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Identification of noisy input-output FIR models with colored output noise

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Abstract: This paper deals with the identification of FIR models corrupted by white input noise and colored output noise. An identification algorithm that exploits the properties of both the dynamic Frisch scheme and the high-order Yule-Walker (HOYW) equations is proposed. It is shown how the HOYW equations allow to define a selection criterion for identifying the input noise variance (and then the FIR coefficients) within the Frisch locus of solutions. The proposed approach does not require any *a priori* knowledge about the input and output noise variances. The algorithm performance is assessed by means of Monte Carlo simulations.

Keywords: System identification, FIR models, errors-in-variables models, Frisch scheme, high-order Yule-Walker equations

1. INTRODUCTION

The identification of finite impulse response (FIR) models plays an important role in many signal processing and control applications (Haykin, 1991; Goodwin and Sin, 1984; Schafer and Oppenheim, 1989; Kalouptsidis, 1997; Davila, 1994; Xu et al., 1995; Diversi et al., 2005; Cerone et al., 2013; Aljanaideh and Bernstein, 2017; Aljanaideh et al., 2018). When both the input and output of the FIR model are corrupted by noise, classical identification algorithms like least squares and prediction error methods lead to biased estimates (Söderström, 2018).

Noisy input-output models can be consistently identified by means of the instrumental variable (IV) method (Söderström, 2018). In this case, the vector of instruments is based on delayed input and output samples leading to a set of high-order Yule-Walker (HOYW) equations. Despite its computational efficiency, this approach may lead to a low estimation accuracy because of the poor estimates of the high-lag autocorrelations involved in the HOYW equations (Söderström, 2018). The total least squares (TLS) scheme can also be used but the ratio of the input noise variance to the output noise variance is assumed as *a priori* known (Davila, 1994; Feng et al., 1998; Feng and Zheng, 2006).

An effective way to counteract the noise-induced bias in the Least Squares estimate consists in using the bias compensation principle (Stoica and Söderström, 1982; Söderström, 2018). Starting from this principle, various bias-compensated least squares (BCLS) algorithms have been proposed to identify noisy FIR models (Zheng, 2003; Feng and Zheng, 2007; Diversi et al., 2008; Diversi, 2008, 2009; Kang and Park, 2013; Arablouei et al., 2014; Jung and Park, 2017). Many BCLS methods assumes that both the input and the output noise are white processes. In (Bertrand et al., 2011; Arablouei et al., 2014), the input noise is a colored process and is correlated with the white output noise. However, the input noise covariance matrix and the cross-covariance between the input and output noise are assumed *a priori* known or already estimated.

This paper deals with the identification of FIR models corrupted by white input noise and colored output noise. The estimation of the FIR coefficients is performed relying on the properties of the dynamic Frisch scheme (Guidorzi et al., 2008). The idea underlying the Frisch scheme consists in finding the estimation of the input-output noise variances within a locus of solutions compatible with the covariance matrix of the noisy data. The search for a single solution inside the locus requires the definition of a suitable selection criterion. For single-input single-output infinite impulse response (IIR) models, the Frisch locus is described by a curve in the first quadrant of $\mathbb{R}^2$. Nevertheless, for the FIR case, such locus can be described by a segment of $\mathbb{R}^+$, as shown in (Diversi et al., 2008).

The use of the Frisch scheme in FIR identification was already proposed in (Diversi et al., 2008) but the criterion employed to select a single solution inside the Frisch locus allows to consider only the case of white input-output noise. In order to deal with colored output noise, the selection criterion proposed in this paper exploits the properties of the HOYW equations. In particular, the HOYW equations allow to define a selection criterion for identifying the input noise variance (and then the FIR coefficients) within the Frisch locus of solution. It is worth noting that the proposed approach does not require any *a priori* knowledge about the input and output noise variances. The effectiveness of the identification algorithm is tested by means of Monte Carlo simulations, where a comparison with the approach in (Diversi et al., 2008) and the TLS method is also considered.

The paper is organized as follows. In Section 2 we describe the general problem of FIR models estimation when only corrupted measurements of its input and output are available. Then, in Section 3, we show how to reformulate the problem into the Frisch scheme framework highlighting the related locus of solutions. In Section 4 we point out how to use HOYW equations to get a unique solution within the solution set and the related identification algorithm. In Section 5, the performance of the proposed procedure is shown by means of numerical

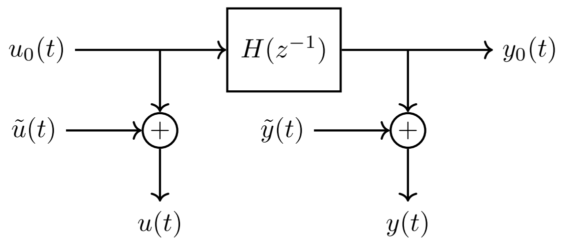

Fig. 1. Noisy input-output FIR model

simulations. Finally, some conclusions are reported in Section 6.

2. PROBLEM STATEMENT

Let us consider the linear, time-invariant FIR system depicted in figure 1. Its input u_0 and output y_0 are linked by the difference equation

$$y_0(t) = H(z^{-1})u_0(t), \tag{1}$$

where $H(z^{-1})$ is the following polynomial in the backward shift operator z^{-1}:

$$H(z^{-1}) = h_0 + h_1 z^{-1} + \ldots + h_{n-1} z^{1-n}. \tag{2}$$

Let us assume that the available measurements of the true input and output are corrupted by the presence of additive noise $\tilde{u}(t)$ and $\tilde{y}(t)$, namely

$$u(t) = u_0(t) + \tilde{u}(t) \tag{3}$$

$$y(t) = y_0(t) + \tilde{y}(t). \tag{4}$$

The following assumptions are introduced.

- **A1.** The FIR length n is known.
- **A2.** The noise-free input u_0 is either a zero–mean ergodic process or a quasi–stationary bounded deterministic signal, i.e. such that the limit
$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} u_0(t)\, u_0(t-\tau) \tag{5}$$
exists $\forall\tau$ (Ljung, 1999). Moreover, $u_0(t)$ is persistently exciting of sufficiently high order.
- **A3.** The input noise $\tilde{u}(t)$ is a zero-mean ergodic white process with unknown variance $\sigma^{2*}_{\tilde{u}}$.
- **A4.** The output noise $\tilde{y}(t)$ is an arbitrarily autocorrelated zero–mean ergodic process with unknown autocorrelation function. Its variance is denoted by $\sigma^{2*}_{\tilde{y}}$.
- **A5.** The additive noises $\tilde{u}(t)$, $\tilde{y}(t)$ are mutually uncorrelated and uncorrelated with the noise-free input $u_0(t)$.

By defining the vectors

$$\varphi_0(t) = [-y_0(t)\; u_0(t)\; u_0(t-1)\; \ldots\; u_0(t-n+1)]^T, \tag{6}$$

$$\varphi(t) = [-y(t)\; u(t)\; u(t-1)\; \ldots\; u(t-n+1)]^T = \left[-y(t)\; \varphi_u^T(t)\right]^T, \tag{7}$$

$$\tilde{\varphi}(t) = [-\tilde{y}(t)\; \tilde{u}(t)\; \tilde{u}(t-1)\; \ldots\; \tilde{u}(t-n+1)]^T, \tag{8}$$

and the vector of FIR coefficients

$$\bar{\theta}^* = [1\; h_0\; h_1\; \ldots\; h_{n-1}]^T = \left[1\; \theta^{*T}\right]^T \tag{9}$$

it is possible to rewrite the model (1)–(4) in the following way:

$$\varphi_0^T(t)\bar{\theta}^* = 0, \tag{10}$$

$$\varphi(t) = \varphi_0(t) + \tilde{\varphi}(t). \tag{11}$$

Finally, the identification problem turns out to be the following:

Problem 1. *Given a set of N measurements of the corrupted input $u(t)$ and output $y(t)$, estimate the FIR coefficients $h_0, h_1, \ldots, h_{n-1}$, i.e. the coefficient vector θ^*.*

3. THE FRISCH LOCUS OF COMPATIBLE SOLUTIONS

The noisy input-output FIR model (1)–(4), belongs to the family of errors-in-variables models so that it can be estimated by exploiting the properties of the Frisch scheme, as done in (Diversi et al., 2008). The main idea behind the Frisch scheme consists in finding the solution of the identification problem within a locus of solutions that are compatible with the covariance matrix of the noisy data (Guidorzi et al., 2008). To this end, let us consider the covariance matrix

$$R = E\left[\varphi(t)\varphi^T(t)\right], \tag{12}$$

where $E[\cdot]$ denotes the expectation operator, and, similarly, the covariance matrices $R_0 = E\left[\varphi_0(t)\varphi_0^T(t)\right]$, $\tilde{R}^* = E\left[\tilde{\varphi}(t)\tilde{\varphi}^T(t)\right]$ of the noise-free data and noise respectively. From (10), (11) and Assumptions A2–A5 we get

$$R_0\bar{\theta}^* = 0, \tag{13}$$

$$R = R_0 + \tilde{R}^*. \tag{14}$$

From (13) and (14) we finally obtain:

$$\left(R - \tilde{R}^*\right)\bar{\theta}^* = 0, \tag{15}$$

where the noise covariance matrix is

$$\tilde{R}^* = \begin{bmatrix} \sigma^{2*}_{\tilde{y}} & 0 \\ 0 & \sigma^{2*}_{\tilde{u}} I_n \end{bmatrix}. \tag{16}$$

and I_n denotes the $n \times n$ identity matrix. Consider now the set of couples $\left(\sigma^2_{\tilde{u}}, \sigma^2_{\tilde{y}}\right)$ belonging to the first quadrant of $\mathbb{R}^2$ and satisfying

$$\left(R - \tilde{R}\right) \geq 0 \;\text{ and }\; \det\left(R - \tilde{R}\right) = 0, \tag{17}$$

where

$$\tilde{R} = \begin{bmatrix} \sigma^2_{\tilde{y}} & 0 \\ 0 & \sigma^2_{\tilde{u}} I_n \end{bmatrix}. \tag{18}$$

Because of (17), any couple of the set can be associated with a parameter vector $\bar{\theta}\left(\sigma^2_{\tilde{u}}, \sigma^2_{\tilde{y}}\right)$ satisfying

$$\left(R - \tilde{R}\right)\bar{\theta}\left(\sigma^2_{\tilde{u}}, \sigma^2_{\tilde{y}}\right) = 0. \tag{19}$$

Note that $\bar{\theta}\left(\sigma^2_{\tilde{u}}, \sigma^2_{\tilde{y}}\right)$ can be computed from any basis of the null space of $R - \tilde{R}$ by normalizing the first element to 1, see (9).

At this point, it is possible to further develop the set of equations (19) in order to restrict the locus of solutions, by exploiting the peculiarities of FIR models. First, the matrix R is partitioned into the following blocks

$$R = \begin{bmatrix} \sigma^2_y & r^T_{yu} \\ r_{yu} & R_u \end{bmatrix} \tag{20}$$

where

$$\sigma^2_y = E\left[y^2(t)\right] \tag{21}$$

$$r_{yu} = -E\left[\varphi_u(t)y(t)\right] \tag{22}$$

$$R_u = E\left[\varphi_u(t)\varphi_u^T(t)\right] \tag{23}$$

Then, (19) may be rewritten as

$$\begin{bmatrix} \sigma^2_y - \sigma^2_{\tilde{y}} & r^T_{yu} \\ r_{yu} & R_u - \sigma^2_{\tilde{u}} I_n \end{bmatrix} \begin{bmatrix} 1 \\ \theta \end{bmatrix} = 0 \tag{24}$$

from which we obtain the following two sets of equations

$$\sigma^2_y - \sigma^2_{\tilde{y}} + r^T_{yu}\theta = 0, \tag{25}$$

$$r_{yu} + \left(R_u - \sigma^2_{\tilde{u}} I_n\right)\theta = 0. \tag{26}$$

Finally, by solving (25) and (26) accounting for conditions (17), we state the following theorem.

Theorem 1 (Diversi et al. (2008)). *The set of all diagonal matrices $\tilde{R}$ satisfying* (17) *is defined by the couples $\left(\sigma^2_{\tilde{u}}, \sigma^2_{\tilde{y}}\right)$ belonging to the first quadrant of $\mathbb{R}^2$ such that*

$$\sigma^2_{\tilde{u}} \in \left[0, \sigma^2_{\tilde{u}_{max}}\right], \tag{27}$$

$$\sigma^2_{\tilde{y}} = \sigma^2_y + r^T_{yu}\theta, \tag{28}$$

with

$$\theta = -\left(R_u - \sigma^2_{\tilde{u}} I_n\right)^{-1} r_{yu}, \tag{29}$$

$$\sigma^2_{\tilde{u}_{max}} = \lambda_{min}\left(R_u - \frac{r_{yu} r^T_{yu}}{\sigma^2_y}\right). \tag{30}$$

Corollary 1 (Diversi et al. (2008)). *The locus of solutions described by Theorem 1 contains the couple $(\sigma^{2*}_{\tilde{u}}, \sigma^{2*}_{\tilde{y}})$ that can be associated with the true coefficients of the FIR model θ^* by means of* (29).

It is important to note that this formulation of the solution set of Problem 1 leads to a locus of solutions which is a function of $\sigma^2_{\tilde{u}}$ only, with $\sigma^2_{\tilde{y}}$ and θ depending on it, namely

$$\theta(\sigma^2_{\tilde{u}}) = -\left(R_u - \sigma^2_{\tilde{u}} I_n\right)^{-1} r_{yu}, \tag{31}$$

$$\sigma^2_{\tilde{y}}(\sigma^2_{\tilde{u}}) = \sigma^2_y + r^T_{yu}\theta(\sigma^2_{\tilde{u}}). \tag{32}$$

with $\sigma^2_{\tilde{y}}(\sigma^{2*}_{\tilde{u}}) = \sigma^{2*}_{\tilde{y}}$ and $\theta(\sigma^{2*}_{\tilde{u}}) = \theta^*$ corresponding to the true solution. In the asymptotic case, the identification problem may thus be solved by looking for $\sigma^{2*}_{\tilde{u}}$ within the interval $\left[0, \sigma^2_{\tilde{u}_{max}}\right]$. To this end it is necessary to introduce a suitable selection criterion. The criterion proposed in (Diversi et al., 2008), which exploits the statistical properties of the residuals of the EIV FIR model, can be applied only when both the input and output noise are white processes. The next section describes a criterion based on the properties of the high-order Yule-Walker equations that allows dealing with colored output noise.

4. COMBINING THE FRISCH SCHEME AND HIGH-ORDER YULE-WALKER EQUATIONS

The search for the true input noise variance $\sigma^{2*}_{\tilde{u}}$ within the Frisch locus can be performed by relying on the high-order Yule-Walker (HOYW) equations, as shown in the following.

Firstly, we define the $q \times 1$ vector of delayed input samples

$$\varphi^q(t) = \left[u(t-n)\; u(t-n-1)\; \ldots\; u(t-n-q+1)\right]^T, \tag{33}$$

with $q \geq 1$. Because of (3), $\varphi^q(t)$ can be decomposed as

$$\varphi^q(t) = \varphi^q_0(t) + \tilde{\varphi}^q(t), \tag{34}$$

where the elements of $\varphi^q_0(t)$ and $\tilde{\varphi}^q(t)$ are vectors of delayed samples of $u_0(t)$ and $\tilde{u}(t)$ respectively. Define the following $q \times (n+1)$ matrix

$$R^q = E\left[\varphi^q(t)\varphi^T(t)\right]. \tag{35}$$

From (34), (11) and Assumptions A3, A5 we get

$$\begin{aligned} R^q &= E\left[\left(\varphi^q_0(t) + \tilde{\varphi}^q(t)\right)\left(\varphi_0(t) + \tilde{\varphi}(t)\right)^T\right] \\ &= E\left[\varphi^q_0(t)\varphi^T_0(t)\right] = R^q_0. \end{aligned} \tag{36}$$

From (10) and (36) we can finally derive the following set of q HOYW equation:

$$R^q\bar{\theta}^* = 0 \tag{37}$$

Note that (37) does not involve the output noise variance $\sigma^{2*}_{\tilde{y}}$. In this context, it is possible to search for $\sigma^{2*}_{\tilde{u}}$ within its range of possible values $\left[0, \sigma^2_{\tilde{u}_{max}}\right]$ by means of an optimization problem. For this purpose, let us define the cost function $J(\sigma^2_{\tilde{u}})$ as

$$J(\sigma^2_{\tilde{u}}) = \left|\left|R^q\,\bar{\theta}(\sigma^2_{\tilde{u}})\right|\right|^2_2, \quad \sigma^2_{\tilde{u}} \in \left[0, \sigma^2_{\tilde{u}_{max}}\right] \tag{38}$$

where $\bar{\theta}(\sigma^2_{\tilde{u}}) = \left[1\;\theta(\sigma^2_{\tilde{u}})\right]^T$ and $\theta(\sigma^2_{\tilde{u}})$ is given by (31). This function has the following properties:

$$\begin{aligned} J(\sigma^2_{\tilde{u}}) &\geq 0 \\ J(\sigma^{2*}_{\tilde{u}}) &= 0. \end{aligned} \tag{39}$$

Then, $\sigma^{2*}_{\tilde{u}}$ and consequently $\theta(\sigma^{2*}_{\tilde{u}}) = \theta^*$ can be obtained by solving

$$\underset{\sigma^2_{\tilde{u}} \in \left[0, \sigma^2_{\tilde{u}_{max}}\right]}{\arg\min}\; J(\sigma^2_{\tilde{u}}). \tag{40}$$

Remark 1. *Relation* (37) *represents a set of high order Yule–Walker equations that could be directly used to obtain the parameter vector θ^* provided that $q \geq n$. This approach can also be viewed as an instrumental variable method that uses delayed inputs as instruments (Söderström, 2018). However, the accuracy of IV methods is often poor since they require the estimation of high–lag autocorrelations (Söderström, 2018).*

In practice, the number of data available from the system is finite. This means we may only rely upon an estimate of the matrices R and R^q based on the N measurements of $u(t)$ and $y(t)$ at our disposal:

$$\hat{R} = \frac{1}{N}\sum_{t=1}^{N}\varphi(t)\varphi^T(t) = \begin{bmatrix} \hat{\sigma}^2_y & \hat{r}^T_{yu} \\ \hat{r}_{yu} & \hat{R}_u \end{bmatrix} \tag{41}$$

$$\hat{R}^q = \frac{1}{N}\sum_{t=q+n+1}^{N}\varphi^q(t)\varphi^T(t). \tag{42}$$

In this case, an estimate $\hat{\sigma}^2_{\tilde{u}}$ of $\sigma^{2*}_{\tilde{u}}$ is found by minimizing the loss function

$$J(\sigma^2_{\tilde{u}}) = \left|\left|\hat{R}^q\,\bar{\theta}(\sigma^2_{\tilde{u}})\right|\right|^2_2, \quad \sigma^2_{\tilde{u}} \in \left[0, \hat{\sigma}^2_{\tilde{u}_{max}}\right] \tag{43}$$

where

$$\hat{\sigma}^2_{\tilde{u}_{max}} = \lambda_{\min}\left(\hat{R}_u - \frac{\hat{r}_{yu}\hat{r}^T_{yu}}{\hat{\sigma}^2_y}\right), \tag{44}$$

The solution of Problem 1 is thus given by

$$\hat{\theta} = -\left(\hat{R}_u - \hat{\sigma}^2_{\tilde{u}} I_n\right)^{-1}\hat{r}_{yu}, \tag{45}$$

with

$$\hat{\sigma}^2_{\tilde{u}} = \underset{\sigma^2_{\tilde{u}} \in \left[0, \hat{\sigma}^2_{\tilde{u}_{max}}\right]}{\arg\min}\; J(\sigma^2_{\tilde{u}}). \tag{46}$$

Remark 2. *Due to the ergodicity assumption (see A2-A4) it follows that*

$$\lim_{N\to\infty}\hat{R} = R, \quad \lim_{N\to\infty}\hat{R}^q = R^q, \quad \text{w. p. 1} \tag{47}$$

so that

$$\underset{\sigma^2_{\tilde{u}} \in \left[0, \hat{\sigma}^2_{\tilde{u}_{max}}\right]}{\arg\min}\; J(\sigma^2_{\tilde{u}}) \underset{N\to\infty}{\longrightarrow} \underset{\sigma^2_{\tilde{u}} \in \left[0, \sigma^2_{\tilde{u}_{max}}\right]}{\arg\min}\; J(\sigma^2_{\tilde{u}}) \quad \text{w. p. 1} \tag{48}$$

The whole identification procedure is summarized by the following algorithm.

Algorithm 1. *Starting from the noisy observations $u(1), \ldots, u(N)$ and $y(1), \ldots, y(N)$:*

(1) Compute the estimate $\hat{R}$ as in (41)*, and extract the highlighted blocks. Compute also the estimate $\hat{\sigma}^2_{\tilde{u}_{max}}$ as in* (44).

(2) Choose the number q of HOYW equations (see (33)*) and compute the estimates $\hat{R}^q$ as in* (42)
(3) Initialize to a generic value of $\sigma^2_{\tilde{u}} \in \left[0, \hat{\sigma}^2_{\tilde{u}_{max}}\right]$.
(4) Compute the relative expression for the parameter vector $\theta(\sigma^2_{\tilde{u}})$ as in (31) *and form $\bar{\theta} = \left[\, 1\, \theta(\sigma^2_{\tilde{u}})^T \,\right]^T$.*
(5) Compute the cost function $J(\sigma^2_{\tilde{u}})$ (43).
(6) Update $\sigma^2_{\tilde{u}}$ to a new value $\in \left[0, \hat{\sigma}^2_{\tilde{u}_{max}}\right]$ corresponding to a decrease in $J(\sigma^2_{\tilde{u}})$.
(7) Repeat the steps 4-6 until the value $\hat{\sigma}^2_{\tilde{u}}$ corresponding to the minimum of $J(\sigma^2_{\tilde{u}})$ is found. The obtained estimation $\hat{\theta}$ of the FIR coefficients is then given by $\hat{\theta} = \theta(\hat{\sigma}^2_{\tilde{u}})$.

Remark 3. *Steps 4-6, corresponding to the search for the solution of the optimization problem* (46)*, may be performed by means of any numerical search procedure. In particular, we deployed the Newton-Raphson method with parameter projection, as described in Appendix A.*

Remark 4. *Once that the estimates $\hat{\sigma}^2_{\tilde{u}}$, $\hat{\theta}$ of the input noise variance and the FIR coefficients have been obtained, an estimate of the output noise variance can be computed by means of* (32)*:*

$$\hat{\sigma}^2_{\tilde{y}} = \hat{\sigma}^2_y + \hat{r}^T_{yu}\hat{\theta}. \tag{49}$$

From $\hat{\sigma}^2_{\tilde{u}}$ and $\hat{\sigma}^2_{\tilde{y}}$ it is possible to evaluate the signal to noise ratio on both the input and the output.

5. NUMERICAL RESULTS

Algorithm 1 is tested by means of Monte Carlo simulations. Two different experiments have been considered. The first refers to a synthetic FIR whereas the second one concerns a FIR model representing the truncated impulse response of an asymptotically stable IIR system. A comparison with the Frisch scheme-based algorithm (Diversi et al., 2008) and the total least squares (TLS) method (Davila, 1994) is also considered.

5.1 Experiment 1

Firstly, to test Algorithm 1, we select a synthetic FIR model of length $n = 5$ already used in (Davila, 1994; Feng et al., 1998; Feng and Zheng, 2006; Zheng, 2003; Feng and Zheng, 2007; Diversi et al., 2008; Diversi, 2008, 2009) with the following coefficients:

$$\theta^* = \left[\, -0.3 \;\, -0.9 \;\, 0.8 \;\, -0.7 \;\, 0.6 \,\right]^T. \tag{50}$$

The input $u_0(t)$ is an ARMA process defined by

$$u_0(t) = \frac{1 - 0.3z^{-1}}{1 - 0.9z^{-1}}\, e(t), \tag{51}$$

where $e(t)$ is a unit variance white noise. A first Monte Carlo simulation of 1000 runs has been carried out by considering additive white noise on both the input and the output. The noise variances $\sigma^{2*}_{\tilde{u}}, \sigma^{2*}_{\tilde{y}}$ have been selected in order to set the signal to noise ratio (SNR) to 5 dB on both the input and the output. A second Monte Carlo simulation of 1000 runs has then been performed by considering additive white noise $\tilde{u}(t)$ on the input and additive colored noise $\tilde{y}(t)$ on the output. In particular, the output noise is the following autoregressive process:

$$\tilde{y}(t) = \frac{1}{1 - 0.8z^{-1}}\, w(t). \tag{52}$$

The variance of $\tilde{u}(t)$ and of the driving white process $w(t)$ have been set to get an input and output SNR of 5 dB. In both Monte Carlo simulations the number of HOYW equations is set to $q = 2$ and the number of samples used is $N = 10000$.

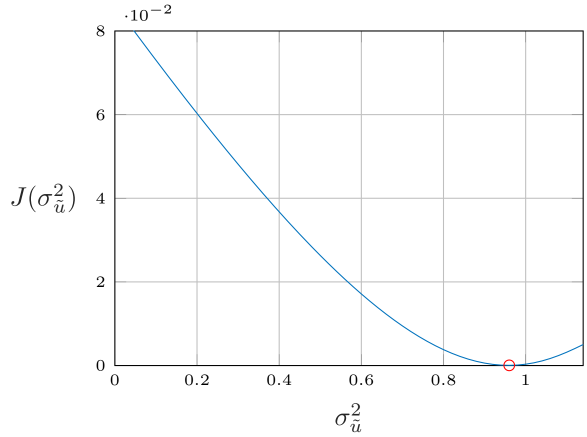

Fig. 2. Values of the cost function $J(\sigma^2_{\tilde{u}})$ cost function over the interval $\left[0, \hat{\sigma}^2_{\tilde{u}_{max}}\right]$ in one of the Monte Carlo runs. The red circle corresponds to the minimum value.

The performance of Algorithm 1 has been compared to that of the Frisch scheme-based algorithm introduced in (Diversi et al., 2008) and of the TLS method. The results of both the Monte Carlo simulations are summarized in Table I that reports the true values of FIR coefficients and noise variances, the mean of their estimates, and the corresponding standard deviations. As expected, when the input and output noise are white, all the algorithms lead to unbiased estimates. In this case, the approach in (Diversi et al., 2008) has a slightly better performance because of the smaller standard deviations of the estimated parameters. The TLS approach outperforms the other methods but requires the a priori knowledge of the noise variance ratio $\sigma^{2*}_{\tilde{y}}/\sigma^{2*}_{\tilde{u}}$. When the output noise is colored, Algorithm 1 still gives good results whereas the algorithm in (Diversi et al., 2008) clearly leads to biased estimates. Again, the TLS method leads to better results but is based on information that are seldom available in practice.

In order to show the effectiveness of the cost function $J(\sigma^2_{\tilde{u}})$, Fig. 2 reports its values over the interval $\left[0, \hat{\sigma}^2_{\tilde{u}_{max}}\right]$ in a run of the second Monte Carlo simulation (colored noise). It is worth noting that the minimum of $J(\sigma^2_{\tilde{u}})$ occurs at a value of $\sigma^2_{\tilde{u}}$ very close to the true one. All other runs of the Monte Carlo simulations exhibit a similar behavior.

5.2 Experiment 2

The aim of the second experiment is to identify a FIR model representing the truncated impulse response of an asymptotically stable IIR system corrupted by white input noise and colored output noise. We consider the following IIR model

$$y_0(t) = \frac{(z+0.6)(z+0.5)}{(z-0.5)(z^2+0.6z+0.58)}\, u_0(t), \tag{53}$$

where the input $u_0(t)$ is the ARMA process (51), the additive input noise $\tilde{u}(t)$ is a white process and the additive output noise $\tilde{y}(t)$ is the colored process (52). Starting from $N = 10000$ samples of the noisy input and output, $u(t)$ and $y(t)$, the truncated impulse response of the system (a FIR model with length $n = 24$) has been identified by means of Algorithm 1 and the algorithm in (Diversi et al., 2008). The hyperparameter q has been set to 24. Two different values of the input and output noise variances have been considered to get SNR$=$ 10 dB and

Table 1. True and estimated values of the FIR coefficients and of the input and output noise variances. Monte Carlo simulation of 1000 runs performed with $N = 10000$. Synthetic FIR.

	h_0	h_1	h_2	h_3	h_4	$\sigma_{\tilde{u}}^{2*}$	$\sigma_{\tilde{y}}^{2*}$
True	-0.3	-0.9	0.8	-0.7	0.6	0.96	0.67
$\tilde{u}(t)$ white, $\tilde{y}(t)$ white							
Algorithm 1	-0.302 ± 0.026	-0.892 ± 0.085	0.794 ± 0.115	-0.694 ± 0.095	0.594 ± 0.054	0.949 ± 0.065	0.678 ± 0.142
Algorithm (Diversi et al., 2008)	-0.303 ± 0.024	-0.895 ± 0.033	0.797 ± 0.043	-0.696 ± 0.034	0.597 ± 0.024	0.956 ± 0.031	0.674 ± 0.043
TLS	-0.301 ± 0.026	-0.898 ± 0.030	0.799 ± 0.029	-0.699 ± 0.033	0.599 ± 0.022	0.959 ± 0.013	0.669 ± 0.009
$\tilde{u}(t)$ white, $\tilde{y}(t)$ colored							
Algorithm 1	-0.302 ± 0.025	-0.891 ± 0.089	0.792 ± 0.121	-0.693 ± 0.099	0.594 ± 0.061	0.947 ± 0.069	0.681 ± 0.153
Algorithm (Diversi et al., 2008)	-0.309 ± 0.015	-0.666 ± 0.020	0.492 ± 0.019	-0.446 ± 0.019	0.435 ± 0.017	0.713 ± 0.026	1.077 ± 0.024
TLS	-0.301 ± 0.025	-0.899 ± 0.027	0.800 ± 0.027	-0.700 ± 0.031	0.599 ± 0.020	0.959 ± 0.012	0.669 ± 0.008

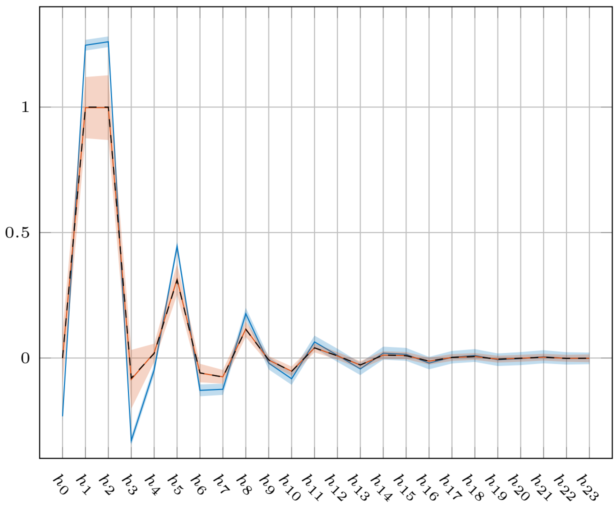

Fig. 3. True truncated impulse response of the IIR system (53) (dashed black), mean of its estimate obtained with Algorithm 1 in red and with Algorithm (Diversi et al., 2008) in blue and associated standard deviations in transparency. Monte Carlo simulation of 1000 runs performed with $N = 10000$ and SNR$= 10$dB.

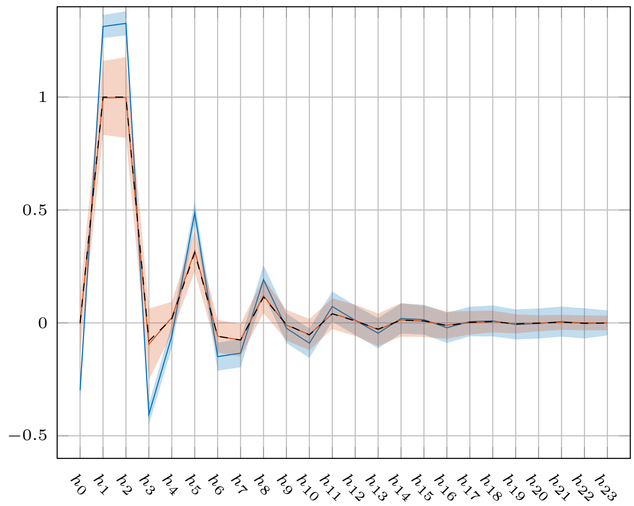

Fig. 4. True truncated impulse response of the IIR system (53) (dashed black), mean of its estimate obtained with Algorithm 1 in red and with Algorithm (Diversi et al., 2008) in blue and associated standard deviations in transparency. Monte Carlo simulation of 1000 runs performed with $N = 10000$ and SNR$= 5$dB.

SNR$= 5$ dB respectively. For each value of the SNR a Monte Carlo simulation of 1000 runs has been performed. The results are shown in Figs. 3 and 4, reporting the first 24 coefficients of the impulse response, the mean of the obtained estimates, and the associated standard deviations. These figures confirm the good performance of the proposed approach. Again, the estimates obtained by using the approach in (Diversi et al., 2008) are clearly biased.

Notice that in both experiments the number of HOYW equations q used to achieve consistent estimation is very small. This, together with the use of the Newton-Raphson method, leads to a relatively low computational load of the proposed approach.

6. CONCLUSIONS

An algorithm for the estimation of FIR models in presence of corrupting white input noise and colored output noise has been presented. The method takes advantage of the properties of both the dynamic Frisch scheme and the high-order Yule-Walker equations. The proposed identification algorithm is shown to achieve consistency in the presence of colored output noise both when identifying a synthetic FIR, and when estimating the actual impulse response of an asymptotically stable IIR system. Moreover, the estimator requires a small number of HOYW equations to attain consistency, that combined with the use of Newton-Raphson, results in a low computational load. Future work may exploit the iterative nature of the optimization method involved in the identification procedure to derive a recursive formulation of Algorithm 1.

Appendix A. NEWTON-RAPHSON ALGORITHM

The Newton-Raphson method with parameter projection has been used to solve the optimization problem defined in (46). Namely, starting from Step 2 in Algorithm 1, iteratively compute

$$\sigma_{\tilde{u}}^2(k+1) = \Pi\left[\sigma_{\tilde{u}}^2(k) - \alpha\frac{J'(\hat{\sigma}_{\tilde{u}}^2)}{J''(\hat{\sigma}_{\tilde{u}}^2)}\right] \tag{A.1}$$

until

$$|\sigma_{\tilde{u}}^2(k+1) - \sigma_{\tilde{u}}^2(k)| \leq \epsilon \text{ with } \epsilon > 0. \tag{A.2}$$

For the sake of clarity in the following discussion we drop the $\hat{\cdot}$ symbol and θ dependency on $\sigma_{\tilde{u}}^2$ is hidden when not necessary. Equation (A.1) is obtained starting from (43) in which, by developing the 2$-$norm, we obtain

$$J(\sigma_{\tilde{u}}^2) = \bar{\theta}^T \hat{R}^{q^T} \hat{R}^q \bar{\theta} = \bar{\theta}^T \hat{\bar{\Sigma}}^q \bar{\theta} = \bar{\theta}^T \begin{bmatrix} \hat{\sigma} & \hat{\rho}^T \\ \hat{\rho} & \hat{\Sigma} \end{bmatrix} \bar{\theta}. \tag{A.3}$$

where $\hat{\bar{\Sigma}}^q = \hat{R}^{q^T} \hat{R}^q$ is a positive definite symmetric matrix and is divided in the above mentioned blocks. Then, by inserting (9) we get

$$J(\sigma_{\tilde{u}}^2) = \theta^T \hat{\Sigma} \theta + 2\hat{\rho}^T \theta + \hat{\sigma}. \tag{A.4}$$

In this form it is easier to compute its derivatives, that are:

$$\frac{\partial J(\sigma_{\tilde{u}}^2)}{\partial \sigma_{\tilde{u}}^2} = 2\left(\theta^T \hat{\Sigma}^T + \hat{\rho}^T\right)\left(\hat{R}_u - \sigma_{\tilde{u}}^2 I_n\right)^{-1}\theta, \tag{A.5}$$

$$\frac{\partial^2 J(\sigma_{\tilde{u}}^2)}{\partial (\sigma_{\tilde{u}}^2)^2} = 2\theta^T \left(\hat{R}_u - \sigma_{\tilde{u}}^2 I_n\right)^{-T} \hat{\Sigma} \left(\hat{R}_u - \sigma_{\tilde{u}}^2 I_n\right)^{-1}\theta \tag{A.6}$$

in which the derivative of θ is already plugged in and is the following

$$\frac{\partial \theta(\sigma_{\tilde{u}}^2)}{\partial \sigma_{\tilde{u}}^2} = \left(\hat{R}_u - \sigma_{\tilde{u}}^2 I_n\right)^{-1}\theta. \tag{A.7}$$

Finally, the projection function $\Pi[\cdot]$ is nothing but the following case function:

$$\Pi[\cdot] = \begin{cases} 0 & \text{if } \sigma_{\tilde{u}}^2 < 0 \\ \hat{\sigma}_{\tilde{u}_{max}}^2 & \text{if } \sigma_{\tilde{u}}^2 \geq \hat{\sigma}_{\tilde{u}_{max}}^2 \end{cases} \tag{A.8}$$

which keeps $\sigma_{\tilde{u}}^2$ into the solution set. The hyperparameter α was set to 1 during simulations.

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