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A notion of equivalence for linear complementarity problems Applications to the design of nonsmooth bifurcations ⋆

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Abstract: Many systems of interest to control engineering can be modeled by linear complementarity problems. We introduce a new notion of equivalence between linear complementarity problems that sets the basis to translate the powerful tools of smooth bifurcation theory to this class of models. Leveraging this notion of equivalence, we introduce new tools to analyze, classify, and design nonsmooth bifurcations in linear complementarity problems and their interconnection.

Keywords: Linear complementarity problems, bifurcations, topological equivalence, piecewise linear equations.

1. INTRODUCTION

Bifurcation theory is one of the most successful tools for the analysis of nonlinear dynamical systems that depend on a control parameter. The theory is firmly grounded on the classical implicit function theorem (Golubitsky and Schaeffer, 1985) and, therefore, it requires smoothness of the maps under study. However, from a practical viewpoint, it is common to approximate complicated nonlinear maps by simpler models. In such situations, the resulting approximation may be nonsmooth.

Linear complementarity problems are nonsmooth (but continuous) models that arise in fields of science such as economics (Nagurney, 1999), electronics (Acary et al., 2011), mechanics (Brogliato, 1999), mathematical programming (Murty, 1988), general systems theory (van der Schaft and Schumacher, 1998), etc. They serve as a departing point in the analysis of systems with unilateral constraints, and also arise as piecewise linear approximations of nonlinear models (Leenaerts and Bokhoven, 1998).

Recently, there have been some attempts to extend bifurcation theory towards the nonsmooth setting, see e.g. Di Bernardo et al. (2008); Leine and Nijmeijer (2004). However, the emphasis has been directed towards analysis of discontinuous systems, and very little is known on bifurcations in complementarity systems.

The purpose of this paper is to provide a methodology for the realization of equilibrium bifurcations in linear complementarity problems. The proposed framework mimics, up to certain extent, the smooth program proposed by Arnold et al. (1985) and relies on tools from nonsmooth analysis and linear algebra. To achieve this, the concept of topological equivalence in complementarity systems is introduced. We focus on static models that arise as the steady-state equations of continuous piecewise linear dynamical systems. Thanks to the piecewise linear structure, the introduced equivalence is global, which constitutes a major difference with respect to smooth bifurcation theories. This fundamental concept allows us to provide a complete classification of planar complementarity problems.

The paper is organized as follows. Section 2 describes the linear complementarity problem and related concepts. Section 3 constitutes the main body of the paper and addresses the problem of topological equivalence between LCPs. Afterwards, an interconnection approach for the realization of bifurcations is presented, together with an example applied to the nonsmooth pleat and the pitchfork singularity. Because of space constraints, proofs are omitted but can be found in (Castaños et al., 2019).

2. PRELIMINARIES

2.1 Linear Complementarity Problems

The linear complementarity problem (LCP) is defined as follows.

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Definition 1. Given a vector $q \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$, the LCP (M, q) consists in finding vectors $z, w \in \mathbb{R}^n$ such that

$$w = Mz + q \,, \qquad \mathbb{R}^n_+ \ni w \perp z \in \mathbb{R}^n_+ \,, \tag{1}$$

where the second relation, called the *complementary condition*, is the short form of the following three conditions: $w \in \mathbb{R}^n_+$, $z \in \mathbb{R}^n_+$, and $w^\top z = 0$.

In what follows, we introduce some concepts that will be useful for studying the geometric structure of LCPs. Given M and an index set $\alpha \subseteq \{1, \dots, n\}$, we define the *complementary matrix* $C_M(\alpha)$ as

$$C_M(\alpha)_{\cdot j} = \begin{cases} -M_{\cdot j} & \text{if } j \in \alpha \\ I_{\cdot j} & \text{if } j \notin \alpha \end{cases} ,$$

where the subscript $\cdot j$ denotes the j-th column. Now define the continuous piecewise-linear function

$$f_M(x) = C_{-M}(\alpha)x \,, \quad x \in \operatorname{pos} C_I(\alpha) \,, \tag{2}$$

where $\operatorname{pos} C_I(\alpha)$ is the cone generated by the columns of $C_I(\alpha)$. Note that the cones $\operatorname{pos} C_I(\alpha)$ are simply the 2^n orthants in $\mathbb{R}^n$ indexed by $\alpha \subseteq \{1, \dots, n\}$, and that

$$f_M(\operatorname{pos} C_I(\alpha)) = \operatorname{pos} C_M(\alpha) \,.$$

Proposition 2. (Cottle et al. (2009)). Let $\operatorname{Proj}_S(x)$ be the projection of x onto the set S and let $z \in \mathbb{R}^n$ be a solution of the LCP (M, q). Then, $x = w - z \in \mathbb{R}^n$ is a solution of

$$f_M(x) = q \,. \tag{3}$$

Conversely, let $x \in \mathbb{R}^n$ be a solution of (3), then $z = \operatorname{Proj}_{\mathbb{R}^n_+}(-x) \in \mathbb{R}^n_+$ is a solution of the LCP (M, q).

Henceforth, we treat the LCP (M, q) and (3) as equivalent problems, in the sense that we only need to know the solution of one of them in order to know the solution of the other.

The solutions of the LCP (M, q) depend on the geometry of the *complementary cones* $\operatorname{pos} C_M(\alpha)$. More precisely, there exists at least one solution x of (3) for every α such that $q \in \operatorname{pos} C_M(\alpha)$. If $C_M(\alpha)$ is nonsingular, the solution is unique, whereas there exists a continuum of solutions if $C_M(\alpha)$ is singular. Thus, for a given q, there can be no solutions, there can be one solution, multiple isolated solutions, or a continuum of solutions, depending on how many complementary cones q belongs to and which properties these cones have.

2.2 Bifurcations in LCPs

In practical applications, the vector q depends on a control or *bifurcation* parameter $\lambda \in \mathbb{R}$. The bifurcation parameter can be an applied voltage or current in electrical circuits, a force or a torque in a mechanical system, or the amount of capital injection in an economic system. The goal of bifurcation theory is to understand how the number of solutions changes as the bifurcation parameter is varied. In LCPs we let $q = \bar{q}(\lambda)$, where $\bar{q} : \mathbb{R} \to \mathbb{R}^n$ is at least continuous, although more regularity constraints can be imposed as needed. The mapping $\bar{q}$ defines a continuous curve, or *path* in $\mathbb{R}^n$. As λ lets q move along this path, the number of solutions to the LCPs might change. Points where the number of solutions changes are called *bifurcation points*.

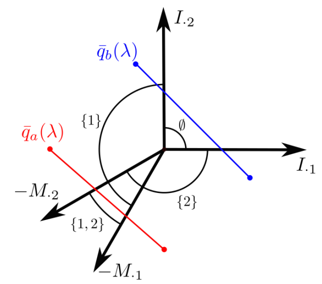

Fig. 1. Cone configuration for matrix M in (4). The thick black lines depict the generators of the complementary cones $\operatorname{pos} C_M(\alpha)$, $\alpha \subseteq \{1, 2\}$, whereas the arcs denote the complementary cones.

Example 3. Let us illustrate this idea in the simple case where the path is a line segment joining two distinct points $q_i \in \mathbb{R}^2$, $i \in \{0, 1\}$, that is, $\bar{q}(\lambda) = (1 - \lambda)q_0 + \lambda q_1$ with $\lambda \in [0, 1]$. In addition, let us set the matrix M as

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{4}$$

and proceed to analyze the two cases shown in Fig. 1.

Case a) We take the path $\bar{q}_a(\lambda)$ given by

$$\bar{q}_a(\lambda) = (1 - \lambda)\begin{bmatrix} -4 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ -4 \end{bmatrix} , \quad \lambda \in [0, 1] \,. \tag{5}$$

According to Proposition 2, solving the LCP $(M, \bar{q}_a(\lambda))$ is equivalent to finding $x \in \mathbb{R}^2$ satisfying

$$C_{-M}(\alpha)x = \bar{q}_a(\lambda), \quad x \in \operatorname{pos} C_I(\alpha) \,, \tag{6}$$

for $\alpha \subseteq \{1, 2\}$. Noting that $C_M(\alpha) = C_{-M}(\alpha)C_I(\alpha)$, it follows that the solutions to (6) are given by $\bigcup_{\alpha \subseteq \{1,2\}} S_\alpha$, where

$$S_\alpha = \big\{(x, \lambda) \in \mathbb{R}^2 \times [0, 1] \mid \exists\, p_\lambda(\alpha) \in \mathbb{R}^2_+ \;: \\ x = C_I(\alpha)p_\lambda(\alpha) \text{ and } \bar{q}_a(\lambda) = C_M(\alpha)p_\lambda(\alpha)\big\} \tag{7}$$

Roughly speaking, in order to solve the parametrized LCP $(M, \bar{q}(\lambda))$ we need to find $p_\lambda(\alpha)$ (the representation of $\bar{q}(\lambda)$ in terms of the generators of the α-th complementary cone). Computing these explicitly and taking $\alpha = \emptyset \subset \{1, 2\}$ we get

$$p_\lambda(\emptyset) = C_M(\emptyset)\bar{q}_a(\lambda) = \begin{bmatrix} 4\lambda - 4 \\ -4\lambda \end{bmatrix} ,$$

and it follows that $p_\lambda(\emptyset) \notin \mathbb{R}^2_+$ for any $\lambda \in \mathbb{R}$. Therefore, $S_\emptyset = \emptyset$. Now, for $\alpha = \{1\} \subset \{1, 2\}$ we get that

$$p_\lambda(\{1\}) = C_M(\{1\})\bar{q}_a(\lambda) = \begin{bmatrix} 4 - 4\lambda \\ 8 - 12\lambda \end{bmatrix} .$$

It follows that $p_\lambda(\{1\}) \in \mathbb{R}^2_+$ for $\lambda \in (-\infty, 2/3]$. Hence,

$$S_{\{1\}} = \left\{(x, \lambda) \in \mathbb{R}^2 \times [0, 2/3] \mid x = \begin{bmatrix} 4\lambda - 4 \\ 8 - 12\lambda \end{bmatrix}\right\}$$

Following a similar procedure for $\alpha = \{2\}$ and $\alpha = \{1, 2\}$ one gets the bifurcation diagrams shown on the left-hand side of Fig. 2.

Case b) We take the path

$$\bar{q}_b(\lambda) = (1 - \lambda)\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -1 \end{bmatrix} . \tag{8}$$

As in the previous case, we need to solve a family of constrained linear problems. Simple computations lead us to the diagram on the right-hand side of Fig. 2.

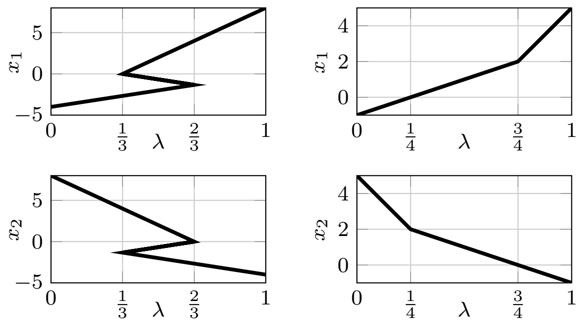

Fig. 2. Solutions to problem (6) for M given by (4) and paths $\bar{q}_a$ as in (5) (left), and $\bar{q}_b$ as in (8) (right), both with $\lambda \in [0, 1]$.

It is clear that, as long as the path $\bar{q}(\lambda)$ lies in the interior of the same cone, or set of cones, the number of solutions cannot change. Exiting and/or entering a cone, that is, crossing a cone face is thus a necessary condition for a bifurcation to occur. It is not sufficient though. For instance, in Example 3 Case b) above, the path $\bar{q}_b(\lambda)$ crosses different cones at the points $\lambda \in \{\frac{1}{4}, \frac{3}{4}\}$. However, there is no change in the number of solutions, see Fig. 2, right. This last observation poses the following question: How can we characterize the face at which bifurcations occur?

The nonsmooth Implicit Function Theorem (see Corollary at page 256 of Clarke (1990)) provides an answer to the last question. Let Ω_f be the set of measure zero where the Jacobian $Df(x)$ of a Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ does not exist.

Definition 4. (Clarke generalized Jacobian). The generalized Jacobian of f at x is the set

$$\partial f(x) = \mathrm{co}\left\{ \lim_{i\to\infty} Df(x_i) \mid x_i \to x,\ x_i \notin S,\ x_i \notin \Omega_f \right\} ,$$

where S is any set of measure zero and co denotes convex closure.

Definition 5. $\partial f(x)$ is said to be of *maximal rank* if every M in $\partial f(x)$ is non-singular.

For a function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $F : (x, y) \to F(x, y)$, the generalized Jacobian with respect to the first argument, denoted by $\partial_x F(x, y)$, is the set of all $n \times n$ matrices M such that $[M\ \ N]$ belongs to $\partial F(x, y)$ for some $n \times m$ matrix N.

Theorem 6. Suppose that $F(x_0, y_0) = 0$ and its generalized Jacobian $\partial_x F(x_0, y_0)$ is of maximal rank. Then there exist a neighborhood U of y_0 and a Lipschitz function $\bar{x} : U \to \mathbb{R}^n$ such that $F(\bar{x}(y), y) \equiv 0$ for all $y \in U$.

By specializing this theorem to (3) with $F(x, q) = f_M(x) - q$, it follows that a solution (x_0, q_0) to an LCP can be a bifurcation point only if $\partial f_M(x^*)$ is not of maximal rank, that is, if there exists a singular matrix M_0 belonging to the set $\partial f_M(x^*)$. This motivates the following definition.

Definition 7. A solution point (x_0, q_0) of (3) such that $\partial f_M(x_0)$ is not maximal rank is called a nonsmooth singularity.

Observe that $\partial f_M(x) = \mathrm{co}\{C_{-M}(\alpha) \mid x \in \mathrm{pos}\, C_I(\alpha)\}$. Thus, $\partial f_M(x)$ is a singleton if x belongs to the interior of an orthant or the convex closure of a (finite) set of matrices if x belongs to the face between two or more orthants.

The following proposition helps in finding nonsmooth singular points.

Proposition 8. Let x_0 be a solution of the LCP (M, q_0). If there exists $M_+ \in \partial f_M(x_0)$ such that $\det(M_+) > 0$ and $M_- \in \partial f(x_0)$ such that $\det(M_-) < 0$, then $\partial f_M(x_0)$ is not maximal rank.

As an application of Proposition 8, let us consider Example 3 above. Note that ∂f_M can be set-valued only for $\{x \in \mathbb{R}^n \mid f_M(x) \in \mathrm{bdr}\,\mathrm{pos}\, C_M(\alpha),\ \alpha \subseteq \{1, \ldots, n\}\}$. With M as in (4), the generalized Jacobian at the coordinate axes is single-valued and nonsingular for all points $x \in \mathbb{R}^n$ not on the coordinate axis. For $x \in \mathrm{pos}\, C_I(\emptyset) \cap \mathrm{pos}\, C_I(\{1\})$ and $x \in \mathrm{pos}\, C_I(\emptyset) \cap \mathrm{pos}\, C_I(\{2\})$ we have

$$\partial f_M(x) = \left\{ \begin{bmatrix} 1 & 0 \\ 2 - 2\mu & 1 \end{bmatrix}, \mu \in [0, 1] \right\}$$

and

$$\partial f_M(x) = \left\{ \begin{bmatrix} 1 & 2 - 2\mu \\ 0 & 1 \end{bmatrix}, \mu \in [0, 1] \right\},$$

respectively. The generalized Jacobians are multivalued but of maximal rank. For $x \in \mathrm{pos}\, C_I(\{1, 2\}) \cap \mathrm{pos}\, C_I(\{1\})$ and $x \in \mathrm{pos}\, C_I(\{1, 2\}) \cap \mathrm{pos}\, C_I(\{2\})$, we have

$$\partial f_M(x) = \left\{ \begin{bmatrix} 1 & 2\mu \\ 2 & 1 \end{bmatrix}, \mu \in [0, 1] \right\}$$

and

$$\partial f_M(x) = \left\{ \begin{bmatrix} 1 & 2 \\ 2\mu & 1 \end{bmatrix}, \mu \in [0, 1] \right\},$$

which are not of maximal rank. It follows from Proposition 8 that solutions of $f_M(x) - q = 0$ satisfying $x_1 = 0$ or $x_2 = 0$ are nonsmooth singular points (cf. the left-hand side of Fig. 2). In contrast, it follows directly from Definition 7 that all solutions of Case b) in Example 3 are regular, see the right-hand side of Fig. 2. It is worth remarking that, in order to have a singularity it is not necessary that $\det(C_{-M}(\alpha)) = 0$ for some α.

When $\det(C_{-M}(\alpha)) = 0$ for some α such that $q_0 \in \mathrm{pos}\, C_{-M}(\alpha)$, another source of singularities appears. In this case, the cone $\mathrm{pos}\, C_{-M}(\alpha)$ is *degenerate*, in the sense that its n-dimensional interior is empty (Danao, 1994). We expect the crossing of degenerate cones to induce nonsmooth bifurcations because, at the crossing of degenerate cones, there is necessarily a continuum of solutions. Indeed, if $\det(C_{-M}(\alpha)) = 0$, the full orthant $\mathrm{pos}\, C_I(\alpha)$ is mapped by f_M onto the lower-dimensional degenerate cone $\mathrm{pos}\, C_{-M}(\alpha)$. Thus, given $q \in \mathrm{pos}\, C_{-M}(\alpha)$, there must exist a (locally linear) subset of $\mathrm{pos}\, C_I(\alpha)$ that is mapped by f_M to q (Danao, 1994).

Example 9. Let us consider the degenerate matrix

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and the path $\bar{q}_a$ as in (5). For $\alpha = \{1, 2\}$, solutions of (6) are characterized by the expression

$$\begin{bmatrix} 4\lambda - 4 \\ -4\lambda \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x \in \mathrm{pos}\, C_I(\{1, 2\})$$

Note that the above equation has a nonempty solution set $S_{\{1,2\}}$ if and only if $4\lambda - 4 = -4\lambda$, that is, if and only if $\lambda = \frac{1}{2}$. Hence, for $\lambda = \frac{1}{2}$ the solution set is given by

$$S_{\{1,2\}} = \Big\{(x, \lambda) \in \mathbb{R}^2 \times \Big\{\frac{1}{2}\Big\} \mid x = \begin{bmatrix} \mu \\ -2-\mu \end{bmatrix}, \mu \in [-2, 0]\Big\},$$

Therefore, the solution set $S_{\{1,2\}}$ has an infinite number of solutions for a single value of λ, which corresponds to the situation in which the path $\bar{q}_a$ intersects the degenerate cone $C_M(\alpha)$.

We summarize the results of this section as follows.

- Nonsmooth bifurcations can happen when the path defined by $q = \bar{q}(\lambda)$ crosses a face of non-degenerate cones, or at the crossing of degenerate cones.
- Crossing a degenerate cone always leads to bifurcations.
- The presence and nature of a bifurcation when crossing a face of a non-degenerate cone depends on the nature and disposition of the other cones that share that face.

It follows that nonsmooth bifurcations in LCPs are essentially determined by: i) the complementary cone configuration; ii) how the path moves across them.

3. MAIN RESULTS

Similarly to smooth bifurcation theory, it is possible to use equivalence relations to provide an exhaustive list of the possible bifurcation phenomena. We start here this program by deriving a notion of *equivalence between LCPs*, which will provide equivalence classes of cone configurations. The relevance of this notion in classifying nonsmooth bifurcation problems will be then illustrated.

3.1 Equivalence between cone configurations

Our notion of equivalence between LCPs (M, q) and (N, r) has topological and algebraic components. The algebraic component captures the relations among the complementary cones that M and N generate. The relevant algebraic structure is that of a Boolean algebra, a subject that we now briefly recall (see Sikorski (1969) for more details).

Let X be a set and $\mathcal{P}(X)$ the power set on X. A *field of sets* is a pair $(X, \mathcal{F})$ where $\mathcal{F} \subset \mathcal{P}(X)$ is closed under intersections of pairs of sets and complements of individual sets (this implies closure under union of pairs of sets).

Let $\mathcal{G}$ be a subset of $\mathcal{P}(X)$. The field of sets *generated* by $\mathcal{G}$ is the intersection of all the fields of sets that contain $\mathcal{G}$.

A field of sets is a concrete example of a Boolean algebra and, as such, the usual algebraic concepts apply to them.

Definition 10. A *Boolean homomorphism* from the field $(X, \mathcal{F})$ onto the field $(X', \mathcal{F}')$ is a mapping $h : \mathcal{F} \to \mathcal{F}'$ such that $h(P_1 \cap P_2) = h(P_1) \cap h(P_2)$ and $h(-P_1) = -h(P_1)$ for all $P_1, P_2 \in \mathcal{F}$. Here, $-P_1$ denotes the complement of P_1. A one-to-one Boolean homomorphism h is called a *Boolean isomorphism.* An isomorphism of a field onto itself is called a *Boolean automorphism.*

Definition 11. A Boolean mapping $h : \mathcal{F} \to \mathcal{F}'$ is said to be *induced* by a mapping $\varphi : X' \to X$ if $h(P) = \varphi^{-1}(P)$ for every set $P \in \mathcal{F}$.

Corollary 12. (Sikorski (1969)). Let $\mathcal{F}$ be a field generated by $\mathcal{G}$. If a bijection $g : \mathcal{G} \to \mathcal{G}'$ is induced by a bijection $\varphi : X' \to X$, then g can be extended to a Boolean isomorphism $h : \mathcal{F} \to \mathcal{F}'$.

Now, consider the collection $\mathcal{G}_M = \{\text{pos}\, C_M(\alpha)\}_\alpha$, and let $(\mathbb{R}^n, \mathcal{F}_M)$ be the field of sets generated by $\mathcal{G}_M$. We are now ready to state our main definition.

Definition 13. Two matrices $M, N \in \mathbb{R}^{n\times n}$ are said to be *LCP equivalent*, $M \sim N$, if there exists topological isomorphisms (i.e., homeomorphisms) $\phi, \psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_M = \varphi \circ f_N \circ \psi$, where ψ induces a Boolean automorphism on $\mathcal{F}_I$.

The commutative diagram $f_M = \varphi \circ f_N \circ \psi$ is standard in the literature of singularity theory and ensures that we can continuously map solutions of the problem $f_M(x) = q$ into solutions of the problem $f_N(x') = \varphi^{-1}(q)$. The requirement on ψ being a Boolean automorphism implies that ψ maps orthants into orthants, intersections of orthants into intersections of orthants, and so forth.

Theorem 14. The matrices $M, N \in \mathbb{R}^n$ are LCP equivalent if, and only if, there exists a bijection $g : \mathcal{G}_M \to \mathcal{G}_N$ induced by a homeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$.

Remark 15. It follows from Corollary 12 that a necessary condition for $M \sim N$ is the existence of a bijection $g : \mathcal{G}_M \to \mathcal{G}_N$ that extends to an isomorphism $h : \mathcal{F}_M \to \mathcal{F}_N$.

Example 16. Consider the matrices

$$M = \begin{bmatrix} -1 & 1 \\ 0.9 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} -1 & 1 \\ 1.1 & -1 \end{bmatrix} \quad \text{and} \quad O = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

Their cone configurations are shown in Fig. 3. By Remark 15, M and N are not equivalent. This is intuitively clear since, depending on the location of q, there can be none, two, or four solutions to the LCP (M, q); whereas, depending on the location of r there can be either one or three solutions to the LCP (N, r).

Although N and O are fairly 'distant' from each other, they are LCP equivalent, as they satisfy the conditions of Theorem 14 (see Fig. 3).

In the example, M and N are not equivalent, even though they are 'close' to each other. This issue takes us to the following concept.

Definition 17. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be *LCP stable* if it is LCP equivalent to every matrix that is sufficiently close to it.

3.2 Classification of LCPs on the plane

The following results will provide a characterization of equivalence classes of stable matrices in $\mathbb{R}^{2\times 2}$.

Lemma 18. Let $M \in \mathbb{R}^{2\times 2}$. If $M_{12}, M_{21} \neq 0$ and $\det(M_{\alpha\alpha}) \neq 0$, for all $\alpha \subseteq \{1, 2\}$, then M is stable.

Theorem 19. Two matrices $M, N \in \mathbb{R}^{2\times 2}$ are equivalent if $M_{12} \cdot N_{12} > 0$, $M_{21} \cdot N_{21} > 0$ and $\det(M_{\alpha\alpha}) \cdot \det(N_{\alpha\alpha}) > 0$ for $\alpha \subseteq \{1, 2\}$.

Corollary 20. Let $M \in \mathbb{R}^{2\times 2}$ with $\det(M_{\alpha\alpha}) = 0$ for some $\alpha \subseteq \{1, 2\}$, then M is not stable.

The results of this section provides a list of "normal forms" to explore equivalence classes of stable matrices in $\mathbb{R}^{2\times 2}$. The matrices

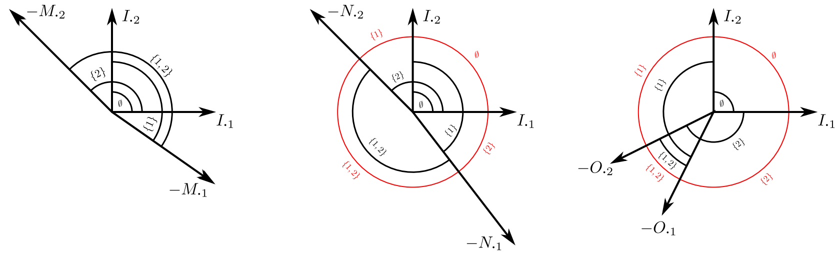

Fig. 3. Complementary cones of the matrices M, N and O in Example 16, depicted by black arcs. The cones generated by $C_{\bar{N}}(\alpha)$ and $C_{\bar{O}}(\alpha)$ are depicted by red arcs. The matrices M and N are not equivalent, but N and O are, as their complementary cones have the same Boolean structure.

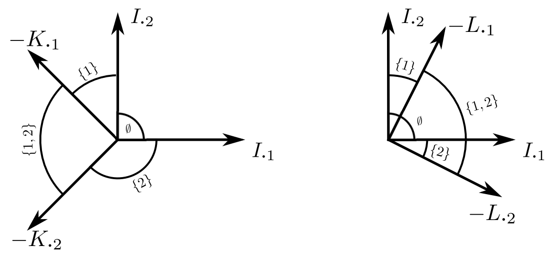

Fig. 4. Complementary cones of the matrices K (left) and L (right) defined in (9), depicted by black arcs.

$$M_\delta = \begin{bmatrix} \delta_1 & \delta_3 \\ -\delta_3(2\delta_0 - \delta_1\delta_2) & \delta_2 \end{bmatrix}$$

and

$$N_\delta = \begin{bmatrix} \delta_1 & \delta_3 \\ -\delta_3(0.5\delta_0 - \delta_1\delta_2) & \delta_2 \ , \end{bmatrix}$$

with $\delta_i \in \{-1, 1\}$, $i = 0, \dots, 3$ span all possible sign combinations in the statement of Theorem 19. Thus, any stable matrix satisfying the condition of the theorem is equivalent to M_δ, for some combination of $\delta_i \in \{-1, 1\}$, $i = 0, \dots, 4$. By varying the parameters of M_δ, we can construct an explicit list of equivalence classes of stable bidimensional matrices. The constructed list might not be exhaustive, but by Corollary 20 what is left out from this classification is the zero-measure set of matrices satisfying $\det(M_{\alpha\alpha}) \neq 0$, for all $\alpha \subseteq \{1, 2\}$, but $M_{12}M_{21} = 0$. Stability and equivalence class of matrices in this zero-measure set are assessed *a posteriori* on a case-by-case basis in the normal form matrix

$$O_\delta = \begin{bmatrix} \delta_1 & \delta_3 \\ \delta_4 & \delta_2 \end{bmatrix}$$

with $\delta_1, \delta_2 \in \{-1, 1\}$ and $\delta_3, \delta_4 \in \{-1, 0, 1\}$, $\delta_3\delta_4 = 0$.

After studying the cone structure of each of these matrices, we conclude that there are only four classes of LCP stable matrices in $\mathbb{R}^{2\times 2}$. Representative members of two different classes are the matrices M and N, defined in Example 16. Two more representative matrices are

$$K = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} -0.5 & -1 \\ -1 & 0.5 \end{bmatrix} . \tag{9}$$

Since $\mathcal{G}_K$ partitions $\mathbb{R}^2$ (see Fig. 4), the LCP (K, q) has a unique solution for every q. A matrix with this property is called a P-matrix (Cottle et al., 2009). The complementary cones of L are also shown in Fig. 4. Depending on q, the LCP (L, q) may either have two or no solutions.

3.3 Bifurcation realization via LCP interconnection

The strong link between piecewise linear functions and LCPs, pointed out in Proposition 2, motivates us to restrict ourselves to piecewise linear paths through cone configurations. In this setting, the path itself can be generated from the solution set of another LCP (Garcia et al., 1983). This approach naturally leads us towards an interconnection framework reminiscent of circuit theory, in the sense that an intricate high-dimensional LCP is treated as the result of the *interconnection* of simpler LCPs. Proceeding in this way we prove that, by selecting appropriate *inputs* and *outputs*, the *feedback* interconnection of LCPs is again an LCP. Afterwards, we use this decomposition approach to obtain the unfoldings of the pitchfork singularity.

We start by considering two linear complementarity problems in their z-coordinates, that is,

$$w_k = M_k z_k + \bar{q}_k, \quad \mathbb{R}^{n_k}_+ \ni w_k \perp z_k \in \mathbb{R}^{n_k}_+ \ .$$

where $M_k \in \mathbb{R}^{n_k \times n_k}$ and $\bar{q}_k \in \mathbb{R}^{n_k}$, for $k \in \{a, b\}$. Let $z_k \in \mathbb{R}^{n_k}$ be the *output* of the k-th LCP and let $\bar{q}_k \in \mathbb{R}^{n_k}$ take the role of *input*. Additionally, consider the interconnection rule

$$\bar{q}_a = H_a z_b + \bar{\theta}_a \ , \quad \bar{q}_b = H_b z_a + \bar{\theta}_b \ , \tag{10}$$

where $H_a \in \mathbb{R}^{n_a \times n_b}$, $H_b \in \mathbb{R}^{n_b \times n_a}$ and $\bar{\theta}_k \in \mathbb{R}^{n_k}$ are additional inputs available for further interconnection. With this convention we have the following result.

Proposition 21. The interconnection of linear complementarity problems under the pattern (10) is again a linear complementarity problem.

3.4 Realization of some nonsmooth bifurcations and their unfolding: A nonsmooth pleat

Let us consider the class of LCPs represented by the matrix O in Fig. 3. This class gives rise to the nonsmooth pleat shown in Fig. 5. The pleat is given by

$$\left\{ \begin{bmatrix} y_1 & y_2 & x_1 \end{bmatrix}^\top \in \mathbb{R}^3 \mid \exists\, x_2 \in \mathbb{R} \text{ such that } f_O(x) = y \right\} ,$$

where $f_O : \mathbb{R}^2 \to \mathbb{R}^2$ is the piecewise linear map defined in Proposition 2. It is worth to remark that the nonsmooth pleat is stable in the sense that the matrix O is LCP-stable.

In complete analogy with the smooth case, see e.g. (Golubitsky and Schaeffer, 1985, Chapter III.12), one can recover a large family of bifurcations from the pleat by selecting appropriate paths through it. We illustrate this with the pitchfork singularity and its unfoldings, but it is also possible to obtain the hysteresis and the cusp

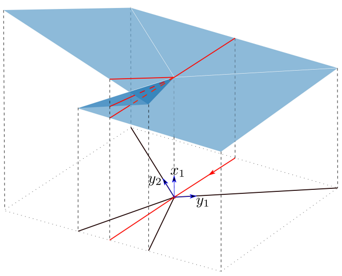

Fig. 5. Nonsmooth pleat, pitchfork path, and their projection to the plane (y_1, y_2). The black lines on the plane are generators of complementary cones.

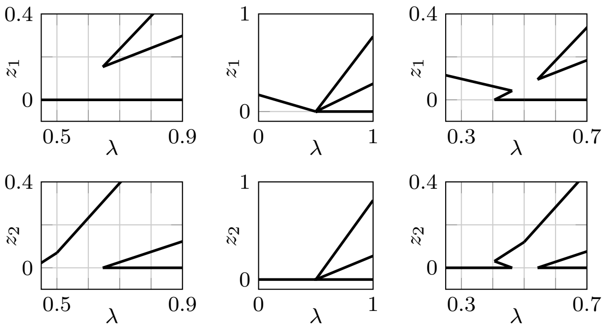

Fig. 6. The pitchfork singularity and its unfoldings.

singularities and their unfoldings by changing the path in a suitable way.

Let us consider the LCP $(M_b, \bar{q}_b)$ associated to the nonsmooth pleat shown in Fig. 5 with matrix $M_b = 2O$ and O as in Example 16. We consider the auxiliary LCP $(M_a, \bar{q}_a)$ with $M_a = 1$ and path $\bar{q}_a(\lambda) = 2\lambda - 1$. The LCP $(M_a, \bar{q}_a)$ has a unique solution for every $\lambda \in \mathbb{R}$ which is computed easily as $z_a(\lambda) = 0$ for $\lambda < \frac{1}{2}$ and $2\lambda - 1$ for $\frac{1}{2} \leq \lambda$. We thus set the path $\bar{q}_b(\lambda)$ as

$$\bar{q}_b(\lambda) = R_s \begin{bmatrix} z_a(\lambda) \\ \lambda \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} ,$$

where R_s is a rotation matrix, s is the angle of rotation and the parameters μ_1, μ_2 are extra degrees of freedom that will allow us to change the path $\bar{q}_b(\lambda)$ on the pleat. Equivalently, the resulting LCP can be seen as the interconnection between LCP $(M_a, \bar{q}_a)$ and LCP $(M_b, \bar{q}_b)$ under the interconnection rule (10) with

$$\begin{aligned} H_a &= 0 \,, & H_b &= [\cos s \;\; \sin s]^\top \,, \\ \bar{\theta}_a &= \mu_1 - \lambda \sin s \,, & \bar{\theta}_b &= \mu_2 + \lambda \cos s \,. \end{aligned}$$

Let us fix $s = \frac{10}{9}\pi$. By varying the parameters μ_1 and μ_2 we are able to displace the path $\bar{q}_b(\lambda)$ on the pleat. The associated bifurcation diagrams are shown in Fig. 6. Note that the central bifurcation diagram corresponds to the pitchfork organizing center, whereas perturbations of this path lead to any of the left or right-hand side diagrams.

4. DISCUSSION AND FURTHER DEVELOPMENTS

We have presented a notion of global equivalence between LCPs that allows us to make a classification of this problems in the planar case. In addition, an interconnection approach for the realization of nonsmooth bifurcations was presented. These tools are thought to be handful for many applications as, for instance, the analysis and design of neuromorphic circuits (Castaños and Franci, 2017), the study of economic equilibria in competitive markets, and the analysis of elastic-plastic structures in engineering, just to name a few. This work also opens the path towards the analysis of behaviors in dynamical linear complementarity systems.

REFERENCES

Acary, V., Bonnefon, O., and Brogliato, B. (2011). *Nonsmooth modeling and simulation for switched circuits*. Lecture Notes in Electrical Engineering. Springer.

Arnold, V.I., Varchenko, A.N., and Gusein-Zade, S.M. (1985). *Singularities of Differentiable Maps*, volume 1. Birkhäuser.

Brogliato, B. (1999). *Nonsmooth mechanics: models, dynamics and control*. Springer-Verlag, London, second edition.

Castaños, F. and Franci, A. (2017). Implementing robust neuromodulation in neuromorphic circuits. *Neurocomputing*, 233, 3–13.

Castaños, F., Miranda-Villatoro, F., and Franci, A. (2019). A notion of equivalence for linear complementarity problems with application to the design of non-smooth bifurcations. ArXiv:1911.03774.

Clarke, F. (1990). *Optimization and nonsmooth analysis*. Society for Industrial and Applied Mathematics.

Cottle, R.W., Pang, J.S., and Stone, R.E. (2009). *The Linear Complementarity Problem*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics.

Danao, R.A. (1994). Q-matrices and boundedness of solutions to linear complementarity problems. *Journal of Optimization Theory and Applications*, 83(2), 321–332.

Di Bernardo, M., Budd, C.J., Champneys, A.R., Kowalczyk, P., Nordmark, A.B., Tost, G.O., and Piiroinen, P.T. (2008). Bifurcations in nonsmooth dynamical systems. *SIAM review*, 50(4), 629–701.

Garcia, C.B., Gould, F.J., and Turnbull, T.R. (1983). Relations between PL maps, complementary cones, and degree in linear complementarity problems. In *Homotopy Methods and Global Convergence*, 91–144. Plenum Publishing Corporation.

Golubitsky, M. and Schaeffer, D. (1985). *Singularities and Groups in Bifurcation Theory*, volume I of *Applied Mathematical Sciences*. Springer, New York.

Leenaerts, D.M. and Bokhoven, W.M.G. (1998). *Piecewise linear modeling and analysis*. Kluwer Academic. Springer, New York.

Leine, R.I. and Nijmeijer, H. (2004). *Dynamics and bifurcations on non-smooth mechanical systems*. Springer, Berlin.

Murty, K.G. (1988). *Linear complementarity, linear and nonlinear programming*. Helderman Verlag, Berlin.

Nagurney, A. (1999). *Network economics: A variational inequality approach*. Advances in Computational Economics. Springer-Science+Business Media.

Sikorski, R. (1969). *Boolean Algebras*. Springer-Verlag, Berlin, Germany.

van der Schaft, A.J. and Schumacher, J. (1998). Complementarity modeling of hybrid systems. *IEEE Trans. Autom. Control*, 43, 483 – 490.