

Maximal Delay Range for Robust Consensus Achieved by PID-Type Control Protocol with Time-Varying Delay^{*}

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Abstract: In this paper, we examine the maximal delay range for robust consensus by using PID-type control protocol for linear first-order multi-agent systems subject to time-varying delays. We derive explicit lower bounds for guaranteed robust consensus of first-order unstable agents collaborated with each other under PID-type control protocol with time-varying delays, which provide *a priori* the range of delay over which the multi-agent system is guaranteed to obtain robustly consensus by proportional (P) and proportional-derivative (PD) protocols for undirected graphs respectively. The results show how the agent dynamics and graph connectivity may fundamentally limit the range of delay tolerable. They also indicate that the derivative control protocol provides an added benefit to increase the allowable delay range by incorporating the delay variation rate. Finally, the numerical examples are used to illustrate the effectiveness of the proposed theoretical results.

Keywords: Maximal delay range, robust consensus, time-varying delay, PID-type protocol, undirected graph.

1. INTRODUCTION

Multi-Agent Systems (MAS) have increasing consideration in the distributed coordination and optimization in recent years. Consensus problem, as a central problem in the analysis and design of MAS, has been studied extensively. However, due to the agents communication to their neighbors via a communication network with limited bandwidth, the information transmission will be inherent delayed. Another delay comes from the computation and the execution time of the agents. These delays are often unknown and time-varying. To account for the potential degrading effect of such time delays, one must incorporate delays into consensus protocol design. Meanwhile, from the control design point of view, it is necessary to address the robustness of the consensus protocol. As such, it is of interest to develop conditions such that the MAS may achieve consensus robustly despite the presence of possibly uncertain and time-varying delays, and to provide estimation on the delay range so that the consensus can be guaranteed *a priori*.

Delay robustness of MAS consensus has been addressed in the recent literature. Notably, upper bounds on homogeneous delay are obtained in Olfati-Saber et al. (2004) to guarantee the consensus robustness of first-order single-integrator agents. Heterogeneous delays are considered in Münz et al. (2010), where a frequency-sweeping method was proposed to estimate the delay range for consensus robustness. The method also gives rise to explicit bounds for single-integrator agents. For second-order agents, robust consen-

suability problems were studied in Münz et al. (2010), Yang et al. (2010) and Wang et al. (2014), which result in conditions that ensure double-integrator agents to achieve consensus robustly against delays varying within a range. More generally, in the presence of time-varying delays, some sufficient conditions on bounded uniform delays with arbitrarily fast time-varying are proposed in Chen et al. (2017) by analyzing the delay-dependent gains. Another maximal admissible upper bound on robust consensus of MAS with bounded uniform delays is required to have the eigenvalues on the imaginary axis (Wang et al. (2014)). In addition, it was shown in Münz et al. (2011) that with a nonlinear, adaptive control protocol, consensus can be maintained for arbitrarily long delays. The majority of the existing results mainly concern with the design of consensus protocols of varying complexities, ranging from simple static state feedback to dynamic, nonlinear and adaptive output feedback. In this paper we consider agents under a time-honored and the favored control law (Åström et al. (1995)) PID-type control protocol with uniform time-varying delays, under undirected network topologies. Previous results on PID-type consensus protocols can be found in, e.g., Ma, and Chen (2019), though delay consensus robustness has been seldom addressed. Our objective is to find the maximal allowable delay range of first-order MASs, within which the MAS can achieve and maintain consensus robustly. On the other hand, motivated by the recent theoretical studies of PID control for delay systems (e.g., Silva et al. (2002); Ma, and Chen (2019)), we focus on the restricted structure and complexity PID-type control protocols, and especially examine the maximal delay range for robust consensus achievable by P-type and PD-type control protocols. From the important discovery that we found, PID-type and PD-

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type protocols achieve the same delay robustness (Chen et al. (2019); Ma, Chen, Lu, & Chen (2019)), it suffices to consider PD-type protocols to investigate the delay robustness.

In this paper, we study the robustness ability of the feedback control protocol to the time-varying delay and obtain the explicit lower bound for the rate-independent delay in light of the small-gain theorem. A sufficient and easy-to-check condition for the guaranteed robust consensus range of the MAS is established. Furthermore, we try to find that the derivative feedback protocol offers the extra degree of freedom to obtain an increased delay range by incorporating the delay variation rate. As expected, the bounds show how the agent dynamics and graph connectivity may fundamentally limit the allowable range of delay under undirected graphs, if the time-varying delay is variation rate-independent. In addition, the allowable delay range will be increased by incorporating the delay variation rate under the PD-type control protocol with respect to P-type control protocol.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Mathematical Preliminaries

We begin with a brief review of algebraic graph theory. A graph of order N can be represented as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, N\}$ is the node set with each node representing an agent, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an *edge* set of paired nodes, and $\mathcal{A} = [a_{ij}]$ is an $N \times N$ adjacency matrix of the graph \mathcal{G} . If an edge $(i, j) \in \mathcal{E}$, the j th node can obtain information from the i th node. The node i is called a *neighbor* of node j . The set of neighbors of node i is denoted as $\mathcal{N}_i \triangleq \{j | (j, i) \in \mathcal{E}\}$. The graph \mathcal{G} is said to be *undirected* if for all $u, v \in \mathcal{V}$, $(u, v) \in \mathcal{E}$ implies that $(v, u) \in \mathcal{E}$. A *path* from node v_1 to node v_k is a sequence of nodes v_1, \dots, v_k such that for each i , $1 \leq i \leq k-1$, (v_i, v_{i+1}) is an edge. A graph is said to be *connected* if there exists a path from any node to any other node, and *complete* if every pair of distinct vertices is connected by an edge. Throughout this paper, we assume that the graph under consideration is undirected and connected. The adjacency matrix $\mathcal{A} = [a_{ij}]$ of a graph \mathcal{G} satisfies the conditions $a_{ii} = 0$ and $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$, $a_{ij} = 0$ if $(v_j, v_i) \notin \mathcal{E}$ for all $i, j \in \mathcal{V}$. The degree of node i is defined as $d_i = \sum_{j=1}^N a_{ij}$, $i, j \in \mathcal{V}$. Then for the graph \mathcal{G} , the degree matrix of \mathcal{G} is defined by $\mathcal{D} \triangleq \text{diag}\{d_1, d_2, \dots, d_N\}$, and the Laplacian matrix $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Denote by $\mathbf{1}_N = [1 \ 1 \ \dots \ 1]^T$. For a connected graph, it is well-known that \mathcal{L} has zero row sums, or equivalently, \mathcal{L} has an eigenvalue at the origin with the eigenvector $\mathbf{1}_N$, i.e., $\mathcal{L}\mathbf{1}_N = \mathbf{0}$. Furthermore, the Laplacian matrix \mathcal{L} admits a unitary decomposition $\mathcal{L} = \mathcal{W}\Lambda\mathcal{W}^H$, $\mathcal{W} = [w_1 \ w_2 \ \dots \ w_N]$ is a unitary matrix. In particular, for an undirected graph, Λ is in general symmetric and nonnegative definite, so that $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ with $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$.

2.2 Consensus and Protocol

Consider a MAS of continuous-time first-order linear agents

$$\dot{x}_i(t) = \alpha x_i(t) + u_i(t), \quad i = 1, \dots, N, \quad (1)$$

where $\alpha \geq 0$ represents the unstable dynamics of the agents, $u_i(t)$ are the control inputs, and $x_i(t)$ are the states of the agents. Note that for a first-order system, the state coincides with its output. Since the existing communication delay and

the self-delay, with the network graph represented by its adjacency matrix \mathcal{A} , we consider the following consensus protocol

$$u_i(t) = -K(s) \sum_{j=1}^N a_{ij} [x_i(t - \tau(t)) - x_j(t - \tau(t))], \quad (2)$$

where $K(s)$ is a linear time-invariant feedback control, which in general is a dynamic output feedback control law, $\tau(t)$ is a time-varying delay, satisfying that

$$0 \leq \tau(t) \leq \tau_M, \quad (3)$$

$$0 \leq |\dot{\tau}(t)| \leq \rho \leq 1, \quad (4)$$

where τ_M is the maximal delay range and ρ is the maximal variation rate of the delay.

In this paper we are particularly interested in PID-type control protocols. Specifically, we focus on the subclass of proportional-derivative (PD-type) control protocols

$$K_{PD}(s) = k_p + k_d s, \quad (5)$$

and the proportional (P-type) control protocols

$$K_P(s) = k_p. \quad (6)$$

For any finite initial state $x_i(0)$, the MAS (1) achieves consensus over the graph \mathcal{G} if and only if

$$\lim_{t \rightarrow +\infty} \|x_j(t) - x_i(t)\| = 0, \quad \forall i, j = 1, \dots, N. \quad (7)$$

Furthermore, with the delayed control protocol (2), for some delay value $\bar{\tau} \geq 0$, we say that the MAS (1) achieves robust consensus over the graph \mathcal{G} if for any finite $x_i(0)$, $i = 1, \dots, N$, the condition (7) holds for all $\tau(t) \in [0, \bar{\tau}]$.

Define $\delta_i(t) = x_i(t) - x_1(t)$, $i = 1, \dots, N$ and write $\delta(t) = [\delta_1(t), \delta_2(t), \dots, \delta_N(t)]^T$. For the PD-type control protocol (2) with (5), it follows that

$$\dot{\delta}(t) = \alpha \delta(t) - (\mathcal{L} - \mathbf{1}_N \beta^T) (k_p \delta(t - \tau(t)) + k_d \dot{\delta}(t - \tau(t))), \quad (8)$$

where β^T is the first row of \mathcal{L} . Evidently, the consensus condition (7) is satisfied if and only if $\lim_{t \rightarrow +\infty} \delta(t) = 0$. In addition, let $\tilde{\delta}(t) = \mathcal{W}^T \delta(t)$ and $\hat{\delta}(t) = [\tilde{\delta}_2(t), \dots, \tilde{\delta}_N(t)]^T$. It is easy to see that $\lim_{t \rightarrow +\infty} \delta(t) = 0$, i.e., the consensus is achieved if and only if $\lim_{t \rightarrow +\infty} \hat{\delta}(t) = 0$, where by simple algebraic manipulation using the property of \mathcal{W} in the preliminaries, $\hat{\delta}(t)$ is found as

$$\dot{\hat{\delta}}(t) = \alpha \hat{\delta}(t) - \hat{\Lambda} (k_p \hat{\delta}(t - \tau(t)) + k_d \dot{\hat{\delta}}(t - \tau(t))), \quad (9)$$

with $\hat{\Lambda}$ being a diagonal matrix, whose diagonal elements consist of the eigenvalues $\lambda_2, \dots, \lambda_N$. Hence, for the time-varying delay $\tau(t) \geq 0$, the MAS (1) achieves consensus under the protocol (2) with (5) if and only if the system (9) is asymptotically stable. It achieves robust consensus under the protocol (2) with (5) for all $\tau(t) \in [0, \bar{\tau}]$ if and only if (9) is robustly stable for all $\tau(t) \in [0, \bar{\tau}]$.

It is worth pointing out that the equation (9) defines a neutral delay system. The stability of this system (Gu et al. (2003)) can be ensured only if the discrete part of the system

$$\hat{\delta}(t) + k_d \hat{\Lambda} \hat{\delta}(t - \tau(t)) = 0 \quad (10)$$

is stable. Compared with the constant delay case, in the presence of a time-varying delay, the stability of the system (9), albeit more sophisticated, can be analyzed using a small-gain criterion. Toward this end, we introduce the linear time-varying operator

$$\Delta \hat{\delta}(t) = \hat{\delta}(t - \tau(t)).$$

The system (9) can be described by the block diagram shown in Fig. 1. Define $\Gamma = \{2, 3, \dots, N\}$ and by employing the small-gain condition (Zhu et al. (2018)), we assert that the system (10) is stable provided that

$$\rho(k_d \hat{\Lambda}) = |k_d| \max_{i \in \Gamma} |\lambda_i| < 1, \quad (11)$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. Note that to achieve robust consensus, it is necessary that the system achieves consensus for $\tau(t) = 0$; that is, the system (9) with $\tau(t) = 0$ is stable, or equivalently, the polynomials

$$(1 + \lambda_i k_d)s + (\lambda_i k_p - \alpha) = 0, \quad i \in \Gamma \quad (12)$$

are all stable. It follows that with an undirected graph \mathcal{G} , the system (2) are all stable whenever $|k_d| < 1/\lambda_N$ and $k_p > \alpha/\lambda_2$. Hence, the feasible parameter set for (k_p, k_d) under the undirected graph \mathcal{G} is found as

$$\Omega_{u-PD} = \left\{ (k_p, k_d) : k_p > \frac{\alpha}{\lambda_2}, |k_d| < \frac{1}{\lambda_N} \right\}.$$

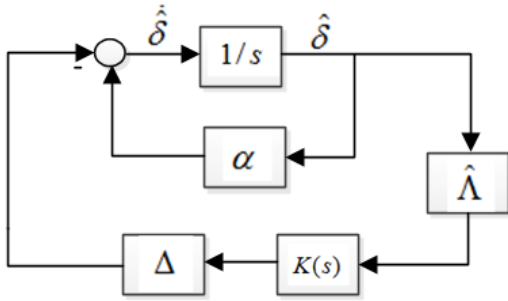


Fig. 1. Diagram of the system (9)

Similarly, the MAS (1) achieves robust consensus by P-type control protocols (2) with (6) if and only if

$$\dot{\hat{\delta}}(t) = \alpha \hat{\delta}(t) - \hat{\Lambda} k_p \hat{\delta}(t - \tau(t)), \quad (13)$$

are all stable.

Evidently, the control parameter set for the P-type control protocol under undirected graphs is found as

$$\Omega_{u-P} = \left\{ k_p : k_p > \frac{\alpha}{\lambda_2} \right\}.$$

3. DELAY CONSENSUS MARGIN AND LOWER BOUND ACHIEVED BY P-TYPE AND PD-TYPE CONSENSUS PROTOCOLS

3.1 Delay Consensus Margin

In this section our primary objective is to determine the maximal delay range so that consensus can be achieved robustly. Firstly, let us revisit the constant but unknown delay (Ma, Chen, Lu, & Chen (2019)). This amounts to determining the delay consensus margin (DCM) achieved by PD-type control protocol (2) with (5)

$$\bar{\tau}_{CD-PD} = \sup \{ \mu \geq 0 : \text{There exists } K_{PD}(s) \text{ such that consensus is achieved for } \forall \tau \in [0, \mu). \}$$

and the DCM achieved by the P-type control protocol (2) with (6)

$$\bar{\tau}_{CD-P} = \sup \{ \mu \geq 0 : \text{There exists } K_P(s) \text{ such that consensus is achieved for } \forall \tau \in [0, \mu). \}.$$

For conventional single-loop feedback systems, the delay margin has been well studied with general linear time-invariant controllers (Qi et al. (2017)) and PID controllers (Silva et al. (2002); Ma, and Chen (2019)). As for the DCMs of first-order agents (Xu et al. (2013)) and second-order agents (Ma, Tian, Zulfiqar, Chen, & Chai (2019)) under the proportional control protocol, the exact expressions have been derived. However, for the time-varying delay, it is a rather difficult problem to obtain the exact DCM. It is generally impossible to obtain necessary and sufficient stability conditions for the time-varying system. As such, in this section we will derive the lower bounds on the allowable delay range $\bar{\tau}_{TV-P}$ and $\bar{\tau}_{TV-PD}$, respectively.

Before giving the main results, we consider the general delay system as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - \tau(t)) \\ y(t) &= Cx(t), \end{aligned} \quad (14)$$

where the time-varying delay $\tau(t)$ satisfies the conditions (3) and (4). Assume that (A, B) is controllable and (C, A) is observable. Let $P_0(s) = C(sI - A)^{-1}B$ be the transfer function matrix of the delay-free system. Denote $K(s)$ to be a LTI output feedback controller, i.e., $u(s) = K(s)y(s)$, that stabilizes $P_0(s)$. Define the complementary sensitivity function of the delay-free system by

$$T_0(s) = P_0(s)K(s)(I + P_0(s)K(s))^{-1}. \quad (15)$$

The following lemma (Ma, and Chen (2019); Qi et al. (2017)) is provided to be used in what follows.

Lemma 3.1. Denote by $\|\cdot\|_\infty$ the \mathcal{H}_∞ norm of a stable transfer function. Let $K(s)$ stabilize $P_0(s)$. Then the system (14) can be robustly stabilized by $K(s)$ for all $\tau(t)$

(i) If $0 \leq \tau(t) \leq \tau_M$ and

$$\|\tau_M s T_0(s)\|_\infty < 1. \quad (16)$$

(ii) If $0 \leq \tau(t) \leq \tau_M$, $0 \leq |\dot{\tau}(t)| \leq \rho < 1$ and

$$\left\| \frac{\tau_M s}{1 + \frac{\tau_M s}{2}} W_{\tau_M}(s) T_0(s) \right\|_\infty < \sqrt{\frac{2 - \rho}{2}}, \quad (17)$$

where $W_{\tau_M}(s)$ is some stable and minimum phase rational function such that

$$\left| \frac{\tau(j\omega)}{1 + \frac{\tau(j\omega)}{2}} W_{\tau_M}(j\omega) \right| \geq \phi_\tau(\omega), \quad \forall \omega \geq 0 \quad (18)$$

with

$$\phi_\tau(\omega) = \begin{cases} 2 \sin(\tau\omega/2), & |\tau\omega| \leq \pi, \\ 2, & |\tau\omega| > \pi. \end{cases}$$

3.2 Lower Bounds Achieved by P-Type and PD-Type Control Protocols

Armed with the small-gain stability conditions given by Lemma 3.1, we now present our main results, which consist of lower bounds on the DCM of the MAS (1) achieved by P-type and PD-type control protocols subject to time-varying delays, respectively. These bounds constitute sufficient conditions for the MAS (1) to achieve robust consensus with respect to time-varying delays subject to the conditions (3) and (4).

Theorem 3.1. For an undirected graph \mathcal{G} , the following statements are true.

(I) The MAS (1) achieves robust consensus by the control protocol (2) for all $\tau(t)$ satisfying (3) with $K_p(s) = k_p$ and $\alpha/\lambda_2 < k_p < 1/(\tau_M \lambda_N)$. Furthermore,

$$\bar{\tau}_{TV-P} \geq \frac{1}{\alpha} \left(\frac{\lambda_2}{\lambda_N} \right). \quad (19)$$

(II) The MAS (1) achieves robust consensus by the control protocol (2) for all $\tau(t)$ satisfying (3), (4) with $K_{PD}(s) = k_p + k_d s$ and $k_p > \alpha/\lambda_2$, $|k_d| < 1/\lambda_N$, and one of the following conditions

$$(i) \quad \tau_M < \frac{2|k_d|}{k_p}, \quad \frac{2\lambda_N|k_d|}{1 + \lambda_N k_d} \|W_{\tau_M}(s)\|_\infty < \sqrt{\frac{2-\rho}{2}},$$

$$(ii) \quad \tau_M \geq \frac{2|k_d|}{k_p}, \quad \frac{\tau_M k_p \lambda_N}{1 + \lambda_N k_d} \|W_{\tau_M}(s)\|_\infty < \sqrt{\frac{2-\rho}{2}}$$

hold. Furthermore,

$$\bar{\tau}_{TV-PD} \geq \left(\frac{2\sqrt{2-\rho}}{2\sqrt{2}\|W_{\tau_M}(s)\|_\infty - \sqrt{2-\rho}} \right) \frac{1}{\alpha} \left(\frac{\lambda_2}{\lambda_N} \right), \quad (20)$$

where $W_{\tau_M}(s)$ is a certain stable, minimum phase rational function satisfying the condition (18).

Proof. Under the undirected graph \mathcal{G} , the system (9) is fully decoupled and its stability reduces to the systems

$$\dot{\delta}_i(t) = \alpha \delta_i(t) - \lambda_i \left(k_p \delta_i(t - \tau(t)) + k_d \dot{\delta}_i(t - \tau(t)) \right), \quad i \in \Gamma. \quad (21)$$

For each $i \in \Gamma$, we define

$$P_{0i}(s) = \frac{\lambda_i}{s - \alpha},$$

$$T_{0i}(s) = \frac{P_{0i}(s)K_{PD}(s)}{1 + P_{0i}(s)K_{PD}(s)},$$

with

$$T_0(s) = \text{diag}\{T_2(s), \dots, T_N(s)\},$$

$$P_0(s) = \text{diag}\{P_2(s), \dots, P_N(s)\}.$$

Consequently, we find that

$$\|\tau_M s T_0(s)\|_\infty = \max_{i \in \Gamma} \|\tau_M s T_{0i}(s)\|_\infty,$$

and

$$\left\| \frac{\tau_M s}{1 + \frac{\tau_M s}{2}} W_{\tau_M}(s) T_0(s) \right\|_\infty = \max_{i \in \Gamma} \left\| \frac{\tau_M s}{1 + \frac{\tau_M s}{2}} W_{\tau_M}(s) T_{0i}(s) \right\|_\infty.$$

With the P-type control protocol (2) with (6), we know that

$$T_{0i}(s) = \frac{\lambda_i k_p}{s - \alpha + \lambda_i k_p}.$$

The \mathcal{H}_∞ norm of $\tau_M s T_{0i}(s)$ is found as

$$\|\tau_M s T_{0i}(s)\|_\infty = \tau_M \lambda_i k_p.$$

As a result, for the undirected graph \mathcal{G} , we have

$$\|\tau_M s T_0(s)\|_\infty = \tau_M \lambda_N k_p.$$

According to Lemma 3.1, the systems in (21) will be stable whenever $\tau_M \lambda_N k_p < 1$. This together with the condition $k_p > \alpha/\lambda_2$ gives the range

$$\frac{\alpha}{\lambda_2} < k_p < \frac{1}{\tau_M \lambda_N}$$

that guarantees the stability of the system (21). In what follows, we try to find a sufficient condition to achieve the consensus. Since

$$\inf_{k_p > \alpha/\lambda_2} \|\tau_M s T_0(s)\|_\infty = \tau_M \lambda_N \left(\frac{\alpha}{\lambda_2} \right),$$

there will exist a $K_p(s) = k_p$ to achieve the consensus provided that $\tau_M \lambda_N \frac{\alpha}{\lambda_2} < 1$. This leads to the condition (19).

In order to establish the bound (20), we follow the idea of Ma, and Chen (2019) to evaluate that

$$\left\| \frac{\tau_M s}{1 + \frac{\tau_M s}{2}} T_0(s) \right\|_\infty = \max_{i \in \Gamma} \left\| \frac{\tau_M s}{1 + \frac{\tau_M s}{2}} T_{0i}(s) \right\|_\infty,$$

where now with a PD-type control protocol (2) with (5). $T_{0i}(s)$ is found to be

$$T_{0i}(s) = \frac{\lambda_i(k_p + k_d s)}{(1 + \lambda_i k_d)s + (\lambda_i k_p - \alpha)}.$$

For this purpose, we notice that

$$\left\| \frac{\tau_M s}{1 + \frac{\tau_M s}{2}} T_{0i}(s) \right\|_\infty \leq \left\| \frac{\lambda_i(k_p + k_d s)}{1 + \frac{\tau_M s}{2}} \right\|_\infty \cdot \left\| \frac{\tau_M s}{(1 + \lambda_i k_d)s + (\lambda_i k_p - \alpha)} \right\|_\infty.$$

Consequently, we find that

$$\left\| \frac{\tau_M s}{(1 + \lambda_i k_d)s + (\lambda_i k_p - \alpha)} \right\|_\infty = \frac{\tau_M}{1 + \lambda_i k_d}$$

and

$$\left\| \frac{\lambda_i(k_p + k_d s)}{1 + \frac{\tau_M s}{2}} \right\|_\infty = \lambda_i \begin{cases} \frac{2|k_d|}{\tau_M}, & |k_d| > \frac{\tau_M}{2} k_p, \\ k_p, & |k_d| \leq \frac{\tau_M}{2} k_p. \end{cases}$$

Hence, we have

$$\left\| \frac{\tau_M s}{1 + \frac{\tau_M s}{2}} T_{0i}(s) \right\|_\infty \leq \frac{\tau_M \lambda_i}{1 + \lambda_i k_d} \begin{cases} \frac{2|k_d|}{\tau_M}, & |k_d| > \frac{\tau_M}{2} k_p, \\ k_p, & |k_d| \leq \frac{\tau_M}{2} k_p. \end{cases} \quad (22)$$

By the monotonicity of the right hand side of (22) in λ_i , we obtain that

$$\left\| \frac{\tau_M s}{1 + \frac{\tau_M s}{2}} T_0(s) \right\|_\infty \leq \frac{\tau_M \lambda_N}{1 + \lambda_N k_d} \begin{cases} \frac{2|k_d|}{\tau_M}, & |k_d| > \frac{\tau_M}{2} k_p, \\ k_p, & |k_d| \leq \frac{\tau_M}{2} k_p. \end{cases} \quad (23)$$

Furthermore, by the monotonicity properties of $|k_d|/(1 + \lambda_N k_d)$ and $1/(1 + \lambda_N k_d)$ with respect to k_d , we know that

$$\inf \left\{ \frac{2|k_d| \lambda_N}{1 + \lambda_N k_d} : |k_d| > \frac{\tau_M}{2} k_p \right\}$$

$$= \inf \left\{ \frac{\tau_M k_p \lambda_N}{1 + \lambda_N k_d} : |k_d| < \frac{\tau_M}{2} k_p \right\}$$

$$= \frac{\tau_M k_p \lambda_N}{1 + \frac{\tau_M}{2} k_p \lambda_N}.$$

It is then easy to find that the right hand side of the inequality (23) achieves its minimum as

$$\frac{\tau_M \alpha \frac{\lambda_N}{\lambda_2}}{1 + (\tau_M/2) \alpha \frac{\lambda_N}{\lambda_2}}$$

at $(k_p, k_d) = (\frac{\alpha}{\lambda_2}, \frac{\alpha \tau_M}{2 \lambda_2})$.

In conclusion, in view of Lemma 3.1, the MAS (1) achieve consensus robustly by some $K_{PD}(s)$ for all $\tau(t)$ satisfying (3) and (4) if

$$\|W_{\tau_M}(s)\|_\infty \frac{\tau_M \alpha \frac{\lambda_N}{\lambda_2}}{1 + (\tau_M/2) \alpha \frac{\lambda_N}{\lambda_2}} < \sqrt{\frac{2-\rho}{2}}.$$

This gives rise to the lower bound (20), hence complete the proof. \blacksquare

Remark 3.1. In light of Lemma 3.1, it is interesting to see that the bound in (19) is always less than or equal to that the case of the constant delay achieved by P-type control protocol in Xu et al. (2013). This, of course, is expected, since

$$\frac{1}{\alpha} \left(\frac{\lambda_2}{\lambda_N} \right) \leq \bar{\tau}_{TV-P} \leq \bar{\tau}_{CD-P} = \frac{1}{\alpha} \left(\frac{\arctan \sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}}{\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}} \right).$$

In the limit, however, when $\lambda_N \rightarrow \lambda_2$, i.e., when the graph tends to be complete, these bounds coalesce to

$$\bar{\tau}_{TV-P} = \bar{\tau}_{CD-P} = \frac{1}{\alpha}.$$

Moreover, the gap between the two cases becomes monotonically decreasing as the ratio λ_N/λ_2 decreases.

Remark 3.2. Similar to its constant-delay counterpart, the lower bound in (20) indicates that inclusion of derivative control action can improve the delay consensus margin. This improvement is gained by incorporating the extra information furnished by the delay variation rate. Indeed, when $W_{\tau_M}(s)$ is appropriately constructed, specifically when

$$\frac{2\sqrt{2-\rho}}{2\sqrt{2}\|W_{\tau_M}(s)\|_{\infty} - \sqrt{2-\rho}} > 1,$$

the bound in (20) always improves that in (19). The improvement can be significant for a small value of ρ . In the limit when $\rho \rightarrow 0$ and $\|W_{\tau_M}(s)\|_{\infty} \approx 1$, the bound in (20) may double that in (19).

Remark 3.3. In practice, most of the derivative control is implemented by using a low-pass filter. As such, the PD-type control protocols can be given in the form of

$$K_{PD}^{T_f}(s) = k_p + \frac{k_d s}{1 + T_f s},$$

where $T_f > 0$ is the filter constant. Similarly, it is to determine the DCM by the corresponding PD-type control protocols with the filter possibly. In general, the DCM is decreasing with the filter constant increasing.

4. SIMULATION RESULTS

In this section we provide an example to illustrate the main results.

Example 4.1. Consider six agents coordinated with an undirected communication topology, whose Laplacian matrix is

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}.$$

The matrix \mathcal{L} has one eigenvalue at the origin and the remaining five eigenvalues are $\lambda_2 = 1, \lambda_3 = 1.5858, \lambda_4 = 3, \lambda_5 = 4, \lambda_6 = 4.4142$. With this undirected graph, the feasible design parameter sets under P-type and PD-type control protocols are found as

$$\Omega_{u-P} = \{k_p > \alpha/1\}$$

and

$$\Omega_{u-PD} = \{k_p > \alpha/1, |k_d| < 1/4.4142\}.$$

respectively. Let us consider the time-varying delay

$$\tau(t) = \alpha(1 - \sin bt),$$

which indicates $\tau_M = 2\alpha$ and $\rho = \alpha b$. Set $\alpha = 0.5$. We first examine the lower bound in (19), which is achieved by P-type control protocols. Based on Theorem 3.1, we know that the MAS (1) can achieve consensus by P-type control protocols regardless of the delay variation rate ρ whenever

$$\tau_M < \frac{1}{\alpha} \cdot \frac{\lambda_2}{\lambda_N} = 0.453.$$

Fig. 2 confirms this assertion: For $\tau_M = 0.3 < 0.453$, the consensus can be achieved for

$$\frac{\alpha}{\lambda_2} < k_p = 0.75 < \frac{1}{0.3\lambda_N}.$$

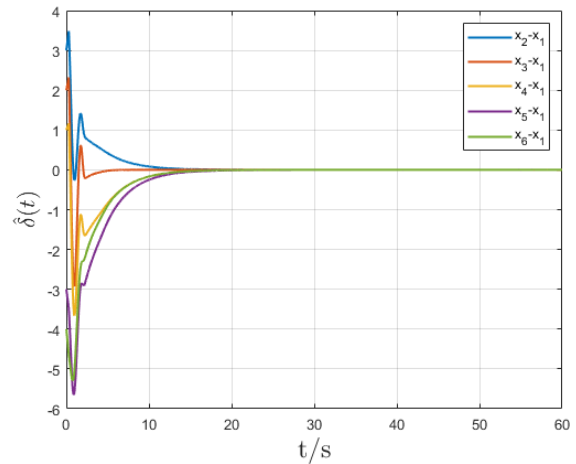


Fig. 2. Consensus achieved at $\tau_M = 0.3$: differences of state versus $x_1(t)$.

Next, we consider the PD-type control protocols. In this case, we choose the following rational functions (Qi et al. (2017) and Ma, and Chen (2019)) satisfying the condition (18),

$$W_{\tau}^{(1)}(s) = 1.216,$$

$$W_{\tau}^{(2)}(s) = \frac{0.1791(\tau s)^2 + 0.7093\tau s + 1}{0.1791(\tau s)^2 + 0.5798\tau s + 1},$$

$$W_{\tau}^{(3)}(s) = \frac{0.02952(\tau s)^4 + 0.210172(\tau s)^3 + 0.70763(\tau s)^2 + 1.3188\tau s + 1}{0.02952(\tau s)^4 + 0.191784(\tau s)^3 + 0.64174(\tau s)^2 + 1.195282\tau s + 1}.$$

By direct computation, we find that $\|W_{\tau}^{(1)}(s)\|_{\infty} = 1.2160$, $\|W_{\tau}^{(2)}(s)\|_{\infty} = 1.0908$, and $\|W_{\tau}^{(3)}(s)\|_{\infty} = 1.0831$, respectively. For $\alpha = 0.5$, choose one pair of $(\rho, \bar{\tau}_{TV}) = (0.6, 0.5702)$ within the consensus region given by the rational function $W_{\tau}^{(3)}(s)$. Fig. 3 shows the relationship between $\bar{\tau}_{TV}$ and ρ with different $\|W_{\tau}^{(i)}(s)\|_{\infty}$. In addition, Fig. 4 indicates that the consensus region achieved by PD-type control protocols can be larger possibly than that achieved by P-type control protocols if the variation rate of the time-varying delay is in a small value. Select $(k_p, k_d) = (0.68, 0.14)$ together with $\tau_M = 0.4 < 0.5702$ satisfying the condition (i) of Theorem 3.1. Fig. 5 gives consensus under PD-type control protocols.

5. CONCLUSION

Based on the small-gain theorem we have derived the explicit lower bounds of delay consensus margin of first-order MAS under PID-type control protocols with time-varying delays under the undirected graph. Unlike the upper bounds obtained elsewhere, which can be used to determine the range of delay where the MAS cannot achieve robustly consensus,

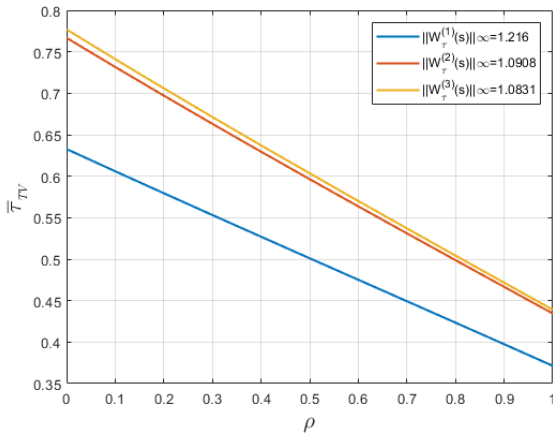


Fig. 3. Relationship between $\bar{\tau}_{TV}$ and ρ

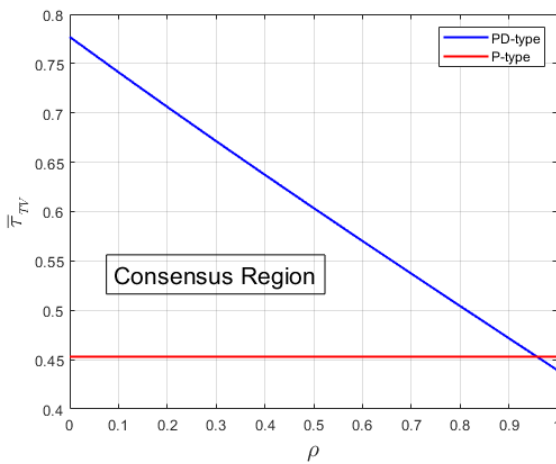


Fig. 4. Consensus region achieved by PD-type control protocols v.s. P-type control protocols.

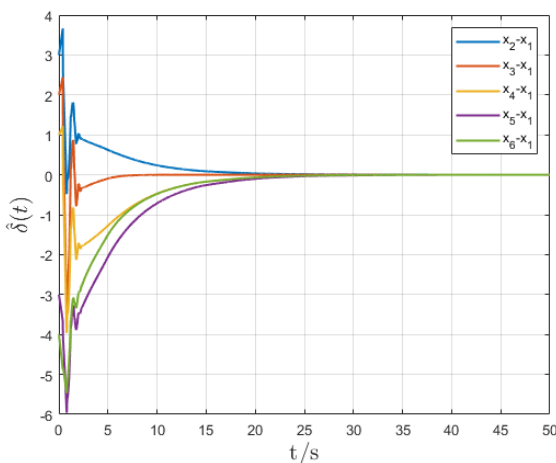


Fig. 5. Consensus achieved at $(\rho, \tau_M) = (0.6, 0.4)$: differences of state versus $x_1(t)$.

the lower bounds obtained herein serve an opposite purpose: they provide *a priori* the ranges of delay over which the MAS is guaranteed to obtain robustly consensus by a PID control protocol. The lower bounds show explicitly the consensus robustness depend on the graph connectivity (λ_2/λ_N) and the agent dynamics. It is more general to consider the

different time-varying delays between the communication delay and the self delay. This will be investigated in our future work.

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