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Proving Routh's Theorem using the Euclidean Algorithm and Cauchy's Theorem ⋆

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Abstract:
This paper presents a proof of Routh's theorem for polynomials with real coefficients, determining the number of roots in the right half plane (RHP). The proof exploits the relationship of the Routh array to the Euclidean algorithm and applies Cauchy's theorem in an analogous way to that of applying the Nyquist criterion to investigate the stability of a control system. While a number of papers have been published over the years with different proofs of Routh's stability criterion or Routh's theorem, the aim in this paper is to present a proof that may offer most insight to undergraduate students of engineering. Routh's theorem and his array are introduced without any proof in most undergraduate texts on control theory, whereas the Nyquist criterion is typically treated quite extensively in such texts. As well as presenting a proof for the regular case when all the coefficients in the first column of the Routh array are non-zero, analogous proofs are given for the singular cases when some of the leading coefficients in a row, or the coefficients of the entire row, become zero. In the first case, these result in a statement on the number of roots in the RHP, more explicit than those typically presented in papers on Routh's theorem. In the second case, the only case where there may be roots on the imaginary axis, use is made of the modified array introduced by Routh, often referred to as the Q-method, to determine the number of such roots, differentiating between simple and multiple roots. One can thus distinguish between exponential stability, marginal stability and polynomial instability, when there are no roots in the RHP, with these results.

Keywords: Education, Routh's theorem, Nyquist's criterion, stability criteria, polynomials, roots.

1. INTRODUCTION

Routh's original proof of his theorem on how to determine the number of roots of a polynomial in the RHP from the first column in the Routh array is based on results on Cauchy indices and Sturm sequences, see Routh (1877), Gantmacher (1959) and Barnett et al. (1977). Wall (1945) proved Routh's theorem by applying analytic theory of continued fractions and Parks (1962) gave a direct algebraic proof of Routh's stability criterion using the second method of Lyapunov, stating that all the roots are in the left half plane (LHP), if there are no sign changes in the first column of the Routh array. Fairly recently, a number of different proofs of Routh's theorem or stability criterion have been published that do not depend on these results and have thus been characterized as "elementary" or "simple," see, e.g., Lepschy et al. (1988), Chapellat et. al (1990), Anagnost et al. (1991), Meinsma (1995), Ho et al. (1998), Ferrante et al. (1999) and Matsumoto (2001).

This paper presents yet another proof of Routh's theorem for polynomials with real coefficients assumed to have no zero roots, aimed at undergraduate students of engineering and could, e.g., be presented to them as auxiliary material. The paper is organized as follows. We introduce a novel notation for the Routh array in the next section, thus making the proofs more transparent. We then relate the polynomial form of the array to the Euclidean algorithm for finding the greatest common divisor of the even and odd part of the polynomial. From this array, we define the sequence of polynomials, obtained as the sum of every two consecutive rows in the array. This sequence lies at the heart of our proofs as well as most of the more recently published proofs of Routh's theorem. We present the proof for the regular case when all the coefficients in the first column of the Routh array are nonzero in Section 3. We use an analogous approach to the singular case when some of the leading coefficients in a row of the Routh array are zero in Section 4, obtaining an explicit statement on the number of roots in the RHP. Finally, the case when all the coefficients in a row of the Routh array become zero, resulting from the even and odd part of the polynomial having a common divisor, is treated in Section 5. This is the only case when the polynomial may have roots on the imaginary axis and we explicitly find their number and the number of such multiple roots. We conclude with observations on stability relevant to control systems in Section 6.

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2. COMPUTATION OF THE ROUTH ARRAY

Consider the real polynomial

$$a(s) = a_n(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \quad (1)$$

where $a(s)$ has no roots at $s = 0$, i.e., $a_0 \neq 0$. If we start with a polynomial $\tilde{a}(s) = s^j \left(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0\right) = s^j a(s)$, we know this polynomial has a root at $s = 0$ with multiplicity j and can thus restrict our attention to the polynomial $a(s)$.

We now introduce the notation

$$\begin{aligned} a_i(s) \;=\; & a_{i,i}s^i + a_{i-1,i-1}s^{i-1} \\ & + a_{i,i-2}s^{i-2} + a_{i-1,i-3}s^{i-3} + \rightarrow, \\ & i = n, n-1, \ldots, 0 \end{aligned} \quad (2)$$

where the former index of the coefficients denotes the row in which the coefficient will appear in the Routh array and the second index denotes the corresponding power in s. The classical Routh array, see any classical text on control systems, e.g., Dorf et al. (2017) and Ogata (2009), is then given by this coefficient array, valid for odd and even n by the use of the right hand arrows:

$$\begin{array}{llll} s^n & a_{n,n} & a_{n,n-2} & \rightarrow \\ s^{n-1} & a_{n-1,n-1} & a_{n-1,n-3} & \rightarrow \\ s^{n-2} & a_{n-2,n-2} & a_{n-2,n-4} & \rightarrow \\ \vdots & & & \\ s^{i+1} & a_{i+1,i+1} & a_{i+1,i-1} & \rightarrow \\ s^{i} & a_{i,i} & a_{i,i-2} & \rightarrow \\ s^{i-1} & a_{i-1,i-1} & a_{i-1,i-3} & \rightarrow \\ \vdots & & & \\ s^2 & a_{2,2} & a_{2,0} & \\ s^1 & a_{1,1} & & \\ s^0 & a_{0,0} & & \end{array} \quad (3)$$

This novel notational representation of the Routh array, efficiently indicates the exact location of each coeffcient in the Routh array and makes the subsequent proofs more transparent. Here, the well known formula shown in our notation

$$a_{i-1,\cdot} = -\frac{1}{a_{i,i}} \begin{vmatrix} a_{i+1,i+1} & a_{i+1,\cdot} \\ a_{i,i} & a_{i,\cdot-1} \end{vmatrix} \quad (4)$$

is used to compute rows $n-2, n-3, \ldots, 1, 0$, starting with rows n and $n-1$ for s^n and s^{n-1}.

Now returning to polynomial (2), we may split it into a higher order part and a lower order part and write it in the form

$$\begin{aligned} a_i(s) &= a_{i,h}(s) + a_{i,l}(s), \quad i = n, n-1, \ldots, 1 \\ a_0(s) &= a_{0,h}(s) \end{aligned} \quad (5)$$

where

$$\begin{aligned} a_{i,h}(s) &= a_{i,i}s^i + a_{i,i-2}s^{i-2} + \rightarrow \\ &= \sum_{k=0}^{\lfloor i/2 \rfloor} a_{i,i-2k}s^{i-2k} \end{aligned} \quad (6)$$

denotes the higher order polynomial even or odd and

$$\begin{aligned} a_{i,l}(s) &= a_{i-1,i-1}s^{i-1} + a_{i-1,i-3}s^{i-3} + \rightarrow \\ &= a_{i-1,h}(s) \end{aligned} \quad (7)$$

denotes the lower order polynomial odd or even. These two polynomials correspond to two consecutive rows in the Routh array (3), the coefficients of $a_{i,h}(s)$ being those of the row for s^i, i.e., the Routh array may be written as the following polynomial table:

$$\begin{aligned}
&a_{n,h}(s) = a_{n,n}s^n + a_{n,n-2}s^{n-2} + \rightarrow \\
&a_{n-1,h}(s) = a_{n,l}(s) = a_{n-1,n-1}s^{n-1} + a_{n-1,n-3}s^{n-3} + \rightarrow \\
&a_{n-2,h}(s) = a_{n-1,l}(s) = a_{n-2,n-2}s^{n-2} + a_{n-2,n-4}s^{n-4} + \rightarrow \\
&\vdots \\
&a_{i+1,h}(s) = a_{i+2,l}(s) = a_{i+1,i+1}s^{i+1} + a_{i+1,i-1}s^{i-1} + \rightarrow \\
&a_{i,h}(s) = a_{i+1,l}(s) = a_{i,i}s^i + a_{i,i-2}s^{i-2} + \rightarrow \\
&a_{i-1,h}(s) = a_{i,l}(s) = a_{i-1,i-1}s^{i-1} + a_{i-1,i-3}s^{i-3} + \rightarrow \\
&\vdots \\
&a_{2,h}(s) = a_{3,l}(s) = a_{2,2}s^2 + a_{2,0}s^0 \\
&a_{1,h}(s) = a_{2,l}(s) = a_{1,1}s^1 \\
&a_{0,h}(s) = a_{1,l}(s) = a_{0,0}s^0.
\end{aligned} \quad (8)$$

As noted by Routh (1877), this sequence of polynomials corresponds to applying the Euclidean algorithm to $a_{n,h}(s)$ and $a_{n-1,h}(s)$ in order to find their greatest common polynomial divisor, i.e.,

$$\begin{aligned} a_{i-1,h}(s) = a_{i+1,h}(s) - \frac{a_{i+1,i+1}}{a_{i,i}} s a_{i,h}(s), \\ i = n-1, n-2, \ldots, 1. \end{aligned} \quad (9)$$

Here, $a_{i,h}(s)$ is the divisor, $a_{i+1,h}(s)$ the dividend, $\frac{a_{i+1,i+1}}{a_{i,i}}s$ the quotient and $a_{i-1,h}(s)$ is the remainder. If the remainder $a_{i-1,h}(s)$ becomes the zero polynomial, $a_{i,h}(s)$ is the greatest common divisor.

We refer to the case when $a_{i,i} \neq 0$, $i = n-1, n-2, \ldots, 0$ as **the regular case**. We refer to the case when a leading coefficient $a_{i,i}$ becomes zero for some $i < n$, but some other coefficients of the same row remain nonzero, as **the first singular case**. Finally, we refer to the case when all the coefficients $a_{i,\cdot}$ of some row become zero as **the second singular case**. The procedure based on equation (9) has to be modified and treated separately in both the singular cases.

The polynomial sequence (5), introduced in different ways, lies directly or indirectly at the heart of most of the fairly recently published proofs of Routh's theorem. In our proof below, we also focus on this sequence viewed as the sum of every two consecutive rows in (8) by relating it to (9). It follows from (5) and (7) by adding $a_{i,h}(s)$ to both sides of (9), that

$$\begin{aligned} a_i(s) = a_{i+1}(s) - \frac{a_{i+1,i+1}}{a_{i,i}} s a_{i,h}(s), \\ i = n-1, n-2, \ldots, 0, \end{aligned} \quad (10)$$

where the divisor and the quotient remain the same as in (9). This is an approach, very similar to that of Ferrante et al. (1999).

The main part of these proofs is then to establish a relation between the root distribution of $a_i(s)$ to that of $a_{i+1}(s)$, $\;i = n-1, n-2, \ldots, 0$. Here, a wide variety of arguments is applied, such as making use of the root locus technique on additive polynomial decompositions in Lepschy et al. (1988), the boundary crossing theorem in Chapellat et. al (1990), geometric considerations of phase changes in Anagnost et al. (1991), the Hermite-Biehler theorem in Ho et al. (1998), continuity arguments in Meinsma (1995) and Ferrante et al. (1999), and Cauchy's theorem in Matsumoto (2001).

Our approach, guided by the aim of presenting a proof that may be most meaningful to undergraduate students of engineering, is to apply Cauchy's theorem to the polynomial ratio $\frac{a_{i+1}(s)}{a_i(s)}$ in an analogous way to that of applying the Nyquist criterion to investigate the stability of a control system. A proof based on the same idea as the Nyquist criterion both serves the purpose of throwing some light on the Routh array, as well as providing an interesting application of this type of argument. Furthermore, as we show, such a proof can be extended in a natural way to the singular cases, resulting in an explicit statement of the number of roots in the RHP for the first singular case. This is without resorting to the ϵ-argument initially introduced by Routh (1877) as is most commonly done in textbooks, see e.g., Dorf et al. (2017) and Ogata (2009), as well as recently published proofs of Routh's theorem. A notable exception is Ferrante et al. (1999), but the direct results on the first singular case are not as explicit as those presented below. The same applies to a direct procedure deduced by Pal et al. (1992) from a more general approach based on Bezoutians, as well as direct procedures proposed by Yeung (1983) and Benidir et al. (1990). We finally note, that while the proof in Matsumoto (2001) is also based on Cauchy's theorem, it does that in a different way, namely without a reference to the Euclidean algorithm and the singular cases are not considered.

3. ROUTH'S THEOREM – REGULAR CASE

Definition: **Let $v(a_0, a_1, \ldots, a_n)$ denote the number of sign changes in the sequence within the brackets.**

Routh's theorem: **The total number of roots of polynomial (1) in the RHP, is equal to the number of sign changes in the coefficients of the first column of the Routh array (3), i.e.,**

$$v(a_{0,0}, a_{1,1}, \ldots, a_{n,n}) \tag{11}$$

assuming that none of these coefficients are zero. Further, there are no roots on the imaginary axis.

Proof: It follows from (10) that

$$\frac{a_{i+1}(s)}{a_i(s)} = 1 + F_i(s), \tag{12}$$

where

$$F_i(s) = \frac{\frac{a_{i+1,i+1}}{a_{i,i}} s a_{i,h}(s)}{a_i(s)} = \frac{\frac{a_{i+1,i+1}}{a_{i,i}} s}{1 + \frac{a_{i,l}(s)}{a_{i,h}(s)}}. \tag{13}$$

Applying Cauchy's theorem to (12) we have

$$R_{i+1} - R_i = N_{i+1}, \tag{14}$$

where R_{i+1} is the number of roots in the RHP of $a_{i+1}(s)$, R_i is the number of roots in the RHP of $a_i(s)$, and N_{i+1} is the number of encirclements of the contour of $\frac{a_{i+1}(s)}{a_i(s)}$ around the origin as we trace the Nyquist contour clockwise. Equivalently, we can determine the number of encirclements of the contour of $F_i(s)$ around -1.

Divide the Nyquist contour into three parts:

- A: $0 \leq j\omega < jR$, i.e., the positive frequencies, starting at the origin and proceeding along the imaginary axis to jR,
- B: $|s| = R$, then clockwise along the semi circle in the RHP,
- C: $-jR < j\omega < 0$, and finally back along the imaginary axis from $-jR$ to the origin.

We then consider the parts A, B and C separately as $R \to \infty$, but for Cauchy's theorem to hold, we must first establish that $a_i(s)$, $0 \leq i \leq n$, has no roots on A or C. Here, we firstly note that since we have assumed that $a_n(0) \neq 0$, it follows from (10) that $a_i(0) \neq 0$, $i = n-1, n-2, \ldots, 0$. Secondly, assume that $a_i(j\omega) = 0$ for some $0 \leq i \leq n$, $\omega \neq 0$. Then, since the even part of $a_i(j\omega)$ in (5) will be real and the odd part of $a_i(j\omega)$ will be purely imaginary, it follows that $a_{i,h}(j\omega) = 0$ and $a_{i,l}(j\omega) = 0$, i.e., $\left(s^2 + \omega^2\right)$ is a common factor of two consecutive polynomial rows in (8). Hence by the Euclidean algorithm, a zero polynomial row must appear further down in the array. But this is the second singular case rather than the regular case, that is under consideration here, so that $j\omega$ cannot be a root of $a_i(s)$.

On A and C, $s = j\omega$ and since one of the polynomials $a_{i,l}(s)$ and $a_{i,h}(s)$ is even and the other is odd

$$\left.\frac{a_{i,l}(s)}{a_{i,h}(s)}\right|_{s=j\omega} = \alpha(j\omega) = j\hat{\alpha}(\omega) \tag{15}$$

for some real valued function $\hat{\alpha}(\omega)$. Thus

$$F_i(j\omega) = \frac{\frac{a_{i+1,i+1}}{a_{i,i}} j\omega}{1 + j\hat{\alpha}(\omega)} = \frac{(\hat{\alpha}(\omega) + j)\frac{a_{i+1,i+1}}{a_{i,i}}\omega}{1 + \hat{\alpha}^2(\omega)}. \tag{16}$$

Note here, as s traces A, $F_i(s)$ remains in the upper half of the complex plane if $\frac{a_{i+1,i+1}}{a_{i,i}} > 0$ and for $\frac{a_{i+1,i+1}}{a_{i,i}} < 0$, it remains in the lower half of the complex plane, except it may touch the origin if $a_{i,h}(j\omega) = 0$ for some $\omega \neq 0$. As s traces C, we have the mirror image of s tracing A.

On B we have

$$\lim_{s\to\infty} F_i(s) = \lim_{s\to\infty} \frac{a_{i+1,i+1}}{a_{i,i}} \frac{s a_{i,h}(s)}{a_i(s)} \approx \frac{a_{i+1,i+1}}{a_{i,i}} s \tag{17}$$

Therefore, if $\frac{a_{i+1,i+1}}{a_{i,i}} > 0$ corresponding to no sign change between $a_{i,i}$ and $a_{i+1,i+1}$, the contour of $F_i(s)$ remains in the upper half plane as s traces A, it effectively follows B as s traces B and it remains in the lower half plane as s traces C. Thus it cannot encircle -1 and hence $R_{i+1} = R_i$.

On the other hand, if $\frac{a_{i+1,i+1}}{a_{i,i}} < 0$ corresponding to a sign change between $a_{i,i}$ and $a_{i+1,i+1}$, $F_i(s)$ remains in the lower half plane as s traces A, it effectively follows $-B$ as s traces B and it remains in the upper half plane as s traces C. Thus, it crosses the real half-line from -1 to $-\infty$ exactly once and hence encircles -1 exactly once, thus $R_{i+1} = R_i + 1$.

Finally, since $R_0 = 0$, it follows that R_n, i.e., the number of roots of the polynomial (1) in the RHP, is the number of sign changes in the first column of the Routh array (3). *q.e.d.*

Example 1: Consider the polynomial

$$\begin{aligned} a(s) &= s^4 - 2s^3 - 13s^2 + 14s + 24 \\ &= (s+1)(s-2)(s+3)(s-4). \end{aligned} \tag{18}$$

The Routh array becomes

$$\begin{array}{llll} s^4 & 1 & -13 & 24 \\ s^3 & -2 & 14 & \\ s^2 & -6 & 24 & \\ s^1 & 6 & & \\ s^0 & 24 & & \end{array} \tag{19}$$

with two sign changes in the first column. Thus $R_4 = 2$, by the statement of Routh's theorem. Further,

$$\left.\begin{aligned} F_3(s) &= \frac{1}{-2}\frac{s\left(-2s^3+14s\right)}{a_3(s)} \\ a_3(s) &= -2s^3 - 6s^2 + 14s + 24 \\ &= -2(s+4)(s-2.3028)(s+1.3028), \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} F_2(s) &= \frac{-2}{-6}\frac{s\left(-6s^2+24\right)}{a_2(s)} \\ a_2(s) &= -6s^2 + 6s + 24 \\ &= -6(s-2.5616)(s+1.5616), \end{aligned}\right\} \tag{21}$$

$$F_1(s) = \frac{-6}{6}\frac{s\,(6s)}{a_1(s)}, \quad a_1(s) = 6s + 24 = 6(s+4), \tag{22}$$

$$F_0(s) = \frac{6}{24}\frac{s\,(24)}{a_0(s)}, \quad a_0(s) = 24. \tag{23}$$

We show the contours of $F_i(s)$, $i = 3, 2, 1, 0$ in Fig. 1 as s traces the Nyquist contour clockwise. We set $R = \omega_{max} = 5$ in the Nyquist contour so as to be able to show the full contours of $F_i(s)$. This R-value is sufficiently large for the Nyquist contour to enclose all the roots of $a(s)$ in the RHP, for this example. The contours of $F_1(s)$ and $F_3(s)$ first enter the lower half plane and encircle -1 once, thus $N_2 = N_4 = 1$. The contours $F_0(s)$ and $F_2(s)$ enter first the upper half plane and do not encircle -1, thus $N_1 = N_3 = 0$. In summary,

$$\begin{aligned} R_3 = R_4 - N_4 = 1;\ R_2 = R_3 - N_3 = 1 \\ R_1 = R_2 - N_2 = 0;\ R_0 = R_1 - N_1 = 0. \end{aligned} \tag{24}$$

4. THE FIRST SINGULAR CASE

Now consider the first singular case, when some of the leading coefficients $a_{j,j}$, in row j of the Routh array are zero, but not all the coefficients in that row are zero. Thus, assume that the first nonzero element of row j is $a_{j,j-2k}$ for some $k \geq 1$, so that the corresponding polynomial is of degree $j - 2k$, rather than j. We are in fact applying the Euclidean algorithm as before, the only difference being that when we divide $a_{j+2,h}(s)$ by $a_{j+1,h}(s)$, the remainder polynomial is of degree $j - 2k$ rather than j, and hence we denote it by $a_{j-2k,h}(s)$. Then we have

$$a_{j-2k,h}(s) = a_{j+1,l}(s) \tag{25}$$

and instead of (9) we have for $i = j$ that

$$a_{j-2k-1,h}(s) = a_{j+1,h}(s) - \beta(s) s a_{j-2k,h}(s) \tag{26}$$

where $\beta(s)$ is an even polynomial of order $2k$.

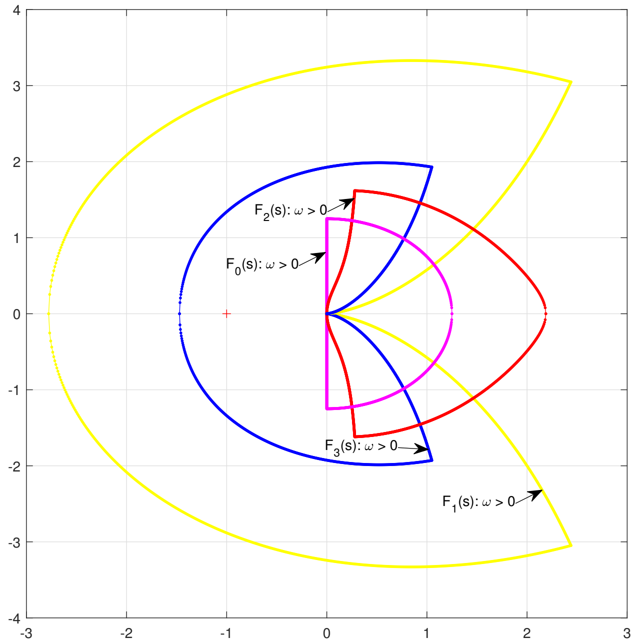

Fig. 1. Example 1. Nyquist plots of $F_i(s)$, $i = 3, 2, 1, 0$, $R = 5$.

Thus, the rows in the Routh array corresponding to s^i, $i = j, j-1, \ldots, j-2k+1$ are simply missing and we proceed from the coefficients of $a_{j-2k,h}(s)$ and $a_{j-2k-1,h}(s)$ as in the regular case. Denoting the intermediate remainder polynomials of the division with $\hat{a}_j(s)$ where

$$\begin{aligned} \hat{a}_{j+1}(s) &= a_{j+1,h}(s) \\ \hat{a}_{i-1}(s) &= \hat{a}_{i+1}(s) - \frac{\hat{a}_{i+1,i+1}}{a_{j-2k,j-2k}} s a_{j-2k,h}(s), \\ & \quad i = j, j-2, \ldots, j-2k \\ a_{j-2k-1,h}(s) &= \hat{a}_{j-2k-1}(s), \end{aligned} \tag{27}$$

we have

$$\beta(s) = \frac{1}{a_{j-2k,j-2k}}\left(\sum_{i=0}^{k-1} \hat{a}_{j-2k+1+2i,j-2k+1+2i} s^{2i} + a_{j+1,j+1} s^{2k}\right). \tag{28}$$

We have a corresponding jump from $a_{j+1}(s)$ to $a_{j-2k}(s)$ and it follows from (5) and by adding $a_{j-2k,h}(s)$ to both sides of (26) that

$$a_{j-2k}(s) = a_{j+1}(s) - \beta(s) s a_{j-2k,h}(s), \tag{29}$$

here $a_{j-2k,h}(s)$ is the divisor, $a_{j+1}(s)$ is the dividend, $\beta(s)s$ is the quotient and $a_{j-2k}(s)$ is the remainder. Equation (10) still holds for $i = n, n-1, \ldots, j+1$ and $i = j - 2k - 1, j - 2k - 2, \ldots, 0$.

The **division (26)** can be carried out in a "Routh-like way" in $k + 1$ steps as follows:

- The coefficients in row s^j are shifted to the left so that the first non-zero coefficient $a_{j,j-2k}$ becomes the leading coefficient in the first column and we have k new trailing zeros. We then compute the coefficients in row s^{j-1} in a "Routh-like way" from rows s^j and s^{j+1}. These will be the coefficients of $\hat{a}_{j-1}(s)$.

- For $i = j-2, j-4, \ldots, j-2k$, we copy the coefficients of the shifted row s^j into row s^i, dropping one trailing zero at each step, and then compute the coefficients in row s^{i-1}, in a "Routh-like way" from rows s^i and s^{i+1}. These will be the coefficients of $\hat{a}_{i-1}(s)$.

After reaching in this way row s^{j-2k-1}, we compute rows s^{j-2k-i}, $i = 2, 3, \ldots, j-2k$ as in the regular case, where for the sake of simplicity of presentation, we assume that leading zero coefficients only occur in a single row in the array.

Thus, the Routh array (3) becomes

$$\begin{array}{rlll}
s^n & a_{n,n} & a_{n,n-2} & \rightarrow \\
s^{n-1} & a_{n-1,n-1} & a_{n-1,n-3} & \rightarrow \\
 & \vdots & & \\
s^{j+2} & a_{j+2,j+2} & a_{j+2,j} & \rightarrow \\
s^{j+1} & a_{j+1,j+1} & a_{j+1,j-1} & \rightarrow \\
s^{j} & [a_{j,j-2k} & a_{j,j-2k-2} & \rightarrow] \\
s^{j-1} & [\hat{a}_{j-1,j-1} & \hat{a}_{j-1,j-3} & \rightarrow] \\
s^{j-2} & [a_{j,j-2k} & a_{j,j-2k-2} & \rightarrow] \\
s^{j-3} & [\hat{a}_{j-3,j-3} & \hat{a}_{j-3,j-5} & \rightarrow] \\
 & \vdots & & \\
s^{j-2k+2} & [a_{j,j-2k} & a_{j,j-2k-2} & \rightarrow] \\
s^{j-2k+1} & [\hat{a}_{j-2k+1,j-2k+1} & \hat{a}_{j-2k+1,j-2k-1} & \rightarrow] \\
s^{j-2k} & a_{j-2k,j-2k} & a_{j-2k,j-2k-2} & \rightarrow \\
s^{j-2k-1} & a_{j-2k-1,j-2k-1} & a_{j-2k-1,j-2k-3} & \rightarrow \\
s^{j-2k-2} & a_{j-2k-2,j-2k-2} & a_{j-2k-2,j-2k-4} & \rightarrow \\
 & \vdots & & \\
s^1 & a_{1,1} & & \\
s^0 & a_{0,0} & &
\end{array} \tag{30}$$

where the bracketed rows are the missing rows that do not enter into the statement of the First Singular Case Theorem, but have been filled in according to the division procedure described above.

First Singular Case Theorem: **The total number of roots of polynomial (1) in the RHP, is equal to**

$$k + v\left\{(-1)^k\ (a_{0,0}, a_{1,1}, \cdots, a_{j-2k-1,j-2k-1}, a_{j-2k,j-2k},)\ a_{j+1,j+1}, a_{j+2,j+2}, \cdots, a_{n,n}\right\} \tag{31}$$

assuming that $a_{j-2k,j-2k}$ **is the first nonzero coefficient in row** j **of the Routh array, but the leading coefficient in all other rows are nonzero. Further, there are no roots on the imaginary axis.**

Proof: Dividing through (29) by $a_{j-2k}(s)$ and rearranging, we have

$$\frac{a_{j+1}(s)}{a_{j-2k}(s)} = 1 + F(s) \tag{32}$$

where

$$F(s) = \frac{\beta(s)s a_{j-2k,h}(s)}{a_{j-2k}(s)} = \frac{\beta(s)s}{1+\alpha(s)}, \tag{33}$$

$$\alpha(s) = \frac{a_{j-2k,l}(s)}{a_{j-2k,h}(s)}. \tag{34}$$

Applying Cauchy's theorem, we now have instead of (14)

$$R_{j+1} - R_{j-2k} = N_{j+1}, \tag{35}$$

where N_{j+1} is the number of encirclements of the contour of F(s) around -1 as we trace the Nyquist contour in the same way as in the proof of the regular case. We first note by the same argument as in the proof of the regular case, it follows from (29) that neither $a_{j+1}(s)$ nor $a_{j-2k}(s)$ have roots on A or C.

On A and C, $s = j\omega$ and since one of the polynomials $a_{j-2k,l}(s)$ and $a_{j-2k,h}(s)$ is odd and the other is even

$$\left.\frac{a_{j-2k,l}(s)}{a_{j-2k,h}(s)}\right|_{s=j\omega} = \alpha(j\omega) = j\hat{\alpha}(\omega) \tag{36}$$

for some real valued function $\hat{\alpha}(\omega)$. Further since $\beta(s)$ is an even polynomial

$$\beta(j\omega) = \hat{\beta}(\omega^2) \tag{37}$$

for some real valued polynomial $\hat{\beta}(\omega^2)$. Thus,

$$F(j\omega) = \frac{\hat{\beta}(\omega^2)j\omega}{1+j\hat{\alpha}(\omega)} = \frac{(\hat{\alpha}(\omega)+j)\,\hat{\beta}(\omega^2)\omega}{1+\hat{\alpha}^2(\omega)}. \tag{38}$$

As s traces A, $F(s)$ must end up in the upper half plane if the leading coefficient of $\hat{\beta}(\omega^2)$, $\frac{(-1)^k a_{j+1,j+1}}{a_{j-2k,j-2k}} > 0$ and in the lower half plane if it is < 0, as

$$\lim_{\omega\to\infty} F(j\omega) = \lim_{\omega\to\infty} j\hat{\beta}(\omega^2)\omega. \tag{39}$$

It may, however, first enter into the opposite half plane up to $\frac{\lceil k \rceil}{2}$ times for values of ω^2 such that $\hat{\beta}(\omega^2) = 0$, since $\hat{\beta}(\omega^2)$ may have up to k distinct positive roots allowing up to k transitions from one half plane to the other. But for such ω, $F(j\omega) = 0 + j*0$, i.e., the contour $F(s)$ will always pass through the origin. Thus, in particular, the contour $F(s)$ cannot encircle -1 as s traces A. But note that the contour $F(s)$ will also touch the origin for ω values such that $a_{j-2k,h}(j\omega) = 0$.

As s traces B, we have

$$\lim_{s\to\infty} F(s) = \frac{a_{j+1,j+1}}{a_{j-2k,j-2k}} s^{2k+1}. \tag{40}$$

Thus, the contour will go clockwise $k + \frac{1}{2}$ circles around -1. If $\frac{(-1)^k a_{j+1,j+1}}{a_{j-2k,j-2k}} > 0$, the contour of $F(s)$ starting in the upper half plane thus ends up in the lower half plane and as s traces C, the contour reaches the origin through the lower half plane having encircled the -1 point $N_{j+1} = k$ times.

Before s reaches the end of C, it may, similarly to the start of A, enter into the opposite half plane up to $\frac{\lceil k \rceil}{2}$ times, but since the contour will always pass between the half planes through the origin it cannot encircle -1 any more times.
If $\frac{(-1)^k a_{j+1,j+1}}{a_{j-2k,j-2k}} < 0$, the contour of $F(s)$ ends up in the upper half plane and as s traces C, reaches the origin through this plane, having now encircled the -1 point $N_{j+1} = k+1$ times. As in the case above, it cannot encircle -1 any more times even if it enters the lower half plane up to $\frac{\lceil k \rceil}{2}$ times before s reaches the end of C.

Thus, we conclude that if there is no sign change between $a_{j-2k,j-2k}$ and $(-1)^k a_{j+1,j+1}$, $a_{j+1}(s)$ has k more roots in the RHP than $a_{j-2k}(s)$, and if there is a sign change, $a_{j+1}(s)$ has $k+1$ more roots in the RHP than $a_{j-2k}(s)$. The result now follows by noting that by the result of the regular case, the difference between the number of roots of $a_{j-2k}(s)$ and $a_0(s)$ in the RHP is the number of sign changes in the sequence $a_{0,0}, a_{1,1}, \ldots, a_{j-2k,j-2k}$ and the difference between the number of roots of $a_n(s)$ and $a_{j+1}(s)$ in the RHP is the number of sign changes in the sequence $a_{j+1,j+1}, a_{j+2,j+2}, \ldots, a_{n,n}$.

q.e.d.

Example 2: Consider the polynomial

$$\begin{aligned} a(s) &= s^{10} + 2s^8 + 4s^6 + 6s^4 + 2s^3 + 4s^2 + s + 1 \\ &= (s + 0.8669 \pm j0.9710)(s - 0.9924 \pm j0.8640) \\ &\times (s - 0.1988 \pm j1.1713)(s + 0.3647 \pm j0.5782) \\ &\times (s - 0.0404 \pm j0.7178) \end{aligned} \tag{41}$$

The Routh array becomes, see (30)

$$\begin{array}{ll} s^{10} & 1\ 2\ \ 4\ \ 6\ \ 4\ 1 \\ s^9 & [2\ \ 1\ \ 0\ \ 0\ \ 0] \\ s^8 & [3/2\ \ 4\ \ 6\ \ 4\ \ 1] \\ s^7 & [2\ \ 1\ \ 0\ \ 0] \\ s^6 & [13/4\ \ 6\ \ 4\ \ 1] \\ s^5 & [2\ \ 1\ \ 0] \\ s^4 & [35/8\ \ 4\ \ 1] \\ s^3 & 2\ 1 \\ s^2 & 29/16\ 1 \\ s^1 & -3/29 \\ s^0 & 1 \end{array} \tag{42}$$

where $k = 3$ and the number of roots in the right half plane is

$$\begin{aligned} &k + v\left\{(-1)^k (a_{0,0}, a_{1,1}, a_{2,2}, a_{3,3}), a_{10,10}\right\} \\ &\quad = 3 + v\{-1, 3/29, -29/16, -2, 1\} = 3 + 3 = 6 \end{aligned} \tag{43}$$

by the statement of First Singular Case Theorem. Further,

$$\alpha(s) = \frac{29/16s^2 + 1}{2s^3 + s}, \tag{44}$$

$$\beta(s) = \frac{1}{2}\left(35/8 + 13/4s^2 + 3/2s^4 + s^6\right) \tag{45}$$

and

$$F(s) = \frac{\beta(s)s}{1 + \alpha(s)}. \tag{46}$$

We show the contour of $F(s)$ in Figs. 2 and 3 as s traces the Nyquist contour clockwise. We set $R = \omega_{max} = 2$ in the Nyquist contour, so as to be able to show the full contour of $F(s)$ in Fig. 3. This R-value is sufficiently large for the Nyquist contour to enclose all the roots of $a(s)$ in the RHP for this example. The details of the contour around the origin are shown in Fig. 2. We have

$$\hat{\beta}\left(\omega^2\right) = \frac{1}{2}\left(35/8 - 13/4\omega^2 + 3/2\omega^4 - \omega^6\right) \tag{47}$$

and since the leading coefficient is $-1/2 < 0$, the contour ends up in the lower half plane as s traces A, according to the proof of the theorem. It thus encircles the -1 point $k + 1 = 4$ times as s traces A, B and C, as seen in Fig. 3. However, as seen in Fig. 2 the contour first enters once into the upper half plane since $\hat{\beta}\left(\omega^2\right)$ has one positive root at $\omega^2 = 1.4042$, crossing into the lower half plane at $s = j\sqrt{1.4042} = j1.1850$. While in the upper half plane it touches the origin at $s = j1/\sqrt{2} = j0.7071$ since $a_{j-2k,h}(s) = 2s^3 + s = 0$ for this s-value. As the contour returns to the origin from the upper half plane when s traces C, we get a mirror image of the contour of $F(s)$.

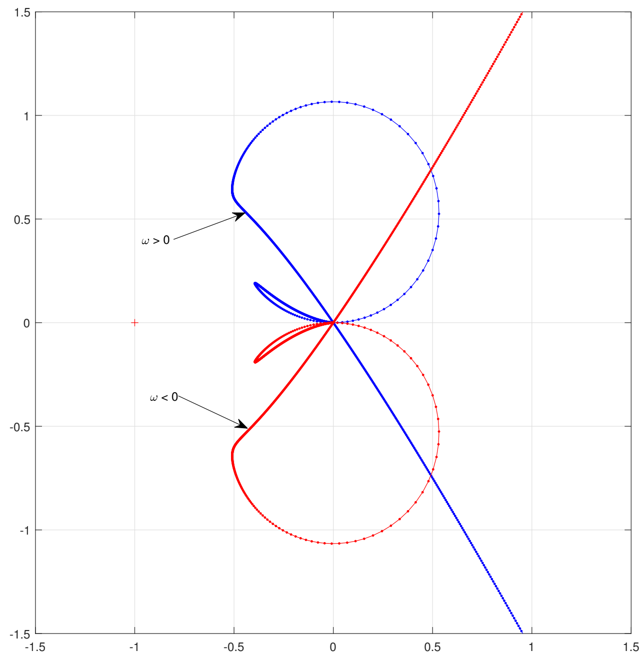

Fig. 2. Example 2. A Nyquist plot of $F(s)$ around the origin, $R = 2$. Note the scales on both axis when compared to Fig. 3.

In summary, $F(s)$ results in $k + 1 = 4$ encirclements of -1, as $\frac{a_{j+1,j+1}}{(-1)^3 a_{j-2k,j-2k}} = -\frac{1}{2}$. In addition, we have 2 signs changes below the jump in the Routh array. This results in $4+2 = 6$ roots in the RHP, in agreement with the result obtained by the First Singular Case Theorem.

5. THE SECOND SINGULAR CASE

Finally, consider the second singular case when the coefficients in some row of the Routh array all become zero. This is the only case where there may be roots on the imaginary axis. Further, it is of interest to determine how many of the roots on the imaginary axis, if any, are multiple roots. As noted by Clark (1992), this is necessary in the case when there are no roots in the RHP, in order to determine accurately, whether the corresponding system is exponentially stable, marginally stable or polynomially unstable. Hence, by the Euclidean algorithm, we have for some $j > 0$, $a_{j,h}(s) \equiv a_{j+1,l}(s) \equiv 0$ and thus $a_{j+1}(s) \equiv a_{j+1,h}(s)$ must be a factor in $a_i(s)$ for $i = j+1, j+2, \ldots, n$. Further, since we assume that $a_n(s)$ has no zero roots, $a_{j+1}(s)$ must be an even polynomial with $j + 1 = 2k$,
i.e., $a_{j+1}(s) = a_{j+1,j+1}s^{2k} + a_{j+1,j-1}s^{2k-2} + \cdots + a_{j+1,0}$.

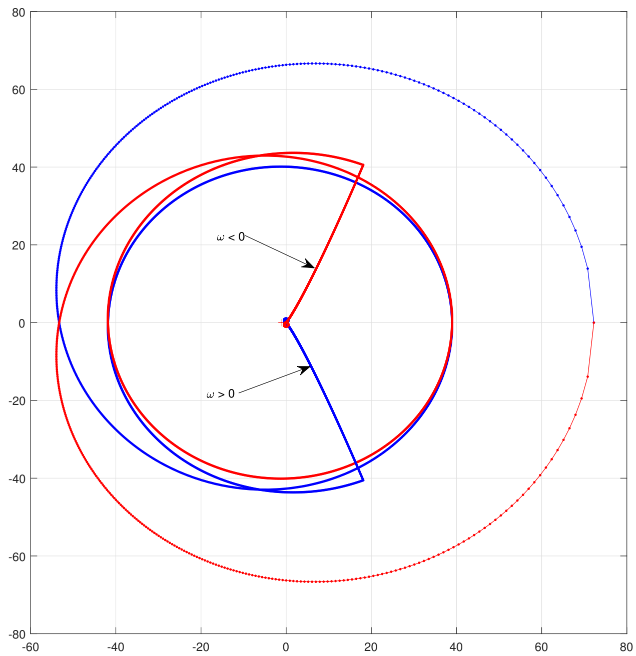

Fig. 3. Example 2. A Nyquist plot of $F(s)$, R=2.

Furthermore, for the sake of simplicity, we assume $a_{i,i} \neq 0, \quad i = j+1, j+2, \ldots, n$.

Following Routh, see also Gantmacher (1959), we now construct the **modified Routh array**, replacing the row of zeros by the derivative of the even polynomial, $a_{j+1}(s)$, above, and then apply the Euclidean algorithm as before. If a new row of zeros occurs the process is repeated. This procedure along with the result (48) below is often referred to as the Q-method.

Second Singular Case Theorem: **The total number of roots of polynomial (1) in the RHP is equal to**

$$v\left(a_{0,0}, a_{1,1}, \cdots, a_{j,j}, a_{j+1,j+1}, \cdots, a_{n,n}\right) \tag{48}$$

where $a_{0,0}, a_{1,1}, \cdots, a_{j+1,j+1}$ **are now the coefficients of the modified Routh array. The total number of roots on the imaginary axis is equal to**

$$j + 1 - 2v\left(a_{0,0}, a_{1,1}, \cdots, a_{j+1,j+1}\right) \tag{49}$$

where $j+1$ **is the number of the row immediately above the first zero row and roots with multiplicity** p **are counted** p **times.**

Proof: By the same argument as in the regular case, the difference between the number of roots in the RHP for $a_n(s)$ and $a_{j+1}(s)$ is the number of sign changes in the sequence $a_{j+1,j+1}, a_{j+2,j+2}, \cdots, a_{n,n}$. Here we note that while $a_n(s)$ may have a root $j\omega$, it follows by the same argument as in the proof of the regular case that $j\omega$ will then also be a root of $a_i(s)$, $i = n-1, n-2, \ldots, j+1$. Therefore, we can still apply Cauchy's theorem to polynomial ratios $\frac{a_{i+1}(s)}{a_i(s)}$, $i = n-1, n-2, \ldots, j+1$ with the Nyquist contour, since the factors of $j\omega$ will cancel out. We next note that the modified Routh array can be justified by the following argument. If $a(s)$ is an even polynomial and $a_\epsilon(s) = a(s+\epsilon) \approx a(s) + \epsilon da(s)/ds$ as $\epsilon \to 0$, then $a(s)$ and $a_\epsilon(s)$ must have the same number of roots in the RHP for a sufficiently small positive value of ϵ. This follows from the observation that if $a(s)$ has a root on the imaginary axis then for a positive perturbation ϵ, such a root will move into the LHP. Thus, applying the regular Routh algorithm to $a(s) + \epsilon da(s)/ds$, the first row in the Routh array has the coefficients of $a(s)$ and the second row the coefficients of $\epsilon da(s)/ds$, or equivalently of $da(s)/ds$, since positive scaling of a row does not affect sign changes in the first column. Therefore the number of roots of $a_{j+1}(s)$ in the RHP will be

$$v\left(a_{0,0}, a_{1,1}, \cdots, a_{j+1,j+1}\right) \tag{50}$$

where $a_{0,0}, a_{1,1}, \cdots, a_{j+1,j+1}$ are the coefficients of the modified array by the same argument as in the regular case. This concludes the proof of the first result. The second result follows directly from the fact that roots of even polynomials are symmetrical with respect to the imaginary axis.
q.e.d.

In order to determine the number of multiple roots on the imaginary axis, we observe that the polynomial $a_{j+1}(s)$ has a multiple root if and only if $a_{j+1}(s)$ and $da_{j+1}(s)/ds$ have a common factor, i.e., if and only if a second row of zeros occurs in the modified Routh array. Denote the polynomial above such a row, i.e., the polynomial whose roots are the multiple roots of $a_{j+1}(s)$, by $a_{j'+1}(s) = a_{2k'}(s)$, $j' < j$.
Second Singular Case Corollary: **The total number of multiple roots on the imaginary axis is equal to**

$$j' + 1 - 2v(a_{0,0}, a_{1,1}, \ldots, a_{j'+1,j'+1}) \tag{51}$$

where $j'+1$ **is the number of the row immediately above the second zero row in the modified Routh array and roots of multiplicity** p **are counted** $p-1$ **times.**
Proof: Follows by the same argument as in the proof of the Second Singular Case Theorem.

Example 3: Consider the polynomial

$$\begin{aligned} &s^{12} - 3s^{11} - 7s^{10} + 37s^9 - 105s^8 + 251s^7 - 281s^6 \\ &+171s^5 - 88s^4 - 248s^3 + 288s^2 - 208s + 192 \\ &= (s \pm 1)(s \pm j)^2(s \pm j2)(s-2)^2(s-3)(s+4). \end{aligned} \tag{52}$$

The modified Routh array becomes

$$\begin{array}{llllllll} s^{12} & 1 & -7 & -105 & -281 & -88 & 288 & 192 \\ s^{11} & -3 & 37 & 251 & 171 & -248 & -208 & \\ s^{10} & \frac{16}{3} & -\frac{64}{3} & -224 & -\frac{512}{3} & \frac{656}{3} & 192 & \\ s^9 & 25 & 125 & 75 & -125 & -100 & & \\ s^8 & -48 & -240 & -144 & 240 & 192 & & \\ s^7 & (\,-384 & -1440 & -576 & 480\,) & & & \\ s^6 & -60 & -72 & 180 & 192 & & & \\ s^5 & -\frac{4896}{5} & -1728 & -\frac{3744}{5} & & & & \\ s^4 & \frac{576}{17} & \frac{3840}{17} & 192 & & & & \\ s^3 & 4800 & 4800 & & & & & \\ s^2 & 192 & 192 & & & & & \\ s^1 & (\,384\,) & & & & & & \\ s^0 & 192 & & & & & & \end{array} \tag{53}$$

The rows in parenthesis are the rows of zeros that have been replaced by the derivative of the polynomial above. Here we have

$$v(a_{0,0}, \ldots, a_{12,12}) = 4 \tag{54}$$

and hence there are 4 roots in the RHP based on the First Singular Case Theorem. Further, the total number of roots on the imaginary axis is

$$j + 1 - 2v\left(a_{0,0}, a_{1,1}, \cdots, a_{j+1,j+1}\right) = 8 - 2 \times 1 = 6. \tag{55}$$

We finally have from Second Singular Case Corollary, that the number of multiple roots on the imaginary axis is

$$j^{'} + 1 - 2v(a_{0,0}, a_{1,1}, \ldots, a_{j'+1,j'+1}) = 2 - 2 \times 0 = 2. \tag{56}$$

We now consider an example from Pal et al. (1992), involving both singular cases.

Example 4: Consider the polynomial

$$\begin{aligned} &s^8 + s^7 + s^6 + s^5 + s^2 + 1 \\ &= (s + 1.1051 \pm j0.42)(s - 0.7455 \pm j0.4825) \\ &\quad \times (s \pm j)(s + 0.1404 \pm j0.9421). \end{aligned} \tag{57}$$

The modified Routh array now becomes where the square brackets and parentheses have the same meaning as in (30), (42) and (53)

$$\begin{array}{llllll} s^8 & 1 & 1 & 0 & 1 & 1 \\ s^7 & 1 & 1 & 0 & 0 & \\ s^6 & [1 & 1 & 0 & 0] & \\ s^5 & [0 & 0 & 0] & & \\ s^4 & [1 & 1 & 0] & & \\ s^3 & [0 & 0] & & & \\ s^2 & 1 & 1 & & & \\ s^1 & (2) & & & & \\ s^0 & 1 & & & & \end{array} \tag{58}$$

Here $k = 2$ and the number of roots in the right half plane is

$$\begin{aligned} &k + v\left\{(-1)^k \left(a_{0,0}, a_{1,1}, a_{2,2}\right), a_{7,7}, a_{8,8}\right\} \\ &\quad = 2 + v\left\{(-1)^2 \left(1, 2, 1\right), 1, 1\right\} = 2 + 0 = 2 \end{aligned} \tag{59}$$

based on the First Singular Case Theorem. Further, we have from the Second Singular Case Theorem

$$j + 1 - 2v(a_{0,0}, a_{1,1}, a_{2,2}) = 2 - 2v(1, 2, 1) = 2 \tag{60}$$

roots on the imaginary axis.

6. STABILITY

We conclude with observations on stability relevant to control systems (see e.g. McNamee et. al. (2013)), that follow directly from the theorems in this paper. We have exponential stability if and only if we have the regular case with no sign changes in the first first column of the Routh array. We have marginal stability if and only if we have the second singular case with no sign changes in the first column of the modified Routh array and no recurring zero row in the modified Routh array. We have polynomial instability if and only if we have the second singular case with no sign changes in the first column and a recurrent zero row in the modified Routh array. Otherwise we have exponential instability. We note that we started by eliminating any possible zero root. It is straightforward to determine the effect of such a root, whether simple or multiple, on the nature of the stability.

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