordering of the Pareto type. The problem is to indicate the dynamics of the Pareto front. The second is that of finding vector-valued controls of the minmax type. Here the internal problem of dynamic maximization is due to a vector criterion with given type of ordering while the external problem is that of vector-valued dynamic minimization under another type of ordering. A similar situation arises for controls of the maxmin type. The paper indicates a variety of solution formulas that describe vector-valued dynamic interrelations for the problems of minmax and maxmin. The solutions are reached by using the Hamiltonian formalism. The suggested vector type of control problem settings are motivated by structure of system dynamics for physical motions, economics, finance, environmental models and related issues. Examples of applications are indicated.

Keywords: Dynamic programming, optimal control, reachability, set-valued minmax, multiobjective optimization.

1. INTRODUCTION

It is well known that many problems in control theory with or without unknown external disturbances involve the application of vector-valued solution criteria. It is equally known that application of vector criteria depends on the type of ordering the scalar participants of each such criterion. In this paper we consider two typical types of such problems as formulated for systems with dynamics.

We begin Parts 1-3 by problems of dynamics and control for discrete-time systems under a vector criterion with ordering of the Pareto type. Formulated here is the related version of the Principle of Optimality with control problem taken for a discretized vector functional of the Mayer-Bolza type. Here the solution is reached by using a discrete-time version of the Hamiltonian Formalism. A set-valued version of the vector-valued Hamilton-Jacobi-Bellman type equation is further indicated.

Next follows a section concerning continuous-time systems. Considered is a nonlinear system with hard bounds on the control and a continuous-time version of the Mayer-Bolza functional. A vector-valued evolution equation of the HJB type in continuous time is then produced. The techniques used here allow to solve the reachability problem for the introduced continuous time system.

The actual second part of the paper is in Part 4 (sections 4.1, 4.2) being devoted to vector-valued maxmin relations. This topic is an innovative issue. Described here in detail are the notions of vector-valued minmax and maxmin. Indicated here is the fact that in the case of these vector-valued notions there exist examples when maxmin is not less or equal to the minmax, but we may even have maxmin > minmax. Here formulated are necessary conditions when this does not happen.

2. DEFINITIONS

Consider here a feasible decision set X and a mapping $\mathbf{F}\colon X \to \mathbf{R}^p$. Let us introduce a Pareto preference order on the set $Y = F(X)$.

Definition 1. Vector $x \in \mathbf{R}^p$ is said to be dominated by vector $y \in \mathbf{R}^p$ in the sense of Pareto if $x \neq y$ and

$$y_i \leq x_i, \;\; i = 1, \ldots, p.$$

We further denote this relation as

$$y \leq x.$$

Definition 2. For any given nonempty $\tilde{X} \subseteq X$ the set-valued minimum for $F(\tilde{X})$ is further defined as

$$\mathbf{Min} F(\tilde{X}) = \left\{ f_* \in F(\tilde{X}) \;\middle|\; not\; \exists\, x \in \tilde{X}\colon F(x) \leq f_* \right\}.$$

The set-valued minimum is also known as the Pareto boundary (the Pareto front).

Definition 3. The set $\tilde{X} \subseteq X$ is said to be bounded from below if there exists an $M \in \mathbf{R}^p$ such that for each $\tilde{x} \in \tilde{X}$

$$M \leq \tilde{x}.$$

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3. THE HAMILTONIAN FORMALISM IN DISCRETE TIME

3.1 The system

Let us consider a dynamic system

$$x_{t+1} = f(t, x_t, u_t), \quad t = t_0, \dots, T-1, \qquad x_{t_0} = x^0 \tag{1}$$

with nonempty geometric control constraints

$$u_t \in \mathcal{P}_t, \quad t = t_0, \dots, T-1 \tag{2}$$

and a vector-valued functional of the Mayer-Bolza type

$$\mathbf{J}(T, x, u) = \sum_{t=t_0}^{T-1} \mathbf{L}(t, x_t, u_t) + \mathbf{\Phi}(x_T) \to \mathbf{Min}. \tag{3}$$

Here $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, and $\mathbf{L}[t], \mathbf{\Phi}[t], \mathbf{J}[t] \in \mathbf{R}^p$.

Following Kurzhanski, Varaiya (2014), we now recall the notion of reachability set.

Definition 4. For the given dynamic system the forward reachability set is defined as

$$\mathbf{X}[T] = \left\{ x \in \mathbf{R}^n | \exists u_t \in \mathcal{P}_t \colon \; x(T; t_0, x^0, u) = x \right\},$$

where $x(T; t_0, x^0, u)$ is a trajectory of system (1) generated by control u.

Let us now introduce an analogue of the reachability set for functional (3).

Definition 5. The reachability set for the mapping (3) under given constraints is further defined as

$$\mathbf{Z}[T] = \left\{ z \in \mathbf{R}^p | \exists u_t \in \mathcal{P}_t \colon \; \mathbf{J}(T, x(\cdot; t_0, x^0, u), u) = z \right\}.$$

Since the minimization of functional (3) in considered in the sense of Pareto, the value function for the problem (1)–(3) has to be a set-valued mapping. Hence, we introduce the following definition.

Definition 6. A set-valued value function for the problem (1)–(3) is a mapping

$$\mathbf{V}(t_0, x^0) = \mathbf{Min}\, \mathbf{Z}[T] = \mathbf{Min}\, \mathbf{Z}(T; t_0, x^0).$$

3.2 An analogue of The Principle of Optimality

This value function is further used to obtain a series of multidimensional analogues for the classic Principle of Optimality and for the Bellman equation. The idea behind both results is based on the following lemma.

Lemma 1. Given any nonempty sets $A, B \subseteq \mathbf{R}^p$, consider their Minkowski sum $C = A + B$. If the set C is closed and bounded from below, the next relation will be true:

$$\mathbf{Min}\, C = \mathbf{Min}\left\{ A + \mathbf{Min}\, B \right\}. \tag{4}$$

This lemma is indicated and proved in Komarov (2019). It describes the conditions that justify bringing one **Min** sign under another. An alternative (and less strict) form for this can be described as follows: assume sets A and B to be nonempty and the Pareto's boundary $\mathbf{Min}\{A + B\}$ to exist. Then the relation (4) is true.

For any given $t = t_0, \dots, T-1$ we may rewrite $\mathbf{Z}[T]$ in the following form:

$$\mathbf{Z}(T; t_0, x^0) = \left\{ \sum_{s=t_0}^{t} \mathbf{L}(s, x_s, u_s) + \mathbf{Z}(T; t+1, x_{t+1}) \middle| u_s \in \mathcal{P}_s \right\},$$

where $x_s = x(s; t_0, x^0, \{u_i\}_{i=t_0}^{s-1})$, $s = t_0, \dots, t$.

Applying lemma 1 to the last relation, we come to a set-valued analogue for the conventional Principle of Optimality .

Proposition 1. Suppose the set $\mathbf{Z}[T]$ is closed and bounded from below. Then, for any given $t = t_0, \dots, T$, the following multiobjective analogue of the Principle of Optimality is true for the set-valued value function $\mathbf{V}(t_0, x^0)$:

$$\mathbf{V}(t_0, x^0) = \mathbf{Min} \left\{ \sum_{s=t_0}^{t} \mathbf{L}(s, x_s, u_s) + \mathbf{V}(t+1, x(t+1; t_0, x^0, u)) \right\}. \tag{5}$$

3.3 An analogue of the Bellman equation

Consider the value function V at an arbitrary point (t, x), where x is supposed to be reachable, i.e. $x \in \mathbf{X}[t]$. Then, following the same scheme, we can introduce a set-valued analogue for the classical Bellman equation.

Proposition 2. Suppose the reachability sets $\mathbf{Z}(T; t, x)$ are closed and bounded from below for each $t = t_0, \dots, T-1$ and $x \in \mathbf{X}[t]$. Then the introduced value function satisfies the following Bellman-type equation:

$$\mathbf{V}(t, x) = \mathbf{Min} \Big\{ \mathbf{L}(t, x_t, u_t) + \mathbf{V}(t+1, x(t+1; t, x_t, u_t)) \Big\}, t = t_0, \dots, T-1 \tag{6}$$

with boundary condition

$$\mathbf{V}(T, \cdot) = \mathbf{\Phi}(\cdot). \tag{7}$$

3.4 On the discrete-time HJB solutions for vector-valued control

The introduced relation (6) (and see also the further (16)) provide necessary, but not sufficient conditions for the set-valued value functions. Thus, an auxiliary analysis is to be done for the solution of corresponding equations in order to find $\mathbf{V}(t, x)$. We now illustrate the absence of sufficiency by an example.

Example 1. Consider the following discrete-time dynamic system:

$$x_{t+1} = x_t + u_t, \; t = 0, 1, 2, \qquad x_0 = 0, \qquad u_t \in [-1; 1] \tag{8}$$

with the vector-valued criterion

$$\mathbf{J}(x, u) = \begin{bmatrix} -x_3 \\ \sum_{t=0}^{2} x_t^2 \end{bmatrix} = \begin{bmatrix} -(u_0 + u_1 + u_2) \\ u_0^2 + (u_0 + u_1)^2 \end{bmatrix} \to \mathbf{Min}. \tag{9}$$

Solving the corresponding Bellman-type equation (6), we get the following minimizers:

$$\begin{cases} u_2 = 1, \\ u_1 \in [0, 1], \\ u_0 \in [-u_1/2, 1]. \end{cases} \tag{10}$$

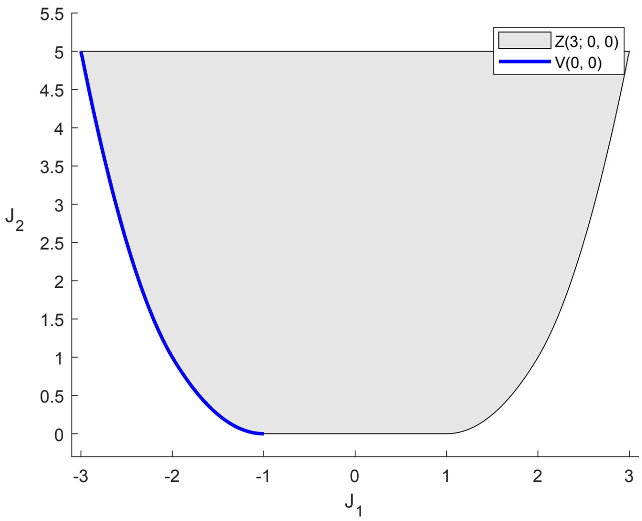

Fig. 1. Pareto boundary for the system (8)

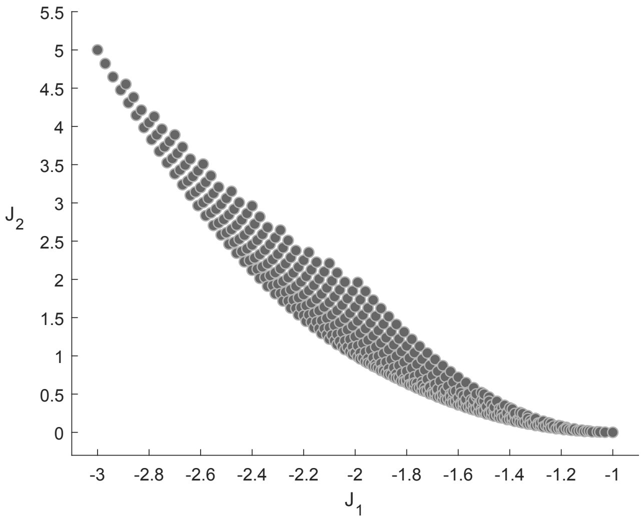

Fig. 2. The values of $\mathbf{J}(x,u)$ that corresponds to minimizers (10)

The true Pareto boundary for the considered problem and the corresponding values of $\mathbf{J}(x,u)$ for the obtained control values are illustrated by Fig. 1 and Fig. 2 respectively.

As one can observe, the set from Fig. 2 is not the expected value of Pareto boundary for $\mathbf{V}(0,0)$ of Fig. 1. Moreover, it cannot be a Pareto boundary for any given set $A \subset \mathbf{R}^p$.

3.5 The continuous-time system

The earlier type of results could be also derived for a system in continuous time.

Consider the following system

$$\begin{aligned} \dot{x} = f(t,x,u),\ t \in [t_0, \vartheta], \\ x(t_0) = x^0 \end{aligned} \tag{11}$$

with nonempty geometric control constraints

$$u(t) \in \mathcal{P}(t),\ t \in [t_0, \vartheta] \tag{12}$$

and functional

$$\mathbf{J}(\vartheta, x(\cdot), u(\cdot)) = \int_{t_0}^{\vartheta} \mathbf{L}(\tau, x(\tau), u(\tau))d\tau + \mathbf{\Phi}(x(\vartheta)). \tag{13}$$

Here $\mathcal{P}(t)$ is a set-valued functional, continuous in the Hausdorff metric.

Definition 7. The reachability set for the described system (11)–(13) is

$$\begin{aligned} \mathbf{X}[\vartheta] = \{x \in \mathbf{R}^n |\ \exists u(t) \in \mathcal{P}(t): \\ x(\vartheta; t_0, x^0, u(\cdot)) = x\}\,. \end{aligned}$$

Definition 8. The reachability set for the introduced mapping (13) under given constraints is

$$\begin{aligned} \mathbf{Z}[\vartheta] = \{z \in \mathbf{R}^p |\ \exists u(t) \in \mathcal{P}(t): \\ \mathbf{J}(\vartheta, x(\cdot; t_0, x^0, u(\cdot)), u(\cdot)) = z\} \end{aligned}$$

Definition 9. A set-valued value function for the problem (11)–(13) is the mapping

$$\mathbf{V}(0, x^0) = \mathbf{Min}\ \mathbf{Z}[\vartheta] = \mathbf{Min}\ \mathbf{Z}(\vartheta; t_0, x^0). \tag{14}$$

For the continuous-time case one also allows to have an analogue for the Principle of Optimality and a related Hamilton-Jacobi-Bellman-type equation. But while the Principle of Optimality has a form, similar to the discrete-time case, the HJB equation is to be constructed in terms of an evolution equation, since it requires to use a derivative for the set-valued value function to be handled.

Proposition 3. Suppose the set $\mathbf{Z}[\vartheta]$ is closed and bounded from below. Then, for any given $t \in [t_0, \vartheta]$, the set-valued function $\mathbf{V}(t_0, x^0)$ satisfies the semigroup property in the following form:

$$\begin{aligned} \mathbf{V}(t_0, x^0) = \mathbf{Min} \Bigg\{ \int_{t_0}^{t} \mathbf{L}(\tau, x(\tau), u(\tau))d\tau + \\ \mathbf{V}(t, x(t; 0, x^0, u(\cdot))) \Bigg\}. \end{aligned} \tag{15}$$

Now again, let us consider this value function at an arbitrary point (t,x), $x \in \mathbf{X}[t]$.

Proposition 4. Suppose the reachability sets $\mathbf{Z}(\vartheta; t, x)$ are closed and bounded from below for each $t \in [t_0, \vartheta]$ and $x \in \mathbf{X}[t]$. Then the value function (14) satisfies the following Hamilton-Jacobi-Bellman-type equation in the form of a set-valued evolution equation:

$$\begin{aligned} \lim_{\sigma \to +0} \frac{1}{\sigma} \mathbf{h} \Bigg(\mathbf{V}(t,x), \mathbf{Min} \Bigg\{ \int_{t}^{t+\sigma} \mathbf{L}(s, x(s), u(s))ds + \\ \mathbf{V}(t+\sigma, x(t+\sigma)) \Bigg| u(s) \in \mathcal{P}(s) \Bigg\} \Bigg) = 0, \end{aligned} \tag{16}$$

with boundary condition

$$\mathbf{V}(\vartheta, \cdot) = \varphi(\cdot). \tag{17}$$

Here $\mathbf{h}(A,B)$ denotes the Hausdorff distance between sets A and B and the $x(t+\sigma)$ in the second part of last relation is understood in the sense of $x(t+\sigma; t, x, u(\cdot))$, where $u(\cdot)$ is an argument of the integral under the sign **Min** .

Remark 1. Given conditions of proposition 4, the set-valued value function $\mathbf{V}(t,x)$ satisfies the evolution equation (16) rewritten in terms of the Hausdorff semi-distance $\mathbf{h}_+(A,B)$.

The introduced approach may be applied to calculation of ellipsoidal approximations for the reachability tubes.

The original method is described in Kurzhanski, Varaiya (2014). One has to solve two optimization problems to find appropriate approximation — the first one provides the trajectory of the center of the ellipsoidal tube while the second one describes the evolution of the matrix of the ellipsoid. These two optimization problems may be rewritten in terms of vector-valued cost minimization and solved with the proposed technique.

3.6 The vector value function for the reachability problem

The introduced vector-valued dynamic programming can also be applied to solve reachability problems in continuous time. Here we describe the value function that provides an evaluation of the forward reachability set for such systems. Similar results may also be obtained for the backward reachability problem in continuous time (and also for systems with discrete-time dynamics).

Consider the following reachability problem: given

$$\begin{gathered} \dot{x} = f(t,x,u),\ t \in [t_0, \vartheta], \\ u(\cdot) \in \mathcal{P}(\cdot), \\ x(t_0) \in \mathcal{X}^0 = \mathcal{X}_1^0 \cap \mathcal{X}_2^0, \end{gathered} \tag{18}$$

find reachability set $\mathbf{X}(\vartheta; t_0, \mathcal{X}_1^0 \cap \mathcal{X}_2^0)$.

Here the mapping $\mathbf{X}(t; t_0, \mathcal{X}^0)$ is understood in the following sense:

$$\mathbf{X}(t; t_0, \mathcal{X}^0) = \bigcup_{x^0 \in \mathcal{X}^0} \mathbf{X}(t, ; t^0, x^0).$$

The classic way of solving such problems is described in details in Kurzhanski, Varaiya (2014). The main idea of the method is to introduce an auxiliary optimization problem with the following value function:

$$\mathbf{V}(t,x) = \min_{x^0, u(\cdot)} \left\{ \mathrm{d}^2(x^0, \mathcal{X}^0) \middle| x(t; t_0, x^0, u(\cdot)) = x \right\}, \tag{19}$$

where $\mathrm{d}(x, Z) = \min \{ \langle x - Z, x - Z \rangle \,|z \in Z \}$ is an Euclidian metric.

Addressing system (18), the following vector-valued optimization problem may be considered:

$$\mathbf{J}(\vartheta, x, u) = \begin{bmatrix} \mathrm{d}^2(x(t_0), \mathcal{X}_1^0) \\ \mathrm{d}^2(x(t_0), \mathcal{X}_2^0) \end{bmatrix} \to \mathbf{Min}.$$

Lemma 2. Consider the following set-valued function for the problem (18):

$$\mathbf{V}(t,x) = \mathbf{Min} \left\{ \begin{bmatrix} \mathrm{d}^2(x^0, \mathcal{X}_1^0) \\ \mathrm{d}^2(x^0, \mathcal{X}_2^0) \end{bmatrix} \middle| x(t; t_0, x^0, u(\cdot)) = x \right\},$$

with control strategies that satisfy constraint $u(\cdot) \in \mathcal{P}(\cdot)$. Then the reachability sets for the system (18) at time $t \in [t_0, \vartheta]$ are the level sets of the introduced value function:

$$\mathbf{X}(\vartheta; t_0, \mathcal{X}_1^0 \cap \mathcal{X}_2^0) = \{ x : \mathbf{V}(t,x) \leq 0 \}.$$

The last inequality is understood in the sense of Pareto.

Corollary 1. The next statement describes the relation between the classic value function (19) for problem (18) and the next set-valued one:

$$\begin{gathered} \mathbf{X}(\vartheta; t_0, \mathcal{X}_1^0 \cap \mathcal{X}_2^0) = \\ \left\{ x \colon \mathbf{Min} \left\{ \begin{bmatrix} \mathrm{d}^2(x^0, \mathcal{X}_1^0) \\ \mathrm{d}^2(x^0, \mathcal{X}_2^0) \end{bmatrix} \middle| x(t; t_0, x^0, u) = x \right\} \leq 0 \right\} = \\ \left\{ x \colon \min \left\{ \mathrm{d}^2(x^0, \mathcal{X}_1^0 \cap \mathcal{X}_2^0) \middle| x(t; t_0, x^0, u) = x \right\} \leq 0 \right\}. \end{gathered}$$

Remark 2. The same results can be obtained for any given p such that

$$\mathcal{X}^0 = \bigcap_{i=1}^{p} \mathcal{X}_i^0.$$

4. THE VECTOR-VALUED MINMAX-MAXMIN RELATIONS

4.1 Additional definitions

In order to prevent confusion, in this section we will not use the notion of Pareto boundary but use the name **Min** operator to be a set-valued minimum.

Definition 10. For any given nonempty $\tilde{X} \subseteq X$ the set-valued maximum for $F(\tilde{X})$ is further defined as

$$\mathbf{Max} F(\tilde{X}) = \left\{ f^* \in F(\tilde{X}) \middle| \, not\ \exists\, x \in \tilde{X} \colon f^* \leq F(x) \right\}.$$

Consider the mapping

$$F(u,v) : U \times V \to \mathbf{R}^p.$$

Sets U, V may be finite-dimensional as well as infinite-dimensional in a space of functions. The given approach doesn't include an analysis of their structure, but assumes some restrictions to be fulfilled. We further assume that every image $F(U,v)$, $F(u,V)$ for any given $u \in U$, $v \in V$ is closed and bounded from both sides:

$$\exists M_*, M^* \colon\ M_* \leq F(U,V) \leq M^*.$$

Let us now introduce the notion of set-valued minmax and maxmin operators.

Definition 11. The vector-valued minmax for the elements of $F(u,v)$ over the set $U \times V$ is defined as

$$\mathbf{Min}_u \mathbf{Max}_v F(u,v) = \mathbf{Min} \left\{ \bigcup_{u \in U} \mathbf{Max} F(u, V) \right\}.$$

Addressing this notion from the perspective of the theory of games, we suppose that the second player knows the strategy u of the first player and chooses one of the optimal solutions (in the sense of Pareto), while the first one aims to minimize his cost.

Definition 12. The vector-valued maxmin for the elements of $F(u,v)$ over the set $U \times V$ will be taken as the mapping

$$\mathbf{Max}_v \mathbf{Min}_u F(u,v) = \mathbf{Max} \left\{ \bigcup_{v \in V} \mathbf{Min} F(U, v) \right\}.$$

The increase in dimensionality of the value function may cause unexpected effects. Namely, while functional $f \colon U \times V \to \mathbf{R}$ always ensures inequality

$$\max \min f(u,v) \leq \min \max f(u,v),$$

the vector-valued $F(u,v)$ does not, as it will be shown further.

In order to compare two set-valued mappings, we introduce the following notation.

Definition 13. We will assume that arbitrary nonempty sets $A, B \subset \mathbf{R}^p$ satisfy the inequality

$$A \leq B,$$

if the next property is true:

$$\forall b \in B \setminus A \Rightarrow \exists a \in A\colon\ a \leq b.$$

Definition 14. The main minmax inequality for the Pareto ordering will be defined as

$$\mathbf{Max}_v\mathbf{Min}_u F(u,v) \leq \mathbf{Min}_u\mathbf{Max}_v F(u,v)$$

and the inverse minmax inequality will be understood as

$$\mathbf{Min}_u\mathbf{Max}_v F(u,v) \leq \mathbf{Max}_v\mathbf{Min}_u F(u,v).$$

We will further consider two common types of functional $F(u,v)$ and discuss the necessary conditions for the existence of both inequalities.

4.2 The functional of type $\mathbf{\Phi}(u) + \mathbf{\Psi}(v)$

Consider the functional with separated inputs u and v:

$$\mathbf{F}(u,v) = \mathbf{\Phi}(u) + \mathbf{\Psi}(v).$$

We can get another representation for the set-valued minmax, namely,

$$\begin{gathered}\mathbf{Min}_u\mathbf{Max}_v\mathbf{F}(u,v) = \mathbf{Min}_u\mathbf{Max}_v[\mathbf{\Phi}(u) + \mathbf{\Psi}(v)] = \\ \mathbf{Min}\left\{\mathbf{\Phi}(\tilde{u}) + \mathbf{\Psi}(V)\middle|\tilde{u} \in U\right\} = \\ \mathbf{Min}\left\{\mathbf{\Phi}(U) + \mathbf{\Psi}(V)\right\}.\end{gathered}$$

Continuing similarly, we get an equality for the maxmin:

$$\begin{gathered}\mathbf{Max}_v\mathbf{Min}_u\mathbf{F}(u,v) = \mathbf{Max}_v\mathbf{Min}_u[\mathbf{\Phi}(u) + \mathbf{\Psi}(v)] = \\ \mathbf{Max}\left\{\mathbf{\Phi}(U) + \mathbf{\Psi}(\tilde{v})\middle|\tilde{v} \in V\right\} = \\ \mathbf{Max}\left\{\mathbf{\Phi}(U) + \mathbf{\Psi}(V)\right\}.\end{gathered}$$

As one can see, if the inputs of functional $\mathbf{F}(u,v)$ are separated, there exists the inverse minmax inequality for the Pareto ordering since

$$\mathbf{Min}\left\{\mathbf{\Phi}(U) + \mathbf{\Psi}(V)\right\} \leq \mathbf{Max}\left\{\mathbf{\Phi}(U) + \mathbf{\Psi}(V)\right\}.$$

However, there are conditions that turn an inequality into an equality which thus makes the main minmax inequality satisfied:

- at least one of $\mathbf{\Phi}(U)$ and $\mathbf{\Psi}(V)$ is presented by a single element;
- $\mathbf{\Phi}(U)$ and $\mathbf{\Psi}(V)$ lies in the same hyperplane.

In the second case the Minkowsky sum $\mathbf{\Phi}(U)+\mathbf{\Psi}(V)$ lies in the same hyperplane and this set is equal to its set-valued maximum and min. The last statement follows from the definition of Pareto ordering.

Example 2. Consider the vector-valued mapping

$$\mathbf{F}(u,v) = \begin{bmatrix} -u \\ u^2 \end{bmatrix} + \begin{bmatrix} v \\ -v^2 \end{bmatrix} \tag{20}$$

defined on $U \times V$ where

$$U = V = [0,1]. \tag{21}$$

The set-valued maxmin and minmax for the given problem is presented by Fig. 3. There are three points where the boundaries intersect and all the other points provide an inverse minmax inequality. However, if we extend the domain to

$$U = V = [-1,1], \tag{22}$$

then both minmax and maxmin will collapse to point $(0,0)$ and thus will be equal, as illustrated by Fig. 4

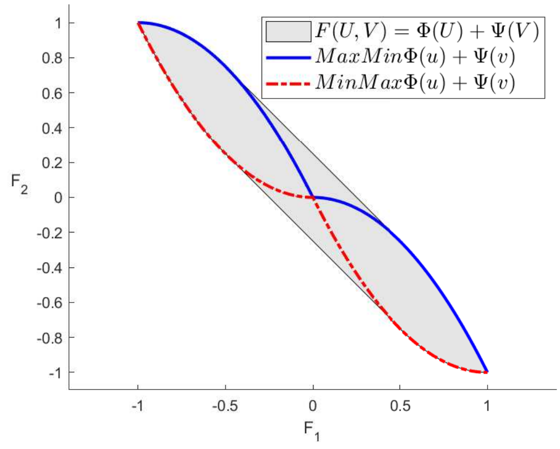

Fig. 3. MinMax and MaxMin boundaries for the functional (20) with domain (21)

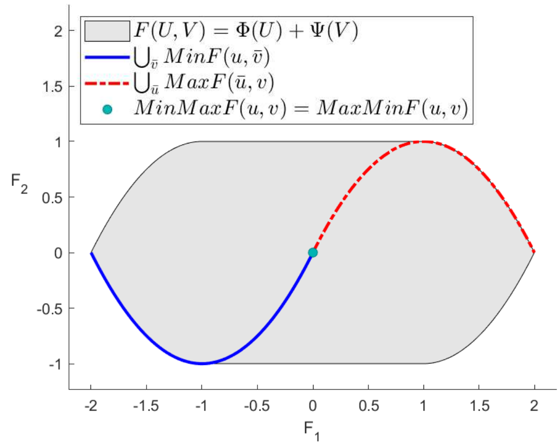

Fig. 4. Equality of MinMax and MaxMin boundaries for the functional (20) with domain (22)

4.3 A necessary condition for the violation of the main minmax inequality

In order to proceed with the second type of vector-valued functional, described in this paper, the next important statement is to be introduced. It provides a necessary condition for violation of the main minmax inequality (for any given functional $\mathbf{F}(u,v)$) and thus requires further research on sufficient conditions for its fulfillment.

Proposition 5. Suppose there exist

$$\begin{aligned} f^* &= \mathbf{F}(u^*,v^*) \in \mathbf{Min}_u\mathbf{Max}_v\mathbf{F}(u,v), \\ f_* &= \mathbf{F}(u_*,v_*) \in \mathbf{Max}_v\mathbf{Min}_u\mathbf{F}(u,v), \end{aligned}$$

such that

$$f^* \leq f_*.$$

Then the value $\hat{f} = \mathbf{F}(u^*,v_*)$ is not comparable with both f^* and f_*.

Remark 3. The last statement is true for any given ordering. However, if considered is the Pareto ordering, then the proposition may be rewritten in the following form.

Corollary 2. Consider a mapping

$$\mathbf{F}(u,v) = [F_1(u,v), \ldots F_p(u,v)]'.$$

Suppose there exist

$$f^* = \mathbf{F}(u^*, v^*) \in \mathbf{Min}_u\mathbf{Max}_v\mathbf{F}(u, v),$$
$$f_* = \mathbf{F}(u_*, v_*) \in \mathbf{Max}_v\mathbf{Min}_u\mathbf{F}(u, v),$$

such that

$$f^* \le f_*.$$

Then there exists an $i \ne j, k \ne l$ such that both products

$$(F_i(u^*, v^*) - F_i(u^*, v_*))\,(F_j(u^*, v^*) - F_j(u^*, v_*))\,,$$
$$(F_k(u_*, v_*) - F_k(u^*, v_*))\,(F_l(u_*, v_*) - F_l(u^*, v_*))$$

are negative.

4.4 The functional of type $F_i(u, v) = u'A_iv$

Consider the following mapping:

$$\mathbf{F}(u, v) = \begin{bmatrix} u'A_1v \\ \cdots \\ u'A_pv \end{bmatrix},$$

where $u \in \mathbf{R}^n$, $v \in \mathbf{R}^m$ and $A_i \in \mathbf{R}^{n\times m}$.

A necessary condition for violating the main minmax inequality in terms of corollary 2 has the form of

$$\begin{cases} \langle u^*, A_i(v^* - v_*)\rangle \langle u^*, A_j(v^* - v_*)\rangle < 0, \\ \langle (u^* - u_*), A_kv_*\rangle \langle (u^* - u_*), A_lv_*\rangle < 0. \end{cases} \quad (23)$$

We now require that for all nonzero $u \in \mathbf{R}^n$, $v \in \mathbf{R}^m$ and $i, j = 0, \ldots, p$ the following condition was fulfilled:

$$(u`A_iv)(u'A_jv) \ge 0.$$

This inequality guarantees that relations (23) never turn out to be true.

Rewrite the last inequality:

$$(u`A_iv)(u'A_jv) = u'(A_ivv'A_j')u \ge 0.$$

We will now look for restrictions that ensure matrix $Q = A_ivv'A_j'$ to be positive semi-definite simultaneously for all $v \in \mathbf{R}^m$.

For this situation we further use the the criterion of Silvester.

Theorem 1. A symmetric matrix $Q = Q' \in \mathbf{R}^{m\times m}$ is positively semi-definite if and only if all its main minor matrices are nonnegative .

Since in general the considered matrix Q, is not symmetrical, the application of Silvester criterion requires the next additional proposition.

Lemma 3. For an arbitrary matrix $Q \in \mathbf{R}^{m\times m}$ the relation $\forall v \ne 0 \Rightarrow v'Qv \ge 0$ is true if and only if the next inequality is true:

$$\frac{Q + Q'}{2} \ge 0.$$

Since

$$\operatorname{rank}\left((A_ivv'A_j') + (A_ivv'A_j')'\right) \le 2,$$

we can relax the conditions of the Silvester criterion and the constraint $v \in \mathbf{R}^m$.

Proposition 6. Suppose for all $i, j = 1, \ldots, p,\ i \ne j$ all the angular minors $\mathcal{M}_k[S]$ of matrix

$$S = A_ivv'A_j' + A_jvv'A_i',$$

of order $k \le 2$ are nonnegative simultaneously for all $v \in (V - V)$. Then the main minmax inequality for Pareto ordering will be true.

If we find an explicit relation for diagonal elements of matrix $S = Q + Q'$, the next corollary may be introduced.

Corollary 3. Suppose conditions of proposition 6 are true. Then $\forall i \ne j = 1, \ldots, r$

$$[A_i]_{kl}[A_j]_{kl} \ge 0,\ \ \forall k = 1, \ldots, n,\ l = 1, \ldots, m.$$

The last proposition provides an intuitive condition to check if the main minmax inequality for a functional of considered type can be true.

Remark 4. Similar conditions for turning the reverse minmax inequality for a functional of the considered type to be true can be obtained if one considers condition

$$(u`A_iv)(u'A_jv) = u'(A_ivv'A_j')u \le 0.$$

5. CONCLUSION

In course of this paper two problems for dynamic systems with multiple criteria were discussed. The first one is of optimal control. Addressing this problem the vector dynamic programming method was introduced and described in details for system with discrete-time dynamics as well as for the one with continuous time. The vector analogues for the Principle of Optimality and the related Hamilton-Jacobi-Bellman equation are introduced and discussed.

Addressing the second problem of finding controls of the minmax type, the notions of set-valued minmax and maxmin were introduced. The possibility of violating the main minmax inequality was illustrated with an example. Vector-valued interrelations for the problems of minmax and maxmin were discussed for two common types of functionals.

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