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Piecewise Affine Feedback Control for Approximate Solution of the Target Control Problem ⋆

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Abstract: This paper presents an approach for approximate solution of the target control problem for a class of nonlinear systems on a finite time interval. The main idea is to use a comparison principle from dynamic programming theory together with a special class of piecewise affine value functions for the piecewise linearized form of the original system of ODEs. Two cases are considered: with continuous value function and with discontinuous one. The resulting feedback control functions are also piecewise affine, continuous or discontinuous respectively. For both cases the theorems on sufficient conditions for solving the feedback control problem are formulated and proved. The appropriate computational procedures complement theoretical results.

Keywords: feedback control, switched system, trajectory tubes, hybridization, comparison principle, piecewise affine functions

1. INTRODUCTION

This work is devoted to the development of methods of approximate construction of feedback control for a system of nonlinear differential equations on a finite time interval. The main idea is to combine the approach of dynamic programming theory (and in particular comparison principle (Kurzhanski (2006), Kurzhanski and Varaiya (2014))) with the techniques of piecewise affine value and control functions defined on a set of simplices in the state space (Habets et al. (2006), Girard and Martin (2012)).

The problem of control synthesis can be solved using the Hamilton-Jacobi-Bellman equation (HJB) for the auxiliary value function (Kurzhanski and Varaiya (2014), Kurzhanski and Varaiya (2007)), with given boundary conditions. In general one should obtain some generalized solution of this equation (Subbotin (1995), Fleming and Soner (2006), Kruzhkov (1966)). The zero level set of the constructed function coincides with the solvability set of the considered control system. It contains all the starting positions from which it is possible to solve the target control problem. The construction of such set, as well as the corresponding set-valued mapping – solvability tube, is a difficult, urgent mathematical problem. This tube or its internal approximations can be used to find the feedback control through the 'aiming' rule (Krasovski and Krasovski (1995)). In this paper the value function is approximated from above using piecewise affine functions of a special kind. This allows to get internal estimates of the desired solvability tube and also piecewise affine feedback control.

In Girard and Martin (2012), an attempt was made to use piecewise affine control functions to solve the problem of transferring an autonomous system from a given initial set to a target set, on an infinite time interval. However, the proposed algorithm did not guarantee the existence of a solution even if it exists, and did not allow to solve the problem for a given finite time interval. In this paper an alternative approach is developed. It focuses on solving the problem in a finite, given time. In addition, the method obtained here allows to transfer the system's trajectory inside a small neighborhood of the target set, even if it is impossible to bring it directly into this set.

The approach proposed in this paper involves the construction of a piecewise affine approximation of the original nonlinear system of differential equations on a given partition of the state space domain into a tuple of simplices. This approach is usually called "hybridization" (Asarin et al. (2007)). Next, using the comparison principle a piecewise affine value function for the resulting switched system (Tochilin (2015)) can be constructed. The main difficulty here is to choose an adequate scheme for calculation of the values of this function at the vertices of simplices. Simultaneously an *admissible* feedback control should be presented. The latter means that the control strategy should generate some well-defined trajectories of the original nonlinear system. These problems are solved in two different ways. First, an algorithm for constructing a continuous piecewise affine value function and corresponding continuous piecewise affine control is described. The results are then generalized to the case when the value and control functions can have finite discontinuities at the boundaries of simplices.

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The work of the algorithms based on the proposed schemes is demonstrated by the example of controlling a system used in mathematical modeling of a population of microorganisms.

2. SYSTEM'S MODEL

Consider a system of nonlinear ordinary differential equations

$$\dot{x} = \mathbf{f}(t, x) + \mathbf{g}(t, x)u, \quad t \in [t_0, t_1], \ x \in \Omega. \tag{1}$$

Here Ω – is a compact set in the space $\mathbb{R}^{n_x}$. Let $\mathbf{f}(t, x)$ be a twice continuously differentiable vector-function, $\mathbf{g}(t, x)$ – continuously differentiable matrix-function of x. Both functions depend continuously on $t \in [t_0, t_1]$. The starting and final time instants t_0, t_1 are fixed. The feedback control function $u(\cdot)$ satisfies the following pointwise restrictions: $u = u(t, x) \in \mathcal{P}$, where $\mathcal{P} \subset \mathbb{R}^{n_u}$ is some convex and compact set.

An *admissible feedback control* $u(\cdot)$ is defined as a set-valued function, which satisfies the mentioned pointwise restrictions and guarantees the existence of solutions to the differential inclusions obtained from (1) for any $x_0 \in \Omega$. A set of all such feedback control strategies is denoted as $\mathcal{U}_f$.

3. FEEDBACK CONTROL PROBLEM

Consider some compact set $\mathcal{X}_1 \subset \Omega$ that can be presented in the following form: $\mathcal{X}_1 = \{x \in \Omega : \varphi(x) \le 0\}$. Here $\varphi(x)$ is some twice continuously differentiable function.

The main problem is to obtain feedback control strategy $u(\cdot) \in \mathcal{U}_f$ that transfers the trajectory of (1) from a given initial position $(\tau, x(\tau))$, $\tau \in [t_0, t_1]$ into the target set $\mathcal{X}_1$. If it's not possible to get inside $\mathcal{X}_1$, then it's necessary to get into it's neighborhood with minimal size.

The *solvability set* $\mathcal{W}(t) = \mathcal{W}(t, t_1, \mathcal{X}_1)$ for a fixed $t \in [t_0, t_1]$ contains all values $x \in \Omega$ for each of which a control strategy $u(\cdot) \in \mathcal{U}_f$ exists that guarantees for any trajectory $x(\tau) = x(\tau, t, x)|_u$, $\tau \in [t, t_1]$, the validity of the following inclusion: $x(t_1) \in \mathcal{X}_1$.

Consider a *value function*

$$V(t, x) = \min_{u(\cdot) \in \mathcal{U}_f} \max_{x(\cdot)} \left\{ \varphi(x(t_1)) \middle| x(t) = x \right\}, \tag{2}$$

where $x(\cdot)$ is a component of system's trajectory that starts from position $\{t, x\}$ with control function $u(\cdot)$ fixed. The value function is related to the solvability set through the following formula:

$$\mathcal{W}(t, t_1, \mathcal{X}_1) = \{x \in \Omega : V(t, x) \le 0\}. \tag{3}$$

We will also consider the μ-neighborhood of the solvability set

$$\mathcal{W}_\mu(t, t_1, \mathcal{X}_1) = \{x \in \Omega : V(t, x) \le \mu\}, \mu \ge 0. \tag{4}$$

If the value function $V(t, x)$ is differentiable in some point (t, x) then it satisfies HJB equation (Fleming and Soner (2006))

$$\min_{u \in \mathcal{P}} V'(t, x; (1, (\mathbf{f}(t, x) + \mathbf{g}(t, x)u)^T)^T) = 0. \tag{5}$$

Here $V'(t, x; l)$ is a directional derivative of $V(t, x)$ at a point (t, x) with direction $l \in \mathbb{R}^{n_x+1}$. At the final time instant

$$V(t_1, x) = \varphi(x), \ \forall x \in \Omega. \tag{6}$$

In general the function $V(t, x)$ can be nondifferentiable, and hence the solution to (5) should be considered in a generalized sense (for example, as a viscosity solution) (Subbotin (1995), Fleming and Soner (2006)).

One of the goals of this paper is to find an approximate solution $V(t, x)$ of (5) and (6), that can be obtained using the comparison principle, in a special class of piecewise affine functions. The problem of control synthesis will be solved simultaneously.

4. PIECEWISE AFFINE SWITCHED SYSTEM

Consider some partitioning of the domain Ω into simplices (Rockafellar and Wets (1998)) $\Omega^{(i)}$, $i = 1, ..., N$, that intersect with each other only on boundary points. Each face of the simplex $\Omega^{(i)}$ that is the convex hull of its n_x vertices is either a part of the boundary of the set Ω itself, or is a face of an adjacent simplex $\Omega^{(k)}, k \ne i$. Also, suppose that $\cup_{i=1}^N \Omega^{(i)} = \Omega$. Consider $g_1, \dots, g_S$ – all vertices of simplices, S is a total number of unique vertices. Here and further, the superscript (i) denotes the correspondence of the considered function, vector or matrix to the simplex $\Omega^{(i)}$.

For any simplex $\Omega^{(i)}$ consider $g_1^{(i)}, \dots, g_{n_x+1}^{(i)}$ – its vertices [1]. Let's make a matrix $G^{(i)}$ from the column vectors $g_1^{(i)}, \dots, g_{n_x+1}^{(i)}$. For each $x \in \Omega^{(i)}$ there is a unique vector $\alpha^{(i)}(x) = (\alpha_1^{(i)}, \dots, \alpha_{n_x+1}^{(i)})^T$ of barycentric coordinates such that $\sum_{k=1}^{n_x+1} \alpha_k^{(i)} = 1$, $\alpha_k^{(i)} \geqslant 0, \forall k$, $G^{(i)} \alpha^{(i)}(x) = x$. Let's extend the vector x to $\widetilde{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}$, and consider $\widetilde{G}^{(i)} = \begin{pmatrix} g_1^{(i)} & \dots & g_{n_x+1}^{(i)} \\ 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{(n_x+1)\times(n_x+1)}$, $(\widetilde{G}^{(i)})^{-1} = \begin{pmatrix} H^{(i)} & h^{(i)} \end{pmatrix}$. Then $\alpha^{(i)}(x) = H^{(i)}x + h^{(i)}$.

Define $c^{(i)}$ as a center of a ball with minimal radius $r^{(i)}$, that includes $\Omega^{(i)}$: $\|x - c^{(i)}\| \le r^{(i)}$, $\forall x \in \Omega^{(i)}$.

For $x \in \Omega^{(i)}$ the right-hand side of (1) can be transformed as follows:

$$\mathbf{f}(t, x) + \mathbf{g}(t, x)u = A^{(i)}(t)x + B^{(i)}(t)u + f^{(i)}(t) + R, \tag{7}$$

where $F^{(i)}(t) = (\mathbf{f}(t, g_1^{(i)}), ..., \mathbf{f}(t, g_{n_x+1}^{(i)})) \in \mathbb{R}^{n_x \times (n_x+1)}$, $A^{(i)}(t) = F^{(i)}(t)H^{(i)} \in \mathbb{R}^{n_x \times n_x}$, $B^{(i)}(t) = \mathbf{g}(t, c^{(i)})$, $f^{(i)}(t) = F^{(i)}(t)h^{(i)} \in \mathbb{R}^{n_x}$, $R = R(t, x)$ is an error of local linearization. The s-th component of this error $(s = 1, ..., n_x)$ can be estimated for any $x \in \Omega^{(i)}$:

$$|R_s(t, x)| \leqslant R_s^{(i)}(t) = M_s^{(i)}(t)d^{(i)} + N_s^{(i)}(t)r^{(i)}, \tag{8}$$

$$M_s^{(i)}(t) = \max\left\{ \rho_{\max}\left(\frac{\partial^2 \mathbf{f}_s(t, x)}{\partial x^2} \right) : x \in \Omega^{(i)} \right\},$$

$$d^{(i)} = \max\Bigg\{ \sum_{k=1}^{n_x+1} \alpha_k \Bigg\| \sum_{r=1}^{n_x+1} \alpha_r (g_r^{(i)} - g_k^{(i)}) \Bigg\|^2 : \alpha_k \in [0, 1], \ \forall k, \ \sum_{k=1}^{n_x+1} \alpha_k = 1 \Bigg\},$$

[1] The presence of a superscript (i) suggests that local numbering for vertices by lower indices is used, rather than global (from 1 to S).

$$N_s^{(i)}(t) = \max_{u\in\mathcal{P}}\Big\{\sum_{k=1}^{n_u}|u_k|\cdot\max\Big\{\Big\|\frac{\partial \mathbf{g}_{sk}(t,x)}{\partial x}\Big\| : x\in\Omega^{(i)}\Big\}\Big\},$$

$\rho_{\max}(R)$ is a maximum singular number of a matrix R. Now let $\mathcal{Q}^{(i)}(t) = [-R_1^{(i)}(t), R_1^{(i)}(t)]\times\ldots\times[-R_{n_x}^{(i)}(t), R_{n_x}^{(i)}(t)]$.

5. CONTINUOUS PIECEWISE AFFINE VALUE FUNCTION

On the set of simplices $\Omega^{(i)}$ consider a piecewise affine function of the following form:

$$V(t,x) = \sum_{k=1}^{n_x+1}\alpha_k^{(i)}(x)v_k^{(i)}(t),\ \text{ if } x\in\Omega^{(i)}. \qquad (9)$$

Here $v_k^{(i)}(t) = V(t, g_k^{(i)})$ are differentiable by $t\in[t_0,t_1]$. For any different simplices $\Omega^{(j_1)}$,...,$\Omega^{(j_r)}$ with a common vertex g_k the corresponding values $v_{k_1}^{(j_1)}(t)$,...,$v_{k_r}^{(j_r)}(t)$ for this vertex will coincide; let's denote them as $v_k(t)$. Thus the function $V(t,x)$ is uniquely defined by a set of values $v_1(t),...,v_S(t)$. Let $v^{(i)}(t) = (v_1^{(i)}(t),...,v_{n_x+1}^{(i)}(t))^T\in\mathbb{R}^{n_x+1}$. Then (9) can be transformed into

$$V(t,x) = (v^{(i)}(t))^T(H^{(i)}x + h^{(i)}),\ \text{ while } x\in\Omega^{(i)}. \qquad (10)$$

The function $V(t,x)$ is differentiable in any direction. However, directional derivatives can have discontinuities at the boundaries of neighboring simplices.

Let's fix some simplex $\Omega^{(i)}$. Consider the expression from (5) while substituting $V(t,x)$ from (10) and using (7) [2]:

$$\Big[(\dot v^{(i)})^T(H^{(i)}x+h^{(i)}) + (\gamma^{(i)}(t))^T(A^{(i)}(t)x + f^{(i)}(t)) -$$
$$\rho(-(B^{(i)}(t))^T\gamma^{(i)}(t)|\mathcal{P})\Big] + (\gamma^{(i)}(t))^T R(t,x) = 0, \qquad (11)$$

$\gamma^{(i)}(t) = (H^{(i)})^T v^{(i)}(t)$. The expression in square brackets is an affine function of x. It takes extreme values at the vertices of $\Omega^{(i)}$. In addition, one can specify an expression for the minimizer by $u\in\mathcal{P}$ in (5), (11):

$$\mathcal{U}^*(t,x) = \begin{cases}\mathcal{P}(l) & , l = -(B^{(i)}(t))^T\gamma^{(i)}(t)\neq 0\\ \mathcal{P} & , l = 0\end{cases}, \qquad (12)$$

where $\mathcal{P}(l) = \mathrm{Argmax}\{l^T z | z\in\mathcal{P}\}$.

Hereinafter piecewise affine control functions of the following form will be used: for any $x\in\Omega^{(i)}$

$$u(t,x) = Y^{(i)}(t)(H^{(i)}x + h^{(i)}) = \sum_{k=1}^{n_x+1}\alpha_k(x)y_k^{(i)}(t). \qquad (13)$$

Here $Y^{(i)}(t)\in\mathbb{R}^{n_u\times(n_x+1)}$ is a matrix with columns $y_1^{(i)}(t),...,y_{n_x+1}^{(i)}(t)$ — values of control in the vertices of $\Omega^{(i)}$. If $y_k^{(i)}(t)\in\mathcal{P}$ for any k, then according to the convexity of $\mathcal{P}$ $u(t,x)\in\mathcal{P}$ for any $x\in\Omega$. The continuous piecewise affine control $u(t,x)$ is uniquely determined by the set of vector functions $y_1(t),...,y_S(t)$ corresponding to the vertices of simplices.

Also consider a set-valued piecewise affine control functions obtained similarly to (13):

$$\mathcal{U}(t,x) = \sum_{k=1}^{n_x+1}\alpha_k(x)\mathcal{Y}_k^{(i)}(t),\ \text{if } x\in\Omega^{(i)}, \qquad (14)$$

where each $\mathcal{Y}_k^{(i)}(t)\subseteq\mathcal{P}$ is a set-valued mapping with convex and compact values. $\mathcal{U}(t,x)$ is continuous with respect to x if $\mathcal{Y}_k^{(i)}(t) = \mathcal{Y}_k^{(j)}(t)$, $\forall i,j$: $g_k^{(i)} = g_k^{(j)}$.

6. INTERNAL APPROXIMATIONS OF SOLVABILITY SET

Let $i(x)$ be any number of simplex such that $x\in\Omega^{(i(x))}$. Consider $\eta_1,...,\eta_S$ — such values that $\forall x\in\Omega^{(i(x))}$

$$\varphi(x)\le(\eta^{(i(x))})^T(H^{(i(x))}x + h^{(i(x))}) = \sum_{k=1}^{n_x+1}\alpha_k(x)\eta_k^{(i(x))}, \qquad (15)$$

where $\eta^{(i(x))} = (\eta_{j_1}^{(i(x))},...,\eta_{j_{n_x+1}}^{(i(x))})^T$, $g_{j_1}^{(i(x))},...,g_{j_{n_x+1}}^{(i(x))}$ – the vertices of $\Omega^{(i(x))}$.

For some fixed $i = 1,...,N$ consider the Taylor series for $\varphi(x)$ centered at $x\in\Omega^{(i)}$:

$$\varphi(g_k^{(i)}) = \varphi(x) + \Big(\frac{\partial\varphi}{\partial x}(x)\Big)^T(g_k^{(i)} - x) + (g_k^{(i)} - x)^T\frac{\partial^2\varphi}{\partial x^2}(\xi_k)(g_k^{(i)} - x),$$

$\xi_k = \xi_k(x, g_k^{(i)})\in\Omega^{(i)}$. Let's combine the obtained formulas for different values of $k = 1,...,n_x+1$, multiplying them by the corresponding functions $\alpha_k(x)$:

$$\sum_{k=1}^{n_x+1}\alpha_k(x)\varphi(g_k^{(i)}) = \varphi(x) + \sum_{k=1}^{n_x+1}\alpha_k(x)(g_k^{(i)} - x)^T\frac{\partial^2\varphi}{\partial x^2}(\xi_k)(g_k^{(i)} - x).$$

Besides, $\forall x\in\Omega^{(i)}$

$$\Big|(g_k^{(i)} - x)^T\frac{\partial^2\varphi}{\partial x^2}(\xi_k)(g_k^{(i)} - x)\Big|\le K_k^{(i)}$$
$$= \max_{\xi\in\Omega^{(i)}}\rho_{\max}\Big(\frac{\partial^2\varphi}{\partial x^2}(\xi)\Big)\cdot\max_{j=1,...,n_x+1}\|g_k^{(i)} - g_j^{(i)}\|^2. \qquad (16)$$

Hence if for any $k = 1,...,S$ $\eta_k = \varphi(g_k) + \max_{i,s}\{K_s^{(i)} : g_s^{(i)} = g_k\}$, then the relation (15) is valid, and the function $\sum_{k=1}^{n_x+1}\alpha_k(x)\eta_k^{(i(x))}$ is continuous for $x\in\Omega$.

Theorem 1. Consider any piecewise continuous set-valued mappings $\mathcal{Y}_k(t)\subseteq\mathcal{P}$, $t\in[t_0,t_1]$, $k = 1,...,S$. Let

$$\zeta_k(t) = \max_i\Big\{(\gamma^{(i)}(t))^T(A^{(i)}(t)g_k + f^{(i)}(t)) + \rho\left(\gamma^{(i)}(t)|\mathcal{Q}^{(i)}(t)\right)$$
$$+\rho\left((B^{(i)}(t))^T\gamma^{(i)}(t)|\mathcal{Y}_k(t)\right)\Big| i\in\{1,...,N\} : g_k\in\Omega^{(i)}\Big\},$$

$\gamma^{(i)}(t) = (H^{(i)})^T v^{(i)}(t)$. For some continuous functions $\delta_k(t)$ let $v_k(t)$, $k = 1,...,S$, be solutions of

$$\begin{cases}\dot v_k(t) = -\zeta_k(t) + \delta_k(t) & , t\in[t_0,t_1]\\ v_k(t_1) = \eta_k\end{cases}, \qquad (17)$$

while $V(t,x)$ is defined according to (9). Then the set

$$\mathcal{W}_\mu^{int}(t_0) = \Big\{x\in\Omega\Big|V(t_0,x)\le\mu - \int_{t_0}^{t_1}\max_k\{\delta_k(\tau)\}d\tau\Big\}$$

is an internal approximation of the μ-neighborhood of the solvability set:

$$\mathcal{W}_\mu^{int}(t_0)\subseteq\mathcal{W}_\mu(t_0,t_1,\mathcal{X}_1).$$

Proof. Assume that the set $\mathcal{W}_\mu^{int}(t_0)$ is not empty. For any $t\in[t_0,t_1]$, $x\in\Omega$ consider piecewise affine, continuous over x and piecewise continuous over t control function

$$\mathcal{U}^*(t,x) = \sum_{k=1}^{n_x+1}\alpha_k(x)\mathcal{Y}_k^{(i)}(t),\ x\in\Omega^{(i)},\ i = 1,...,N. \qquad (18)$$

[2] Hereinafter $\rho(l\,|\,\mathcal{P})$ is the value of the support function for a closed convex set $\mathcal{P}$ in direction l.

For any $x_0 \in \mathcal{W}_\mu^{int}(t_0)$ define some solution $x(t) = x(t,t_0,x_0)|_{\mathcal{U}^*(\cdot)}$, $t \in [t_0,t_1]$, of (1). Let $u^*(t,x) = \sum_{k=1}^{n_x+1} \alpha_k(x) y_k^{(i)}(t) = (Y^{(i)}(t))^T (H^{(i)}x + h^{(i)}) \in \mathcal{U}^*(t,x)$ is a continuous selection of this mapping that corresponds to the obtained trajectory. Here matrix $Y^{(i)}(t) \in \mathbb{R}^{(n_x+1)\times n_u}$ is combined from the rows $(y_k^{(i)}(t))^T$: $y_k^{(i)}(t) \in \mathcal{Y}_k^{(i)}(t)$. Let's estimate from above the derivative of $V(t,x)$ along the constructed trajectory [3]:

$$\begin{aligned}
&V'(t,x;(1, A^{(i)}(t)x + B^{(i)}(t)u^*(t,x) + f^{(i)}(t) + R(t,x))^T) \\
&\le (\dot{v}^{(i)})^T (H^{(i)}x + h^{(i)}) + (\gamma^{(i)}(t))^T (A^{(i)}(t)x + B^{(i)}(t)(Y^{(i)}(t))^T \cdot \\
&\qquad (H^{(i)}x + h^{(i)}) + f^{(i)}(t)) + \rho\left(\gamma^{(i)}(t)|\mathcal{Q}^{(i)}(t)\right) \\
&\le \max_k \Big\{ \dot{v}_k(t) + (\gamma^{(i)}(t))^T (A^{(i)}(t) g_k + B^{(i)}(t) y_k(t) + f^{(i)}(t)) : \\
&\qquad g_k \in \Omega^{(i)} \Big\} + \rho\left(\gamma^{(i)}(t)|\mathcal{Q}^{(i)}(t)\right) \\
&\le \max_k \Big\{ \dot{v}_k(t) + \zeta_k(t) : g_k \in \Omega^{(i)} \Big\} \le \max_k \{\delta_k(t)\}. \qquad (19)
\end{aligned}$$

Integrating $V(t,x)$ along the constructed trajectory we obtain:

$$\begin{aligned}
&\varphi(x(t_1)) \le (\eta^{(i(x(t_1)))})^T (H^{(i(x(t_1)))} x + h^{(i(x(t_1)))}) \\
&\quad = V(t_1, x(t_1)) \le V(t_0, x_0) + \int_{t_0}^{t_1} \max_k \{\delta_k(\tau)\} d\tau \le \mu,
\end{aligned}$$

and thus $x_0 \in \mathcal{W}_\mu(t_0, t_1, \mathcal{X}_1)$.

The theorem 1 is formulated for a fixed initial time t_0. However, it is easy to see that a similar statement holds for any $t \in [t_0, t_1]$.

Let $W(t,x) = V(t,x) + \int_{t_0}^{t_1} \max_k\{\delta_k(\tau)\}d\tau$. It follows from the proof of the theorem that if the inequality $W(t,x) \le \mu$ holds, then piecewise affine control (18) solves the problem of transferring the trajectory of the system from a given position (t,x) to a μ-neighborhood of the target set $\mathcal{X}_1$.

7. ALGORITHM FOR PIECEWISE AFFINE FEEDBACK CONTROL

For some fixed vertex g_k consider $i_1, ..., i_m$ – the numbers of all simplices that include this vertex. Let

$$\Delta_k(v_1, ..., v_S) = \sum_{j=1}^{m} \sum_{l=j+1}^{m} \|\gamma^{(i_j)} - \gamma^{(i_l)}\|^2.$$

If $m = 1$, then $\Delta_k = 0$. It's possible to simplify the expression for $\Delta_k(v_1, ..., v_S)$. For that purpose consider the set of all vertices $g_{j_1}, ..., g_{j_r}$ of simplices $\Omega^{(i_1)}, ..., \Omega^{(i_m)}$. Suppose that $j_1 < ... < j_r$, and $j_s = k$ for some $s \in \{1, ..., r\}$. Let $\tilde{v}_k = (v_{j_1}, ..., v_{j_r})^T$, $v^{(i)} = S_k^{(i)} \cdot \tilde{v}_k$, $\forall i = i_1, ..., i_m$, where $S_k^{(i)} \in \mathbb{R}^{(n_x+1)\times r}$. Then

$$\Delta_k(v_1, ..., v_S) = (\tilde{v}_k)^T P_k \tilde{v}_k, \qquad (20)$$

where $P_k = \sum_{j=1}^{m} \sum_{l=j+1}^{m} ((H^{(i_j)})^T S_k^{(i_j)} - (H^{(i_l)})^T S_k^{(i_l)})^T \cdot ((H^{(i_j)})^T S_k^{(i_j)} - (H^{(i_l)})^T S_k^{(i_l)})$. Let $s_k(t) = \text{sign}\left\{2(\mathbf{e}_s)^T P_k \tilde{v}_k\right\}$, where all the elements of $\mathbf{e}_s \in \mathbb{R}^r$ are zeros, except 1 standing at the s-th position. Finally, consider

$$\delta_k(t) = \varepsilon s_k(t) \Delta_k(t). \qquad (21)$$

Here $\varepsilon > 0$ – some fixed value. The use of (21) in (17) allows to avoid divergence of the values of $\gamma^{(i_j)}$ in the neighboring simplices as t decreases from t_1 to t_0. This term corresponds to the "small viscosity" used in the numerical method of solving first order PDEs (see Kruzhkov (1966)).

The theorem 1 also allows to construct a piecewise affine, continuous feedback control that leads the trajectory of the system to a priori known small neighborhood of the target set. When constructing a piecewise affine value function for each $t \in [t_0, t_1]$, the set $\mathcal{Y}_k(t)$, $k = 1, ..., S$, can be defined as

$$\mathcal{Y}_k(t) = \frac{1}{m} \sum_{j=1}^{m} \mathcal{U}^{*,(i_j)}(t, g_k), \qquad (22)$$

where $\mathcal{U}^{*,(i_j)}(t, g_k)$ is obtained from (12) for the simplex $\Omega^{(i_j)}$. Here $i_1, ..., i_m$ – are the numbers of all simplices that include g_k. The choice of control in the form (22) is due to the fact that it is close to the "optimal" control $\mathcal{U}^{*,(i_j)}(t, g_k)$ for each of the simplices $\Omega^{(i_j)}$.

Now the basic algorithm for solving the feedback control problem can be formulated:

(1) Using the constructions of the theorem 1 and also formulas (20) – (22) it's necessary to calculate a continuous piecewise affine function $V(t,x)$, and also an adjusted function $W(t,x) = V(t,x) + \int_t^{t_1} \max_k\{\delta_k(\tau)\}d\tau$, $t \in [t_0, t_1]$, $x \in \Omega$.
(2) For any state (t,x) the value $W(t,x)$ specifies the size of the neighborhood of $\mathcal{X}_1$ that is guaranteed to be reachable. In particular, if $W(t,x) = 0$, then an admissible control exists that transfers the trajectory into a target set.
(3) Feedback control that solves the main problem is given by the following formula:

$$\mathcal{U}(t,x) = \sum_{k=1}^{n_x+1} \alpha_k(x) \mathcal{Y}_k(t), \qquad (23)$$

where $\mathcal{Y}_k(t)$ are defined from (22), $\mathcal{U}^{*,(i_j)}(t, g_k)$ – from (12). Here for any i $\gamma^{(i)}(t) = \frac{\partial W}{\partial x}(t, \xi)$, $\forall \xi \in \text{int}\, \Omega^{(i)}$.

8. DISCONTINUOUS VALUE AND CONTROL FUNCTIONS

The results of the previous sections can be generalized to the case of value and control functions that can have discontinuities on the boundaries of simplices. In this section, we assume that the system (1) is stationary.

Denote $\sigma(i,k)$ as the index in the local numbering for the simplex $\Omega^{(i)}$ corresponding to the vertex g_k, that is $g_{\sigma(i,k)}^{(i)} = g_k$. Suppose that for each simplex $\Omega^{(i)}$ the values $v_{\sigma(i,k)}^{(i)}(t)$ and $\mathcal{Y}_{\sigma(i,k)}^{(i)}$, corresponding to a fixed vertex g_k, can differ for different values of i. Let $i_1, ..., i_m$ be the numbers of all simplices including vertex g_k. The discontinuous piecewise affine value function $V(t,x)$ can still be defined from (9).

For any two adjacent simplices $\Omega^{(i^*)}$ and $\Omega^{(i^{**})}$ having a common face $\mathcal{H}_{i^*, i^{**}}$, which is an $(n_x - 1)$-dimensional simplex, we'll say that $\Omega^{(i^*)}$ is *unreachable* from $\Omega^{(i^{**})}$ if

[3] For brevity of notation the functions' arguments are omitted here: $i = i(t)$, $x = x(t)$.

$$\min_s \Big\{ (n_{i^*,i^{**}})^T (A^{(i^{**})} g_s + f^{(i^{**})}) - \rho(-(B^{(i^{**})})^T n_{i^*,i^{**}} | \mathcal{P}) \\ - \rho(-n_{i^*,i^{**}} | \mathcal{Q}^{(i^{**})}) : g_s - \text{is a vertex of } \mathcal{H}_{i^*,i^{**}} \Big\} > 0. \quad (24)$$

Here $n_{i^*,i^{**}}$ is a normal to $\mathcal{H}_{i^*,i^{**}}$ that points to the half-space that contains $\Omega^{(i^{**})}$. If the condition (24) is not satisfied, then $\Omega^{(i^*)}$ is *reachable* from $\Omega^{(i^{**})}$.

For each vertex g_k, each pair of simplices $\Omega^{(i^*)}$ and $\Omega^{(i^{**})}$ containing this vertex, $i_1 \le i^* < i^{**} \le i_m$, simplex $\Omega^{(i^*)}$ is *reachable* from $\Omega^{(i^{**})}$ if there are such different values $i_{j_1}, ..., i_{j_s}$, $s \in \{2, ..., m\}$, that $i_{j_1} = i^*$, $i_{j_s} = i^{**}$ and $\Omega^{(i_{j_l})}$ is reachable from $\Omega^{(i_{j_{l+1}})}$ for any $l = 1, ..., s-1$. Otherwise, $\Omega^{(i^*)}$ is *unreachable* from $\Omega^{(i^{**})}$. For any $i \in \{i_1, ..., i_m\}$ let $\mathcal{I}(i,k) = \{j \in \{i_1, ..., i_m\},\ j \ne i,\ \Omega^{(j)}$ is reachable from $\Omega^{(i)}\} \cup \{i\}$.

Theorem 2. Consider a set of piecewise continuous set-valued mappings $\mathcal{Y}^{(i)}_{\sigma(i,k)}(t) \subseteq \mathcal{P}$, $t \in [t_0, t_1]$, $k = 1, ..., S$, $i = 1, ..., N$. Let for any k, i^*, i^{**} such that $g_k \in \Omega^{(i^*)} \cap \Omega^{(i^{**})}$ the following condition be valid:

$$\mathcal{Y}^{(i^*)}_{\sigma(i^*,k)}(t) = \mathcal{Y}^{(i^{**})}_{\sigma(i^{**},k)}(t), \text{ if } i^* \in \mathcal{I}(i^{**},k), i^{**} \in \mathcal{I}(i^*,k). \quad (25)$$

For any $k = 1, ..., S$, $i = 1, ..., N$ let

$$\zeta^{(i)}_{\sigma(i,k)}(t) = \max_s \Big\{ (\gamma^{(s)}(t))^T (A^{(s)} g_k + f^{(s)}) + \rho\left(\gamma^{(s)}(t) | \mathcal{Q}^{(s)}\right) \\ + \rho\left((B^{(s)})^T \gamma^{(s)}(t) | \mathcal{Y}^{(s)}_{\sigma(s,k)}(t) \right) \Big| i \in \{1, ..., N\} : g_k \in \Omega^{(s)}, \\ s \in \mathcal{I}(i,k) \text{ and } v^{(s)}_{\sigma(s,k)} \ge v^{(i)}_{\sigma(i,k)} \Big\},\ \gamma^{(i)}(t) = (H^{(i)})^T v^{(i)}(t), \quad (26)$$

$$\begin{cases} \dot v^{(i)}_{\sigma(i,k)}(t) = -\zeta^{(i)}_{\sigma(i,k)}(t) + \delta^{(i)}_{\sigma(i,k)}(t), \quad t \in [t_0, t_1] \\ v^{(i)}_{\sigma(i,k)}(t_1) = \varphi(g_k) + \max\limits_{j,s} \{K^{(j)}_s : g^{(j)}_s = g_k, j \in \mathcal{I}(i,k)\} \end{cases}. \quad (27)$$

Here $\delta^{(i)}_{\sigma(i,k)}(t)$ are some piecewise continuous functions such that for any i^*, i^{**}, $g_k \in \Omega^{(i^*)} \cap \Omega^{(i^{**})}$, if $i^{**} \in \mathcal{I}(i^*,k), v^{(i^{**})}_{\sigma(i^{**},k)} \ge v^{(i^*)}_{\sigma(i^*,k)}$ then

$$\delta^{(i^*)}_{\sigma(i^*,k)}(t) = \delta^{(i^{**})}_{\sigma(i^{**},k)}(t). \quad (28)$$

Define $V(t,x)$ according to (9). Then the set

$$\mathcal{W}^{int}_\mu(t_0) = \Big\{ x \in \Omega \Big| V(t_0, x) \le \mu \\ - \int_{t_0}^{t_1} \max_{k,i} \left\{ \delta^{(i)}_{\sigma(i,k)}(\tau) : g_k \in \Omega^{(i)} \right\} d\tau \Big\} \quad (29)$$

is an internal approximation of the μ-neighborhood of the solvability set:

$$\mathcal{W}^{int}_\mu(t_0) \subseteq \mathcal{W}_\mu(t_0, t_1, \mathcal{X}_1).$$

Proof. First let's prove that for any vertex g_k, any two numbers i^*, i^{**} such that $g_k \in \Omega^{(i^*)} \cap \Omega^{(i^{**})}$, if $i^{**} \in \mathcal{I}(i^*,k)$, then $v^{(i^*)}_{\sigma(i^*,k)}(t) \ge v^{(i^{**})}_{\sigma(i^{**},k)}(t)$, $\forall t \in [t_0, t_1]$. Both functions are continuously differentiable, and $v^{(i^*)}_{\sigma(i^*,k)}(t_1) \ge v^{(i^{**})}_{\sigma(i^{**},k)}(t_1)$ according to (27). Suppose that $\exists t \in [t_0, t_1)$ for which $v^{(i^*)}_{\sigma(i^*,k)}(t) < v^{(i^{**})}_{\sigma(i^{**},k)}(t)$, and let $\theta = \inf\{\tau \in [t, t_1] : v^{(i^*)}_{\sigma(i^*,k)}(\tau) \ge v^{(i^{**})}_{\sigma(i^{**},k)}(\tau)\}$. Then $\theta > t$, $v^{(i^*)}_{\sigma(i^*,k)}(\theta) = v^{(i^{**})}_{\sigma(i^{**},k)}(\theta)$. Therefore $\exists \tau^* \in [t, \theta]$:

$$\begin{cases} \dot v^{(i^*)}_{\sigma(i^*,k)}(\tau^*) > \dot v^{(i^{**})}_{\sigma(i^{**},k)}(\tau^*) \\ v^{(i^*)}_{\sigma(i^*,k)}(\tau^*) < v^{(i^{**})}_{\sigma(i^{**},k)}(\tau^*) \end{cases}.$$

From (28) it follows that $\delta^{(i^*)}_{\sigma(i^*,k)}(\tau^*) = \delta^{(i^{**})}_{\sigma(i^{**},k)}(\tau^*)$. According to (26) $\zeta^{(i^*)}_{\sigma(i^*,k)}(\tau^*) \ge \zeta^{(i^{**})}_{\sigma(i^{**},k)}(\tau^*)$. Hence using (27) we obtain $\dot v^{(i^*)}_{\sigma(i^*,k)}(\tau^*) \le \dot v^{(i^{**})}_{\sigma(i^{**},k)}(\tau^*)$, and that contradicts to the inequality obtained before.

Consider a piecewise affine control function

$$\mathcal{U}^*(t,x) = \sum_{k=1}^{n_x+1} \alpha_k(x) \mathcal{Y}^{(i)}_k(t),\ x \in \Omega^{(i)},\ i = 1, ..., N. \quad (30)$$

The set-valued mapping $\mathcal{U}^*(t,x)$ is continuous by x in each of the simplices, but can have discontinuities at their boundaries. However, such discontinuous control is admissible, as it generates trajectories of (1), since (25) is satisfied. It follows from this condition that the control function's gap at the boundary of some simplices $\Omega^{(i^*)}$ and $\Omega^{(i^{**})}$ is possible only in cases where the trajectory of the system can get from the first simplex to the second, or back, but not there and back at the same time. For any trajectory there will be no more than a finite number of points of discontinuity of the control function while $t \in [t_0, t_1]$. For any $x_0 \in \mathcal{W}^{int}_\mu(t_0)$ consider some solution $x(t) = x(t, t_0, x_0)|_{\mathcal{U}^*(\cdot)}$, $t \in [t_0, t_1]$. Let $u^*(t,x) = \sum_{k=1}^{n_x+1} \alpha_k(x) y^{(i)}_k(t) = (Y^{(i)}(t))^T (H^{(i)} x + h^{(i)}) \in \mathcal{U}^*(t,x)$ – the corresponding single-valued selection of the set-valued mapping. Here $Y^{(i)}(t) \in \mathbb{R}^{(n_x+1)\times n_u}$ is combined from the rows $(y^{(i)}_k(t))^T$: $y^{(i)}_k(t) \in \mathcal{Y}^{(i)}_k(t)$. There exist $t_0 = \tau_1 < \tau_2 < ... < \tau_{K-1} = \tau_K = t_1$ such that for any interval $[\tau_j, \tau_{j+1}]$, $j = 1, ..., K-1$, the trajectory lies inside single simplex $\Omega^{(i)}$, or it moves along the boundary of several simplices $\Omega^{(i_1)}, ..., \Omega^{(i_m)}$. For $t \in (\tau_j, \tau_{j+1})$ it's possible to estimate the total derivative of the value function along the trajectory similarly to (19):

$$V'(t, x(t); (1, A^{(i(t))} x(t) + B^{(i(t))} u^*(t, x(t)) + f^{(i(t))} + R(x(t)))^T) \\ \le \max_k \Big\{ \dot v^{(i(t))}_{\sigma(i(t),k)}(t) + \zeta^{(i(t))}_{\sigma(i(t),k)}(t) : g_k \in \Omega^{(i(t))} \Big\} \\ \le \max_{k,i} \left\{ \delta^{(i)}_{\sigma(i,k)}(t) : g_k \in \Omega^{(i)} \right\}. \quad (31)$$

For $t = \tau_j$, $j = 1, ..., K$, the function $V(t, x(t))$ can have a gap. Let $x(\tau_j) \in \Omega^{(i^*)} \cap \Omega^{(i^{**})}$, and the trajectory $x(t)$ transfers from $\Omega^{(i^*)}$ to $\Omega^{(i^{**})}$ at time instant τ_j. Hence $\Omega^{(i^{**})}$ is reachable from $\Omega^{(i^*)}$. It was proved that $v^{(i^*)}_{\sigma(i^*,k)}(\tau_j) \ge v^{(i^{**})}_{\sigma(i^{**},k)}(\tau_j)$, for any vertex g_k of the facet $\mathcal{H} = \Omega^{(i^*)} \cap \Omega^{(i^{**})}$. The values $V(\tau_j - 0, x(\tau_j - 0))$ and $V(\tau_j + 0, x(\tau_j + 0))$ are convex hulls of $v^{(i^*)}_{\sigma(i^*,k)}(\tau_j)$ and $v^{(i^{**})}_{\sigma(i^{**},k)}(\tau_j)$ for various values of k respectively, with similar coefficients. Then it follows that $V(\tau_j - 0, x(\tau_j - 0)) \ge V(\tau_j + 0, x(\tau_j + 0))$, and any gap of the value function is accompanied by it's decrease.

Taking into account the described constraints on discontinuities and integrating $V(t, x(t))$ along the constructed trajectory on each of the segments $[\tau_j, \tau_{j+1}]$, we obtain

$$\varphi(x(t_1)) \le V(t_0, x_0) + \int_{t_0}^{t_1} \max_{k,i} \left\{ \delta^{(i)}_{\sigma(i,k)}(\tau) : g_k \in \Omega^{(i)} \right\} d\tau \le \mu,$$

and hence $x_0 \in \mathcal{W}_\mu(t_0, t_1, \mathcal{X}_1)$.

As in the case of the continuous value function, it makes sense to define the values of $\delta^{(i)}_{\sigma(i,k)}(t)$ in such a way as to minimize the discrepancies of the values of $\gamma^{(i)}(t)$ for any neighboring simplices, that is similar to (20) – (21). However, it is necessary to take into account (28) and (25).

9. EXAMPLE

Consider a system of differential equations used for modeling bioprocesses in bioreactors (Alt and Markov (2012)):

$$\begin{cases} \dot{x}_1 = -\alpha\mu^* \dfrac{x_1 x_2}{K + x_1} - u(x_1 - S) \\ \dot{x}_2 = \mu^* \dfrac{x_1 x_2}{K + x_1} - u x_2 \end{cases}. \tag{32}$$

Here x_1, x_2 are the concentrations of the substrate and the biomass of bacteria, S is a constant concentration of fresh nutrient, $u \in [u_{min}, u_{max}]$ – the inflow, that can be controlled. The value $\alpha > 0$ is a constant that defines the growth yield, while $\mu^* > 0$ represents the specific growth rate of the biomass. $K > 0$ defines a certain pattern of the biomass growth for a specific type of microorganisms.

The following system parameters were used in the simulation: $\mathcal{P} = [u_{\min}, u_{\max}]$, $u_{\min} = 0.3$, $u_{\max} = 0.4$, $K = 7.1$, $\mu^* = 1.2$, $S = 5.7$, $\alpha = 10.5$. The target set $\mathcal{X}_1 = \{(x_1, x_2)^T : (x_1 - 1.2)^2 + (x_2 - 0.9)^2 \leq 0.01\}$, $t \in [0, 1.5]$.

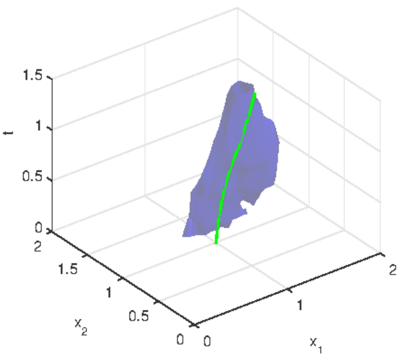

Fig. 1. The case of continuous value function.

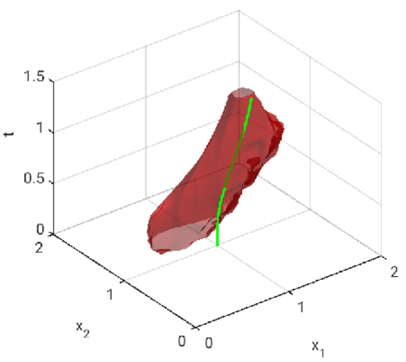

Fig. 2. The case of discontinuous value function.

Fig. 1, 2 demonstrate the internal estimates of the solvability tube, obtained respectively using the continuous and discontinuous value functions. Also the components of the trial trajectories of the system (32) with the continuous and discontinuous piecewise affine feedback control (18) and (30) correspondingly are shown. Here $x_0 = (1;1)^T$. For the first case $V(0, x_0) = 0.1317$, while $V(1.5, x(t_1)) = 0.0135$ (the decreasing distance is due to the "nonoptimality" of the uncertainty). For the second, discontinuous case $V(0, x_0) = 0.1301$, $V(1.5, x(t_1)) = 0.0129$ – that's the effect of the ability to use discontinuous control strategy.

10. CONCLUSION

This paper presents a solution scheme for the target control problem in the class of piecewise affine functions for a rather broad class of nonlinear systems. Two approaches using continuous or discontinuous value functions are presented. The first one is easier to implement, but the second, discontinuous case is less conservative. The proposed approach can be effectively used to solve control problems for nonlinear systems with small dimension of the state space. For example, such problems arise in mathematical biology. The problem of convergence of the proposed approximate solutions to the real value function when the size of simplices tends to zero remains open. It will be considered in the future papers.

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