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Lyapunov-Krasovskii prescribed derivative and the Bellman functional for time-delay systems ⋆

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Abstract: The Dynamic Programming approach for optimal control problem establishes that the necessary condition for optimality is that the minimum cost function (the Bellman functional) must satisfy the Hamilton-Jacobi-Bellman equation. A sufficient condition is that if there exists a functional that satisfies the Hamilton-Jacobi-Bellman equation, then it is the minimum cost function. For linear time-delay systems, Krasovskii proposed the Bellman functional, and an optimal structure for the controller was reported. The Dynamic Programming combined with prescribed derivative functionals leads to an iterative procedure which allows finding suboptimal controls law at each step. There is numerical evidence that shows that these functionals are equivalent. However, their algebraic structure is different. The Bellman functional has only three terms and the iterative functional is composed by thirteen summands. The algebraic relation between both functionals is not easy to see. The present contribution gives a proof of this connection by using Fubini's Theorem.

Keywords: Time-delay systems. Dynamic Programming. Bellman functional. Optimal control.

1. INTRODUCTION

The design of optimal control laws using Dynamic Programming was introduced in Bellman (1957). This approach uses the well-known Bellman functional to find the optimal control. For this reason, the issue of its construction or proposal is fundamental to solve the problem. For time-delay linear systems, the Bellman functional was proposed in Krasovskii (1962) and constructed in Ross and Flügge-Lotz (1969). Subsequently, these works were expanded and extended. For example in Kushner and Barnea (1970), the conditions under which the Riccati equation has at least one solution are analyzed. Some methods are proposed for the solution of this equation in Kim and Lozhnikov (2000). In Santos (2006) and Santos et al. (2009), a Lyapunov-Krasovskii functional with prescribed derivative (which is equal minus the quadratic function under the integral) is introduced to obtain suboptimal control laws.

To establish the connection between both functionals we use variable changes together with Fubini's Theorem (see Thomas and Finney (1996)). We show that these functionals have the same structure. The above correspondance was validated by numerical verification on an example presented in Ross (1971) and in Santos et al. (2009).

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The contribution is organized as follows: the preliminaries are given in section 2. The main results of this research can be found in section 3: there, the two functionals are compared. The conclusions of this contribution are in section 4.

We denote the space of $\mathbb{R}^n$-valued piecewise-continuous functions on $[-h,0]$ by $\mathrm{PC}([-h,0],\mathbb{R}^n)$. For a given initial function $\varphi(\theta)$, $x_t(\varphi)$ denotes the state of the delay system $\{x(t+\theta,\varphi),\theta\in[-h,0]\}$, with delay $h>0$. The Euclidian norm for vectors is represented by $\|\cdot\|$. The set of piecewise continuous functions is equipped with the norm $\|\varphi\|_h=\sup_{\theta\in[-h,0]}\|\varphi(\theta)\|$. The notations $Q>0$ and $R>0$ mean that matrices Q and R are positive definite.

2. PRELIMINARIES AND PROBLEM STATEMENT

Consider time-delay systems of the form

$$\begin{aligned}&\dot{x}(t)=Ax(t)+Bx(t-h)+Du(t),\\&\varphi\in\mathrm{PC}([h,0],\mathbb{R}^n)\end{aligned}\tag{1}$$

where $x(t)$, x_t in $\mathcal{D}\subset\mathbb{R}^n$, the set $\mathcal{D}$ represents a domain for the space of solutions which contains the trivial one, the matrices $A,B\in\mathbb{R}^{n\times n}$, $D\in\mathbb{R}^{n\times r}$ are constant and the vector $u(t)\in\mathbb{R}^r$ with $r\leq n$. The control $u(t)$ is a piecewise continuous function.

Consider the following quadratic performance index:

$$J=\int_0^\infty g(x_t,u(t))dt,\tag{2}$$

with

$$g(x_t, u(t)) = x^T(t)Qx(t) + u^T(t)Ru(t),$$

where $Q \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{r\times r}$, $Q > 0, R > 0$.

The optimal control problem consists in the synthesis of the optimal control $u^0(t)$ that minimizes the quadratic performance index (2). According to Krasovskii (1962), and Ross and Flügge-Lotz (1969), the admissible controls for system (1) are stabilizing controls that depend on the state x_t. The system (1) in closed-loop with an admissible control is an exponentially stable system of the form:

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)d\theta, \quad t \geq 0. \tag{3}$$

Here $x(t)$, $x_t \in \mathcal{D}$, A_0, A_1 are given real $n \times n$ matrices, $h > 0$ is the delay, and $G(\theta)$ is a continuous matrix defined for $\theta \in [-h, 0]$. As system (3) is Lipschitz, for any initial condition $x_0 = \varphi$, $\varphi \in \mathrm{PC}([h, 0], \mathbb{R}^n)$, the solution exists and is unique. The Cauchy formula for system (3) is given in the following theorem.

Theorem 1. (Bellman and Cooke (1963)) The solution $x(t, \varphi)$ of system (3) is given by

$$\begin{aligned} x(t,\varphi) &= K(t)\varphi(0) + \int_{-h}^{0} K(t-h-\zeta)A_1\varphi(\zeta)d\zeta \\ &+ \int_{-h}^{0}\int_{-h}^{0} K(t-\zeta+\theta)G(\theta)d\theta\varphi(\zeta)d\zeta, \quad t \geq 0, \end{aligned} \tag{4}$$

where the $n \times n$ matrix function $K(t)$ satisfies the matrix equation

$$\dot{K}(t) = K(t)A_0 + K(t-h)A_1 + \int_{-h}^{0} K(t+\theta)G(\theta)d\theta, \quad t \geq 0,$$

and the initial condition $K(0) = I$, and $K(\theta) = 0$ for all $\theta \in [-h, 0)$.

The necessary conditions of Dynamic Programming provide the structure of Bellman functional $V(x_t)$ and some of its properties.

Proposition 2. (Ross and Flügge-Lotz (1969)). *If $u_L = u_L(x_t)$, $t \geq 0$, is an admissible linear control, φ is an initial function defined on $[-h, 0]$, then the functional*

$$V(\varphi) = J(\varphi, u_L) = \int_0^{\infty} (x^T(t)Qx(t) + u_L(t)^TRu_L(t))dt, \tag{5}$$

can be expressed as

$$\begin{aligned} V(\varphi) &= \varphi^T(0)\Pi_0\varphi(0) + 2\varphi^T(0)\int_{-h}^{0}\Pi_1(\theta)\varphi(\theta)d\theta \\ &+ \int_{-h}^{0}\int_{-h}^{0}\varphi^T(\xi)\Pi_2(\xi,\theta)\varphi(\theta)d\xi d\theta, \end{aligned} \tag{6}$$

where

i) $\Pi_0 > 0$ *is a symmetric positive matrix.*
ii) $\Pi_1(\theta)$ *is defined on* $[-h, 0]$.
iii) $\Pi_2(\xi, \theta)$ *is defined on* $\xi, \theta \in [-h, 0]$, $\Pi_2^T(\xi, \theta) = \Pi_2(\theta, \xi)$.

It is worthy of mention that there exist no report on the construction of the Bellman functional, but only an indication in Ross and Flügge-Lotz (1969): a Riesz approximation may be used.

In Santos et al. (2009) a method is proposed for the construction of an approximation of the Bellman functional, denoted $V_1(x_t)$. Indeed for a control law

$$u_L(t) = \Gamma_0x(t) + \int_{-h}^{0}\Gamma_1(\theta)x(t+\theta)d\theta, \tag{7}$$

it appears from (5) that the Bellman functional can be obtained by substituting (7) into (5) and carrying out the integration from 0 to ∞ along the system trajectories (4). More precisely

$$V_1(\varphi) = \int_0^{\infty} w(x_t(\varphi))dt, \tag{8}$$

where

$$\begin{aligned} w(x_t) &= x^TW_{00}x(t) + 2x^T(t)\int_{-h}^{0}W_{01}(\theta)x(t+\theta)d\theta \\ &+ 2\int_{-h}^{0}\int_{\theta_1}^{0} x^T(t+\theta_1)W_{11}(\theta_1,\theta_2)x(t+\theta_2)d_1d\theta_2, \end{aligned} \tag{9}$$

and

$$W_{00} = Q + \Gamma_0^TR\Gamma_0,$$

$$W_{01}(\theta) = \Gamma_0^TR\Gamma_1(\theta),$$

$$W_{11}(\theta_1,\theta_2) = \Gamma_1(\theta_1)R\Gamma_1(\theta_2).$$

Here $W_{00} \in \mathbb{R}^{n\times n}$ is a positive definite matrix, and $W_{01}(\theta), W_{11}(\theta_1, \theta_2) \in \mathbb{R}^{n\times n}$ are matrix functions with $\theta, \theta_1, \theta_2 \in [-h, 0]$ respectively, and

$$W_{11}^T(\theta_1,\theta_2) = W_{11}(\theta_2,\theta_1).$$

The final result (see Santos et al. (2009)) is

$$\begin{aligned} V_1(x_t) &= 2x^T(t)\int_{-h}^{0}\int_{0}^{-\theta} K^T(s)W_{01}(\theta)x(t+s+\theta)dsd\theta \\ &+ 2\int_{-h}^{0}\int_{-h}^{0}\int_{0}^{-\theta} x^T(t+\zeta)A_1^TK^T(s-h-\zeta)W_{01}(\theta) \\ &\times x(t+s+\theta)dsd\theta d\zeta + 2\int_{-h}^{0}\int_{-h}^{0}\int_{\theta_1}^{0} x^T(t+\zeta)G^T(\theta_1) \\ &\times \left[\int_{0}^{-\theta} K^T(s-\zeta+\theta_1)W_{01}(\theta)x(t+s+\theta)ds\right] d\zeta d\theta_1 d\theta \\ &+ 2\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1}^{\theta_1-\theta_2} x^T(t+\sigma)W_{11}(\theta_1,\theta_2) \\ &\times x(t+\sigma+\theta_2-\theta_1)d\sigma d\theta_2 d\theta_1 \\ &+ 2\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1-\theta_2}^{0} x^T(t+\sigma)W_{11}(\theta_1,\theta_2) \\ &\times K(\sigma+\theta_2-\theta_1)d\sigma d\theta_2 d\theta_1 x(t) \\ &+ 2\int_{-h}^{0}\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1-\theta_2}^{0} x^T(t+\sigma)W_{11}(\theta_1,\theta_2) \\ &\times K(\sigma+\theta_2-\theta_1-h-\zeta)d\sigma d\theta_2 d\theta_1 A_1x(t+\zeta)d\zeta \\ &+ 2\int_{-h}^{0}\int_{-h}^{\zeta}\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1-\theta_2}^{0} x^T(t+\sigma)W_{11}(\theta_1,\theta_2) \\ &\times K(\sigma+\theta_2-\theta_1-\zeta+\theta_3)d\sigma d\theta_2 d\theta_1 \\ &\times G(\theta_3)x(t+\zeta)d\theta_3 d\zeta \\ &+ x^T(t)F(0, W_{00}, W_{01}(\theta), W_{11}(\theta_1,\theta_2))x(t) \end{aligned}$$

$$
\begin{aligned}
&+2x^T(t)\int_{-h}^{0}F(-h-\zeta,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times A_1x(t+\zeta)d\zeta\\
&+2x^T(t)\int_{-h}^{0}\int_{-h}^{\zeta}F(-\zeta+\theta_3,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times G(\theta_3)x(t+\zeta)d\theta_3d\zeta\\
&+\int_{-h}^{0}\int_{-h}^{0}x^T(t+\zeta_1)A_1^T\\
&\times F(\zeta_1-\zeta_2,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times A_1x(t+\zeta_2)d\zeta_2d\zeta_1+2\int_{-h}^{0}\int_{-h}^{0}\int_{-h}^{\zeta_2}x^T(t+\zeta_1)A_1^T\\
&\times F(h+\zeta_1-\zeta_2+\theta_3,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times G(\theta_3)d\theta_3x(t+\zeta_2)d\zeta_2d\zeta_1\\
&+\int_{-h}^{0}\int_{-h}^{0}\int_{-h}^{\zeta_2}\int_{-h}^{\zeta_1}x^T(t+\zeta_1)G^T(\theta_3)\\
&\times F(\zeta_1-\theta_3-\zeta_2+\theta_4,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times G(\theta_4)x(t+\zeta_2)d\theta_3d\theta_4d\zeta_2d\zeta_1,
\end{aligned}
\tag{10}
$$

with

$$
\begin{aligned}
&F(\tau,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))=U(\tau,W_{00})\\
&+\int_{-h}^{0}\left[U(\theta+\tau,W_{01}(\theta))+U^T(\theta-\tau,W_{01}(\theta))\right]d\theta\\
&+\int_{-h}^{0}\left(\int_{\theta_1}^{0}[U(\theta_2-\theta_1+\tau,W_{11}(\theta_1,\theta_2))\right.\\
&+U^T(\theta_2-\theta_1-\tau,W_{11}(\theta_1,\theta_2))d\theta_2d\theta_1.
\end{aligned}
$$

The Lyapunov matrix $U(\tau,M)$ is defined as

$$U(\tau,M)=\int_0^\infty K^T(t)MK(t+\tau)dt,$$

where M is a real $n\times n$ matrix and $K(t)$ is the fundamental matrix of system (3). The Lyapunov matrix satisfies the conditions

$$
\begin{aligned}
U'(\tau,M)&=U(\tau,M)A_0+U(\tau-h,M)A_1\\
&+\int_{-h}^{0}U(\tau+\theta,M)G(\theta+\tau)d\theta,\quad \tau\geq 0,\\
U(-\tau,M)&=U^T(\tau,M^T),
\end{aligned}
$$

and

$$-M=U'(+0,M)-U'(-0,M).$$

3. FUNCTIONALS COMPARISON

The main purpose of this contribution is to present the structural connection between the Lyapunov-Krasovskii functional (10) constructed via the prescribed derivative functional approach and the proposed form for the Bellman functional (6). Our main motivation was our long lasting concern about the apparent complexity of the thirteen summands of the constructed approximation Santos et al. (2009), when compared to the three terms Bellman functionals introduced in Krasovskii (1962) and Ross and Flügge-Lotz (1969).

Proposition 3. Consider the functional $V_1(x_t)$ given in (10), then it can be expressed as

$$
\begin{aligned}
V_1(x_t)&=x^T\Theta_0x(t)+2x^T(t)\int_{-h}^{0}\Theta_1(\zeta)x(t+\zeta)d\zeta\\
&+\int_{-h}^{0}\int_{-h}^{0}x^T(t+\zeta_1)\Theta_2(\zeta_1,\zeta_2)x(t+\zeta_2)d\zeta_1d\zeta_2,
\end{aligned}
\tag{11}
$$

with

$$\Theta_0=F(0,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2)),\tag{12}$$

$$
\begin{aligned}
\Theta_1(\zeta)&=F(-h-\zeta,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))A_1\\
&+\int_{-h}^{\zeta}F(-\zeta+\theta_3,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))G(\theta_3)d\theta_3\\
&+\int_{-h}^{\zeta}K^T(\zeta-\theta)W_{01}(\theta)d\theta\\
&+\int_{-h}^{\zeta}\int_{\theta_1-\zeta}^{0}K^T(\zeta+\theta_2-\theta_1)W_{11}^T(\theta_1,\theta_2)d\theta_2d\theta_1,
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
&\Theta_2(\zeta_1,\zeta_2)=A_1^TF(\zeta_1-\zeta_2,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))A_1\\
&+2\int_{-h}^{\zeta_2}A_1^TK^T(\zeta_2-\theta-h-\zeta_1)W_{01}(\theta)d\theta\\
&+2\int_{-h}^{\zeta_1}\int_{-h}^{\zeta_2}G^T(\theta_1)K^T(\zeta_2-\theta-\zeta_1+\theta_1)W_{01}(\theta)d\theta d\theta_1\\
&+\int_{-h}^{\zeta_1}W_{11}(\theta_1,\zeta_2-\zeta_1+\theta_1)d\theta_1\\
&+2\int_{-h}^{\zeta_1}\int_{\theta_1-\zeta_1}^{0}W_{11}(\theta_1,\theta_2)K(\zeta_1+\theta_2-\theta_1-h-\zeta_2)\\
&\times A_1d\theta_2d\theta_1\\
&+2\int_{-h}^{\zeta_2}\int_{-h}^{\zeta_1}\int_{\theta_1-\zeta_1}^{0}W_{11}(\theta_1,\theta_2)K(\zeta_1+\theta_2-\theta_1-\zeta_2+\theta_3)\\
&\times G(\theta_3)d\theta_2d\theta_1d\theta_3\\
&+2\int_{-h}^{\zeta_2}A_1^TF(h+\zeta_1-\zeta_2+\theta_3,W_{00},W_{01}(\theta),\\
&W_{11}(\theta_1,\theta_2))G(\theta_3)d\theta_3\\
&+\int_{-h}^{\zeta_2}\int_{-h}^{\zeta_1}G^T(\theta_3)F(\zeta_1-\theta_3-\zeta_2+\theta_4,W_{00},\\
&W_{01}(\theta),W_{11}(\theta_1,\theta_2))G(\theta_4)d\theta_3d\theta_4.
\end{aligned}
\tag{14}
$$

Proof

Our aim is to prove that the Bellman functional (6) and functional (10) have same structure. The proof mainly relies on the use of Fubini's Theorem, together with appropriate changes of variables.

The first integral on the right-hand side of (10) can be transformed as follows:

$$
\begin{aligned}
&2x^T(t)\int_{-h}^{0}\int_{0}^{-\theta}K^T(s)W_{01}(\theta)x(t+s+\theta)dsd\theta\\
&=\langle\zeta=s+\theta\rangle=2x^T(t)\int_{-h}^{0}\int_{\theta}^{0}K^T(\zeta-\theta)W_{01}(\theta)\\
&\times x(t+\zeta)d\zeta d\theta.
\end{aligned}
\tag{15}
$$

The region of integration on the right-hand side of (15) is given by Fig. 1, which helps us to introduce the following equality

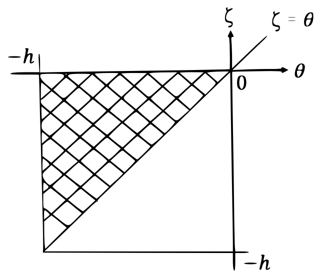

Fig. 1. Region of integration in (15)

$$2x^T(t)\int_{-h}^{0}\int_{0}^{-\theta}K^T(s)W_{01}(\theta)x(t+s+\theta)dsd\theta$$
$$=2x^T(t)\int_{-h}^{0}\left[\int_{-h}^{\zeta}K^T(\zeta-\theta)W_{01}(\theta)d\theta\right]x(t+\zeta)d\zeta. \tag{16}$$

The second summand of (10) can be transformed as follows:

$$2\int_{-h}^{0}\int_{-h}^{0}\int_{0}^{-\theta}x^T(t+\zeta_1)A_1^TK^T(s-h-\zeta_1)W_{01}(\theta)$$
$$\times x(t+s+\theta)dsd\theta d\zeta_1=\langle\zeta_2=s+\theta\rangle$$
$$=2\int_{-h}^{0}\int_{-h}^{0}\int_{\theta}^{0}x^T(t+\zeta_1)A_1^TK^T(\zeta_2-\theta-h-\zeta_1)W_{01}(\theta)$$
$$\times x(t+\zeta_2)d\zeta_2d\zeta_1=2\int_{-h}^{0}R_1(\zeta_1)d\zeta_1,$$

with

$$R_1(\zeta_1)=\int_{-h}^{0}\int_{\theta}^{0}x^T(t+\zeta_1)A_1^TK^T(\zeta_2-\theta-h-\zeta_1)W_{01}(\theta)$$
$$\times x(t+\zeta_2)d\zeta_2d\theta. \tag{17}$$

The region of integration in (17) is similar to the one in Fig. 1 with ζ equal to ζ_2. Then, using Fubini's Theorem, we can find that,

$$2\int_{-h}^{0}\int_{-h}^{0}\int_{0}^{-\theta}x^T(t+\zeta)A_1^TK^T(s-h-\zeta)W_{01}(\theta)$$
$$\times x(t+s+\theta)dsd\theta d\zeta$$
$$=\int_{-h}^{0}\int_{-h}^{0}x^T(t+\zeta_1)\left[2\int_{-h}^{\zeta_2}A_1^TK^T(\zeta_2-\theta-h-\zeta_1)\right.$$
$$\left.\times W_{01}(\theta)d\theta\right]x(t+\zeta_2)d\zeta_1d\zeta_2. \tag{18}$$

The third summand of (10) can be transformed as follows:

$$2\int_{-h}^{0}\int_{-h}^{0}\int_{\theta_1}^{0}\int_{0}^{-\theta}x^T(t+\zeta_1)G^T(\theta_1)K^T(s-\zeta_1+\theta_1)W_{01}(\theta)$$
$$\times x(t+s+\theta)dsd\zeta_1d\theta_1d\theta=\langle\zeta_2=s+\theta\rangle$$
$$=2\int_{-h}^{0}\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta}^{0}x^T(t+\zeta_1)G^T(\theta_1)K^T(\zeta_2-\theta-\zeta_1+\theta_1)$$
$$\times W_{01}(\theta)x(t+\zeta_2)d\zeta_2d\zeta_1d\theta_1d\theta=2\int_{-h}^{0}R_2(\theta)d\theta,$$

with

$$R_2(\theta)=\int_{-h}^{0}\int_{\theta_1}^{0}R_3(\zeta_1,\theta_1,\theta)d\zeta_1d\theta_1 \tag{19}$$

and

$$R_3(\zeta_1,\theta_1,\theta)=\int_{\theta}^{0}x^T(t+\zeta_1)G^T(\theta_1)$$
$$\times K^T(\zeta_2-\theta-\zeta_1+\theta_1)W_{01}(\theta)x(t+\zeta_2)d\zeta_2. \tag{20}$$

The region of integration in (19) is similar to the one in Fig. 1 where ζ is equal to ζ_1 and θ is changed by θ_1. The above implies that

$$R_2(\theta)=\int_{-h}^{0}\int_{-h}^{\zeta_1}R_3(\zeta_1,\theta_1,\theta)d\theta_1d\zeta_1. \tag{21}$$

Then, using equations (20) and (21), the same procedure leads to

$$2\int_{-h}^{0}\int_{-h}^{0}\int_{\theta_1}^{0}x^T(t+\zeta)G^T(\theta_1)\left[\int_{0}^{-\theta}K^T(s-\zeta+\theta_1)\right.$$
$$\left.\times W_{01}(\theta)x(t+s+\theta)ds\right]d\zeta d\theta_1d\theta$$
$$=\int_{-h}^{0}\int_{-h}^{0}x^T(t+\zeta_1)\left[2\int_{-h}^{\zeta_1}\int_{-h}^{\zeta_2}G^T(\theta_1)\right.$$
$$\left.\times K^T(\zeta_2-\theta-\zeta_1+\theta_1)W_{01}(\theta)d\theta d\theta_1\right]x(t+\zeta_2)d\zeta_1d\zeta_2. \tag{22}$$

Now, we analyze the fourth term of (10). First, the variable σ is changed to ζ_1, hence this summand rewrites as

$$2\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1}^{\theta_1-\theta_2}x^T(t+\zeta_1)W_{11}(\theta_1,\theta_2)$$
$$\times x(t+\zeta_1+\theta_2-\theta_1)d\zeta_1d\theta_2d\theta_1. \tag{23}$$

Defining the functional

$$R_4(t,\theta_1)=\int_{\theta_1}^{0}\int_{\theta_1}^{\theta_1-\theta_2}x^T(t+\zeta_1)W_{11}(\theta_1,\theta_2)$$
$$\times x(t+\zeta_1+\theta_2-\theta_1)d\zeta_1d\theta_2, \tag{24}$$

then the term (23) takes the form

$$2\int_{-h}^{0}R_4(t,\theta_1)d\theta_1. \tag{25}$$

We can find the region of integration of (24) depicted on Fig. 2.

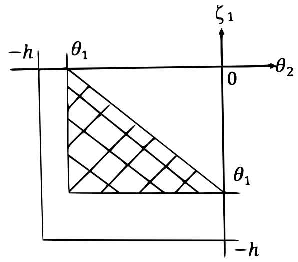

Fig. 2. Region of integration of (24)

Fig. 2 helps to rewrite (24) as

$$R_4(t,\theta_1)=\int_{\theta_1}^{0}\int_{\theta_1}^{\theta_1-\zeta_1}x^T(t+\zeta_1)W_{11}(\theta_1,\theta_2)$$
$$\times x(t+\zeta_1+\theta_2-\theta_1)d\theta_2d\zeta_1. \tag{26}$$

Replacing (26) into (25), (23) reduces to

$$
\begin{aligned}
&2\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1}^{\theta_1-\zeta_1} x^T(t+\zeta_1)W_{11}(\theta_1,\theta_2)\\
&\times x(t+\zeta_1+\theta_2-\theta_1)d\theta_2 d\zeta_1 d\theta_1\\
&=2\int_{-h}^{0}\int_{\theta_1}^{0} R_5(t,\zeta_1,\theta_1)d\zeta_1 d\theta_1,
\end{aligned}
\tag{27}
$$

with

$$
\begin{aligned}
R_5(t,\zeta_1,\theta_1)=&\int_{\theta_1}^{\theta_1-\zeta_1} x^T(t+\zeta_1)W_{11}(\theta_1,\theta_2)\\
&\times x(t+\zeta_1+\theta_2-\theta_1)d\theta_2.
\end{aligned}
$$

The region of integration in the right-hand side of (27) is similar to the one in Fig. 1, with ζ equal to ζ_1, and θ with θ_1, then by Fubini's Theorem, this equality is transformed to

$$
\begin{aligned}
&2\int_{-h}^{0}\int_{-h}^{\zeta_1}\int_{\theta_1}^{\theta_1-\zeta_1} x^T(t+\zeta_1)W_{11}(\theta_1,\theta_2)\\
&\times x(t+\zeta_1+\theta_2-\theta_1)d\theta_2 d\theta_1 d\zeta_1=\langle \zeta_2=\zeta_1+\theta_2-\theta_1\rangle\\
&=2\int_{-h}^{0} R_6(t,\zeta_1)d\zeta_1
\end{aligned}
\tag{28}
$$

with

$$
\begin{aligned}
R_6(t,\zeta_1)=&\int_{-h}^{\zeta_1}\int_{\zeta_1}^{0} x^T(t+\zeta_1)W_{11}(\theta_1,\zeta_2-\zeta_1+\theta_1)\\
&\times x(t+\zeta_2)d\zeta_2 d\theta_1.
\end{aligned}
\tag{29}
$$

The region of integration in (29) is given by

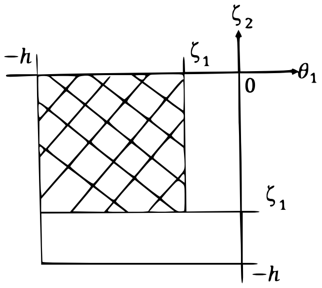

Fig. 3. Region of integration in (29)

Using Fig. 3 to transform (29), it follows that the right-hand side of (28) can be expresed as

$$
2\int_{-h}^{0}\int_{\zeta_1}^{0} x^T(t+\zeta_1)R_7(\zeta_1,\zeta_2)x(t+\zeta_2)d\zeta_2 d\zeta_1,
\tag{30}
$$

with

$$
R_7(\zeta_1,\zeta_2)=\int_{-h}^{\zeta_1} W_{11}(\theta_1,\zeta_2-\zeta_1+\theta_1)d\theta_1.
$$

Now, we can observe that the region of integration in (30) is similar to the one in Fig. 1, with ζ equal to ζ_2, and θ with ζ_1. Hence, the fourth summand of (10) is rewritten as

$$
\begin{aligned}
&2\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1}^{\theta_1-\theta_2} x^T(t+\sigma)W_{11}(\theta_1,\theta_2)\\
&\times x(t+\sigma+\theta_2-\theta_1)d\sigma d\theta_2 d\theta_1\\
&=\int_{-h}^{0}\int_{-h}^{0} x^T(t+\zeta_1)\left[\int_{-h}^{\zeta_1} W_{11}(\theta_1,\zeta_2-\zeta_1+\theta_1)d\theta_1\right]\\
&\times x(t+\zeta_2)d\zeta_1 d\zeta_2.
\end{aligned}
\tag{31}
$$

By carrying out a similar procedure for the fifth to seventh summands in (10), we obtain that these terms rewrite as

$$
\begin{aligned}
&2\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1-\theta_2}^{0} x^T(t+\sigma)W_{11}(\theta_1,\theta_2)K(\sigma+\theta_2-\theta_1)\\
&\times d\sigma d\theta_2 d\theta_1 x(t)\\
&=2x^T(t)\int_{-h}^{0}\Bigg[\int_{-h}^{\zeta}\int_{\theta_1-\zeta}^{0} K^T(\zeta+\theta_2-\theta_1)W_{11}^T(\theta_1,\theta_2)\\
&\times d\theta_2 d\theta_1\Bigg]x(t+\zeta)d\zeta,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
&2\int_{-h}^{0}\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1-\theta_2}^{0} x^T(t+\zeta_1)W_{11}(\theta_1,\theta_2)\\
&\times K(\zeta_1+\theta_2-\theta_1-h-\zeta_2)A_1x(t+\zeta_2)d\zeta_1 d\theta_2 d\theta_1 d\zeta_2\\
&=\int_{-h}^{0}\int_{-h}^{0} x^T(t+\zeta_1)\Bigg[2\int_{-h}^{\zeta_1}\int_{\theta_1-\zeta_1}^{0} W_{11}(\theta_1,\theta_2)\\
&\times K(\zeta_1+\theta_2-\theta_1-h-\zeta_2)A_1 d\theta_2 d\theta_1\Bigg]x(t+\zeta_2)d\zeta_1 d\zeta_2,
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
&2\int_{-h}^{0}\int_{-h}^{\zeta}\int_{-h}^{0}\int_{\theta_1}^{0}\int_{\theta_1-\theta_2}^{0} x^T(t+\sigma)W_{11}(\theta_1,\theta_2)\\
&\times K(\sigma+\theta_2-\theta_1-\zeta+\theta_3)d\sigma d\theta_2 d\theta_1 D(\theta_3)x(t+\zeta)d\theta_3 d\zeta\\
&=\int_{-h}^{0}\int_{-h}^{0} x^T(t+\zeta_1)\Bigg[2\int_{-h}^{\zeta_2}\int_{-h}^{\zeta_1}\int_{\theta_1-\zeta_1}^{0} W_{11}(\theta_1,\theta_2)\\
&\times K(\zeta_1+\theta_2-\theta_1-\zeta_2+\theta_3)D(\theta_3)d\theta_2 d\theta_1 d\theta_3\Bigg]\\
&\times x(t+\zeta_2)d\zeta_1 d\zeta_2.
\end{aligned}
\tag{34}
$$

Now, one can easily verify that by using the same approach the ninth to thirteenth summands in (10) can be rewritten as follows

$$
\begin{aligned}
&2x^T(t)\int_{-h}^{0} F(-h-\zeta,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times A_1x(t+\zeta)d\zeta\\
&=2x^T(t)\int_{-h}^{0}\Bigg[F(-h-\zeta,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times A_1\Bigg]x(t+\zeta)d\zeta,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&2x^T(t)\\
&\times\int_{-h}^{0}\int_{-h}^{\zeta} F(-\zeta+\theta_3,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times G(\theta_3)x(t+\zeta)d\theta_3 d\zeta=2x^T(t)\\
&\times\int_{-h}^{0}\Bigg[\int_{-h}^{\zeta} F(-\zeta+\theta_3,W_{00},W_{01}(\theta),W_{11}(\theta_1,\theta_2))\\
&\times G(\theta_3)d\theta_3\Bigg]x(t+\zeta)d\zeta,
\end{aligned}
\tag{36}
$$

$$\begin{aligned}
&\int_{-h}^{0}\int_{-h}^{0} x^T(t+\zeta_1)A_1^T \\
&\times F(\zeta_1-\zeta_2, W_{00}, W_{01}(\theta), W_{11}(\theta_1,\theta_2)) \\
&\times A_1 x(t+\zeta_2)d\zeta_2 d\zeta_1 \\
&= \int_{-h}^{0}\int_{-h}^{0} x^T(t+\zeta_1) \\
&\times \left[A_1^T F(\zeta_1-\zeta_2, W_{00}, W_{01}(\theta), W_{11}(\theta_1,\theta_2))A_1\right] \\
&\times x(t+\zeta_2)d\zeta_1 d\zeta_2,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&2\int_{-h}^{0}\int_{-h}^{0}\int_{-h}^{\zeta_2} x^T(t+\zeta_1)A_1^T \\
&\times F(h+\zeta_1-\zeta_2+\theta_3, W_{00}, W_{01}(\theta), W_{11}(\theta_1,\theta_2)) \\
&\times G(\theta_3)d\theta_3 x(t+\zeta_2)d\zeta_2 d\zeta_1 \\
&= \int_{-h}^{0}\int_{-h}^{0} x^T(t+\zeta_1)\Bigg[2\int_{-h}^{\zeta_2} A_1^T \\
&\times F(h+\zeta_1-\zeta_2+\theta_3, W_{00}, W_{01}(\theta), W_{11}(\theta_1,\theta_2)) \\
&\times G(\theta_3)d\theta_3\Bigg] x(t+\zeta_2)d\zeta_1 d\zeta_2,
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&\int_{-h}^{0}\int_{-h}^{0}\int_{-h}^{\zeta_2}\int_{-h}^{\zeta_1} x^T(t+\zeta_1)G^T(\theta_3) \\
&\times F(\zeta_1-\theta_3-\zeta_2+\theta_4, W_{00}, W_{01}(\theta), W_{11}(\theta_1,\theta_2)) \\
&\times G(\theta_4)x(t+\zeta_2)d\theta_3 d\theta_4 d\zeta_2 d\zeta_1 \\
&= \int_{-h}^{0}\int_{-h}^{0} x^T(t+\zeta_1)\Bigg[\int_{-h}^{\zeta_2}\int_{-h}^{\zeta_1} G^T(\theta_3) \\
&\times F(\zeta_1-\theta_3-\zeta_2+\theta_4, W_{00}, W_{01}(\theta), W_{11}(\theta_1,\theta_2)) \\
&\times G(\theta_4)d\theta_3 d\theta_4\Bigg] x(t+\zeta_2)d\zeta_1 d\zeta_2.
\end{aligned} \tag{39}$$

Finally, grouping terms with no integral, single integral from $-h$ to 0, and double integral from $-h$ to 0, the functional (11) is obtained, showing that the structure of the Bellman functional is recovered.

■

4. CONCLUSIONS

We have proved that the functional corresponding to the performance index of the suboptimal control constructed at each step of the solution of Bellman's equation has the structure of the general functional proposed by Krasovskii and Ross.

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Appendix A. FUBINI'S THEOREM

For the sake of completness, we remind below Fubini's Theorem (See Thomas and Finney (1996)).

First form

If $f(x,y)$ is continuous on the rectangular region $\mathcal{R}: a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint\limits_{\mathcal{R}} f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

Stronger form

Let $f(x,y)$ be continuous on region $\mathcal{R}$.

1.- If $\mathcal{R}$ is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then

$$\iint\limits_{\mathcal{R}} f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

2.- If $\mathcal{R}$ is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then

$$\iint\limits_{\mathcal{R}} f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy.$$