Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Antiwindup Design Approach to Constrained Primal-Dual Dynamics ⋆

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Abstract: An antiwindup control framework is developed for primal-dual dynamics in convex optimization. The proposed architecture provides for straightforward implementation of the ensuing primal-dual dynamics and for establishing global asymptotic stability using the notion of shifted-passivity with Lyapunov function commonly encountered in antiwindup literature.

Keywords: Primal-Dual Dynamics, Constrained Control, Shifted Passivity, Antiwindup Compensator, Constrained Optimization

1. INTRODUCTION

Primal-dual dynamics belong to the class of continuous-time gradient based algorithms for real-time solution of convex optimization problems (Arrow et al., 1958). The need for fast, efficient and architecturally simple computational methods in resource constrained and in critical infrastructure systems has made the primal-dual dynamics approach a method of choice (Ma and Elia, 2015; Stegink et al., 2015). As a result, the primal-dual dynamics approach has attracted significant research interest seeking to establish provable stability and robustness guarantees (Feijer and Paganini, 2010; Cherukuri et al., 2016; Nguyen et al., 2018; Qu and Li, 2019), and to develop mechanisms for hardware circuit realization (Kose, 1956; Vichik et al., 2016; Levenson and Adegbege, 2016; Skibik and Adegbege, 2018) and for embedding such computational methods in closed loop as a dynamic controller (Jokic et al., 2009; Nicotra et al., 2018; Yoshida et al., 2019).

For equality constrained convex optimization problems, the corresponding primal-dual dynamics can be interpreted as the feedback interconnection of a gradient system and a linear integral controller (Ishizaki et al., 2016; Adegbege, 2019). Such interpretation has been exploited for output regulation of a class of nonlinear problems with convex properties (Jayawardhana et al., 2007; Stegink et al., 2015). However, for problems with inequality constraints, additional nonlinearity must be introduced at the input of the gradient system to account for the violation of the inequality constraint. To address this, previous works have focused on reformulating the primal-dual dynamics has a projected dynamical system Cherukuri et al. (2016) or as a switched system (Feijer and Paganini, 2010; Kosaraju et al., 2018) and invoking tools from hybrid control to establish stability. Others have employed a proximal-like Lagrangian reformulation of the problem, and then establishing stronger stability results using non-diagonal Lyapunov functions (Qu and Li, 2019; Ding and Jovanović, 2019). In Adegbege (2020), an artificial switching element is introduced at the input of the controller to counter the effect of the input nonlinearity and to enforce the optimality of the equilibrium condition of the modified primal-dual dynamics.

In this work, we adopt a different approach. We interpret the primal-dual dynamics within the context of antiwindup control systems (Mulder et al., 2001; Gomes da Silva Jr et al., 2014). We consider the interconnection without input nonlinearity as the nominal control loop for which a suitable linear controller has already been designed. The controller ensures that the equilibrium point of the primal-dual dynamics is the optimal solution of a corresponding equality constrained convex optimization problem. Then an antiwindup augmentation is incorporated into the closed loop to account for the effect of the input nonlinearity. This additional augmentation is constructed such that the input-output passivity of the nominal interconnection is preserved while the equilibrium point solves the original inequality constrained convex optimization problem. By proper characterization of the input nonlinearity, asymptotic stability and convergence of the augmented system can be established using the notion of shifted passivity (Van der Schaft, 2000; Monshizadeh et al., 2019) and employing Lyapunov function similar to those in standard antiwindup control literature (Mulder et al., 2001).

The paper is structured as follows. In section 2, we provide the mathematical background necessary for setting up the proposed modified primal-dual dynamics and we introduce the notion of shifted passivity for constrained gradient dynamics. In section 3, we state our main result and contribution. We show that with antiwindup augmentation, a corrective signal is injected into the loop such that the modified interconnected system recovers the shifted-passivity property of the nominal closed loop. Global asymptotic stability of the equilibrium then follows from applying classical invariance principle. Finally in section 4, we provide a computational example to illustrate the effectiveness of the proposed approach to solving nonlinear optimization problems arising in model predictive control.

⋆ Financial support through TCNJ's Engineering Research Prize is acknowledged.

2. MATHEMATICAL PRELIMINARIES AND PROBLEM SETUP

2.1 Constrained Gradient Dynamics

We consider the gradient system

$$\dot{x} = -\nabla f(x) + g^T v \tag{1a}$$
$$y = gx \tag{1b}$$

where $f(x)$ is a continuously differentiable strongly convex function that maps from $\mathbb{R}^n$ to $\mathbb{R}$ and $\nabla f(x)$ denotes the gradient vector defined as $[\nabla f(x)]_i = (\partial f)\ (\partial x_i)$. The constant matrix $g \in \mathbb{R}^n \to \mathbb{R}^m$ is assumed to be full row-ranked. The vectors $v \in \mathbb{R}^m_+$ (nonnegative orthant) $\subset \mathbb{R}^m$ and $y \in \mathbb{R}^m$ are respectively the system input and the output.

The task is to construct an appropriate controller for (1) such that the equilibrium point x^* is globally asymptotically stable and corresponds to the optimum solution of the convex optimization problem:

$$\min_x f(x) \tag{2a}$$
$$\text{subject to } \; gx \geq w \tag{2b}$$

where $w \in \mathbb{R}^m$ may be time dependent. The classical gradient method for solving (2) involves the construction of the Lagrangian function:

$$L(x, \lambda) = f(x) + \lambda^T (w - gx) \tag{3}$$

where $\lambda \in \mathbb{R}^m_+$ is the vector of Lagrangian multipliers associated with the inequality constraint (2b), and then followed by the saddle-point dynamics (Feijer and Paganini, 2010; Cherukuri et al., 2016):

$$\eta_x \dot{x} = -\nabla_x L(x, \lambda) = -\nabla f(x) + g^T \lambda \tag{4a}$$
$$\eta_\lambda \dot{\lambda} = [\nabla_\lambda L(x, \lambda)]^+_\lambda = [w - gx]^+_\lambda; \tag{4b}$$

where η_x and η_λ are time constants for the x and λ trajectories respectively, and $[P]^+_\lambda$ denote the positive projection of P to ensure that λ remains in $\mathbb{R}^m_+$. It is typically defined elementwise as:

$$[P_i]^+_\lambda = \begin{cases} P_i & ; \ \lambda_i > 0 \text{ or } P_i > 0 \\ 0 & ; \ \text{otherwise.} \end{cases} \tag{5}$$

The presence of projection such as (5) in the primal-dual dynamics (4) renders the gradient flow discontinuous with associated implementation issues (Qu and Li, 2019). By proper characterization of the input nonlinearity, we avoid the use of projection while ensuring feasibility at equilibrium.

2.2 Input Nonlinearity Characterization

We define the input nonlinearity $v = \phi(u)$ with

$$\phi(u) = \begin{cases} \phi_1(u_1) \\ \vdots \\ \phi_m(u_m) \end{cases} \text{ and } \phi_i(u_i) = \begin{cases} u_i & u_i \geq 0 \\ 0 & u_i < 0 \end{cases} \tag{6}$$

to capture the satisfaction or the violation of the constraint. We also define a complementary nonlinearity

$$\psi(u) = \begin{cases} \psi_1(u_1) \\ \vdots \\ \psi_m(u_m) \end{cases} \text{ and } \psi_i(u_i) = \begin{cases} u_i & u_i < 0 \\ 0 & u_i \geq 0 \end{cases} \tag{7}$$

such that $\psi(u) + \phi(u) = u$. As constructed, the nonlinearity $\phi(\cdot)$ satisfies $\phi(0) = 0$, and the incremental sector condition (Desoer and Vidyasagar, 1975):

$$(v - v^*)^T((u - u^*) - (v - v^*)) \geq 0 \tag{8}$$

for all u, u^*, v, v^* satisfying $v = \phi(u)$ and $v^* = \phi(u^*)$.

When $\phi_i(u_i) = u_i$ for all i, the gradient system (1) is unconstrained and we say the nonlinearity is inactive. We will subsequently take advantage of this characterization to establish global asymptotic stability for the primal-dual dynamics.

2.3 Linear Controller

It is well known that the problem of regulating nonlinear problems of the form (1) to a non-zero equilibrium x^* requires integral control (Jayawardhana et al., 2007). Here we consider the proportional-integral controller:

$$\dot{\xi} = K_I(w - y) \tag{9a}$$
$$u = \xi + (w - y) \tag{9b}$$

where K_I is diagonal positive definite matrix. The integral part of the controller ensures that at equilibrium, the equality constraints $y^* = gx^* = w$ holds. To handle inequality constraints, the integral controller must be modified to guarantee $gx^* \geq w$ at equilibrium. We will explore this in section 3.

2.4 Shifted Passivity As Input-Output Property

As we are interested in the forced equilibrium of the constrained gradient dynamic, we rely on the concept of shifted passivity (Van der Schaft, 2000; Monshizadeh et al., 2019). Suppose we define the equilibrium condition of the gradient system (1) as

$$\mathcal{E} = \{(x^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^m_+ |\ \nabla f(x^*) - g^T v^* = 0\} \tag{10}$$

with the corresponding output as $y^* = gx^*$, shifted passivity is defined in terms of the input $v - v^*$ and the output $y - y^*$ of the incremental model

$$\dot{x} = -(\nabla f(x) - \nabla f(x^*)) + g^T(v - v^*), \tag{11a}$$
$$y - y^* = g(x - x^*) \tag{11b}$$

for a fixed (x^*, v^*).

Definition 1. The gradient system (1) is exponentially shifted passive (resp. shifted passive) if for a fixed $(x^*, v^*) \in \mathcal{E}$ and a corresponding $y^* = gx^*$, there exists a positive semidefinite function $V_x(x - x^*)$ such that:

$$\dot{V}_x = (\nabla V_x)^T \dot{x} \leq -\varepsilon V_x + (v - v^*)(y - y^*) \tag{12}$$

with a scalar $\varepsilon > 0$ (resp. $\varepsilon = 0$) for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m_+$.

Lemma 2. Suppose $(x^*, v^*) \in \mathcal{E}$ is fixed for the gradient system (1) with a strongly convex $f(x)$ and a full row-ranked g. Then the gradient system is exponentially shifted passive with the storage function

$$V_x(x - x^*) = \frac{1}{2}(x - x^*)^T(x - x^*). \tag{13}$$

Proof. By strong convexity of $f(x)$, the following monotonicity property (Boyd and Vandenberghe, 2004)

$$(x - x^*)^T(\nabla f(x) - \nabla f(x^*)) \geq \delta(x - x^*)^T(x - x^*) \tag{14}$$

holds for a positive constant δ. Since $(x^*, v^*) \in \mathcal{E}$ and g is full-row ranked, it follows that (cf. (Simpson-Porco, 2018, corollary 3.6), (Ishizaki et al., 2016, lemma 3)):

$$v^* = (gg^T)^{-1} g \nabla f(x^*) \quad \text{and} \quad x^* = \nabla f^{-1}(g^T v^*) \tag{15}$$

and thus, x^* and v^* are uniquely defined. Evaluating the derivative of V_x along (1) gives:

$$\begin{aligned} \dot{V}_x =& -(x - x^*)^T (\nabla f(x) - \nabla f(x^*)) + (x - x^*)^T g^T (v - v^*) \\ \leq& -\delta (x - x^*)^T (x - x^*) + (y - y^*)^T (v - v^*) \qquad (16) \\ =& -\varepsilon V_x + (y - y^*)^T (v - v^*) \qquad (17) \end{aligned}$$

with $\varepsilon = 2\delta$. Condition (17) is exactly the definition of exponentially shifted passivity.

It is well known that for linear systems, passivity implies shifted passivity for any forced equilibrium (Jayawardhana et al., 2007). For the proportional-integral controller (9), the equilibrium condition for a fixed w is $(y^* = w, u^* = \xi^*)$ and hence the incremental form is given by

$$\dot{\xi} = -K_I (y - y^*), \tag{18a}$$

$$u - u^* = (\xi - \xi^*) - (y - y^*). \tag{18b}$$

It is straightforward to show that with the storage function

$$V_k(\xi - \xi^*) = \frac{1}{2} (\xi - \xi^*) K_I^{-1} (\xi - \xi^*), \tag{19}$$

the controller satisfies the following shifted passivity condition with respect to $-(y - y^*)$ and $(u - u^*)$:

$$\dot{V}_k = -(y - y^*)^T (u - u^*) - (y - y^*)^T (y - y^*) \tag{20a}$$

$$\leq -(y - y^*)^T (u - u^*). \tag{20b}$$

For static nonlinearity (6), we define the following notions of shifted passivity (Monshizadeh et al., 2019).

Definition 3. The nonlinear map $v = \phi(u)$ with $0 = \phi(0)$ and satisfying $v^* = \phi(u^*)$ is output shifted passive (resp. shifted passive) if the following inequality holds

$$0 \leq (u - u^*)^T (v - v^*) - \rho (v - v^*)^T (v - v^*) \tag{21}$$

for scalar $\rho > 0$ (resp. $\rho = 0$) for all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^m_+$.

3. CONSTRAINED PRIMAL-DUAL DYNAMICS AS FEEDBACK INTERCONNECTION

We follow the two step antiwindup design approach (Mulder et al., 2001; Gomes da Silva Jr et al., 2014) where a nominal controller is first constructed ignoring the input nonlinearity i.e. setting $\phi(u) = u$ and then an antiwindup augmentation is introduced to account for the effect of the input nonlinearity in closed loop.

3.1 Ignoring Input Nonlinearity-Nominal Interconnection

For the nominal interconnection, we ignore the input nonlinearity such that the output of the linear controller (9) is coupled directly to the gradient system (1) with $v = u$. The resulting interconnection becomes:

$$\dot{x} = -\nabla f(x) + g^T u, \; y = gx \tag{22a}$$

$$\dot{\xi} = K_I (w - y) \tag{22b}$$

$$u = \xi + (w - y) \tag{22c}$$

where w can be considered an exogenous input. This is illustrated in Fig. 1. At equilibrium, we have

$$0 = -\nabla f(x^*) + g^T u^*, \tag{23a}$$

$$0 = w - gx^* \tag{23b}$$

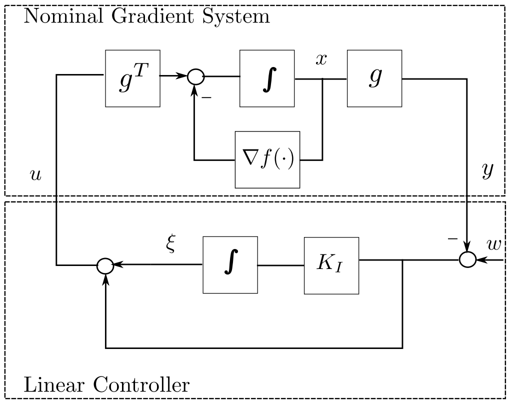

Fig. 1. Closed-Loop Interconnection with Linear Control

This is exactly the Karush Kuhn Tucker (KKT) optimality condition for the equality constrained optimization problem (Bertsekas, 2014):

$$\min_x f(x), \tag{24a}$$

$$\text{subject to} \;\; gx = w. \tag{24b}$$

To study the stability of the equilibrium point (x^*, ξ^*), we express (22) in incremental form relative to (x^*, ξ^*, u^*, y^*) where u^* and $y^* = gx^*$ are the corresponding points satisfying the equilibrium condition (23), and then employ the concept of shifted-passivity (Van der Schaft, 2000).

Defining the displacement variables $\hat{x} = x - x^*$, $\hat{\xi} = \xi - \xi^*$, $\hat{u} = u - u^*$ and $\hat{y} = y - y^*$, the incremental form of (22) is obtained as:

$$\dot{x} = -(\nabla f(x) - \nabla f(x^*)) + g^T \hat{u}, \; \hat{y} = g\hat{x} \tag{25a}$$

$$\dot{\xi} = -K_I \hat{y}, \; \hat{u} = \hat{\xi} - \hat{y} \tag{25b}$$

We state the following result.

Proposition 4. Consider the closed-loop system comprising the feedback interconnection of the gradient dynamics (1) and the proportional-integral controller (9) with $v = u$. Suppose that $f(x)$ is strongly convex and that g is full row ranked. Then the equilibrium point (x^*, ξ^*) is globally asymptotically stable. Moreover x^* is the optimal solution of (24).

Proof. We consider the Lyapunov function candidate

$$V = V_x + V_k = \frac{1}{2} \hat{x}^T \hat{x} + \frac{1}{2} \hat{\xi}^T K_I^{-1} \hat{\xi}. \tag{26}$$

By construction $V > 0$ for all $x \neq x^*$ and $\xi \neq \xi^*$. Evaluating the derivative of V along the trajectories of the incremental model (25) yields:

$$\dot{V} = -\hat{x}^T (\nabla f(x) - \nabla f(x^*)) + \hat{y}^T \hat{u} - \hat{u}^T \hat{y} - \hat{y}^T \hat{y} \tag{27a}$$

$$\leq -\varepsilon V - \hat{y}^T \hat{y} \leq 0. \tag{27b}$$

Note the first inequality follows from the exponential shifted passivity of the gradient dynamics and the shifted passivity of the controller. Since V is radially unbounded, all solution trajectories must be bounded (Wassim and Chellaboina, 20008). Define the invariant set $\mathcal{M} = \{(\hat{x}, \hat{\xi}) \in \mathbb{R}^n \times \mathbb{R}^m | V = 0\}$. Observe that the first term of (27b) vanishes only when $x = x^*$ and by continuity argument $\hat{x} \equiv 0 \to \hat{y} \equiv 0$. It then follows from (25b) that ξ is uniformly continuous and hence $\lim_{t \to \infty} \hat{\xi} = 0$. So the largest invariant set contained in $\mathcal{M}$ is the equilibrium point (x^*, ξ^*). We therefore conclude asymptotic

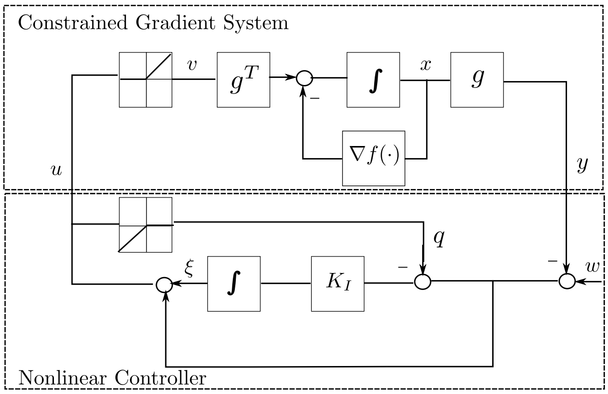

Fig. 2. Closed-Loop Interconnection with Control Augmentation

stability from the LaSalle invariance principle (Khalil, 2002). Global asymptotic stability follows from the radial unboundedness of V. Finally, since the equilibrium point (x^*, ξ^*) is globally asymptotic stable, the KKT condition (23) has a unique solution $(x^*, u^* = \xi^*)$ and hence x^* must be the unique optimal solution of (24).

3.2 Accounting for Input Nonlinearity-Antiwindup Control Augmentation

In the presence of input nonlinearity, the gradient system is no longer guaranteed to be shifted passive and hence the stability argument of Proposition 4 no longer holds. To counteract the effect of the input nonlinearity, we modify the controller by incorporating an antiwindup compensation in the form of a static feedback around the nonlinearity as follows (Gomes da Silva Jr et al., 2014):

$$\dot{x} = -\nabla f(x) + g^T v,\ y = gx, \tag{28a}$$

$$\dot{\xi} = K_I(w - y) + K_I(v - u),\ v = \phi(u), \tag{28b}$$

$$u = \xi + (w - y). \tag{28c}$$

Using the complementary nonlinearity $q = \psi(u)$ as defined in (7), the augmented interconnection can be restructured as depicted in Fig. 2. As shown, the effect of the control augmentation is to inject corrective signal q into the controller whenever $u < 0$. However for $u \in \mathbb{R}^m_+$, the corrective signal $q = 0$, and the interconnection recovers the nominal closed loop system of Fig. 1. We show subsequently that the antiwindup augmentation restores the shifted passivity of the gradient system and the controller, and hence asymptotic stability of the equilibrium can be established using similar augment of Proposition 4.

At equilibrium, we have

$$0 = -\nabla f(x^*) + g^T v^*, \tag{29a}$$

$$0 \geq w - gx^*, 0 \leq v^*, v^{*T}(w - gx^*) = 0. \tag{29b}$$

This is exactly the KKT optimality condition for the inequality constrained optimization problem (2) (Bertsekas, 2014). Expressing (28) in incremental form gives:

$$\dot{x} = -(\nabla f(x) - \nabla f(x^*)) + g^T \hat{v},\ \hat{y} = g\hat{x} \tag{30a}$$

$$\dot{\xi} = -K_I \hat{y} - K_I \hat{q},\ \hat{u} = \hat{\xi} - \hat{y} \tag{30b}$$

where $\hat{v} = \phi(u) - \phi(u^*)$ and $\hat{q} = \psi(u) - \psi(u^*)$.

We now state our main result.

Proposition 5. Consider the closed-loop system (28) comprising the feedback interconnection of the gradient dynamics (1) with $v = \phi(u)$ and the augmented nonlinear controller (28b)-(28c) controller. Suppose that $f(x)$ is strongly convex and that g is full row ranked. Then the equilibrium point (x^*, ξ^*) is globally asymptotically stable. Moreover x^* is the optimal solution of (2).

Proof. We employ the candidate Lyapunov function

$$V = V_x + V_k + V_\psi \tag{31}$$

where V_x and V_k are as defined in (13) and (19) respectively, and V_ψ is introduced to account for incremental sector boundedness of the complementary nonlinearity $\psi(\cdot)$ and it is specified as (cf. Mulder et al. (2001)):

$$V_\psi = \sum_{i=1}^{m} \int_0^t (q_i(\tau) - q_i^*)\left((u_i(\tau) - u_i^*) - (q_i(\tau) - q_i^*)\right) d\tau.$$

Note that V_ψ is nonnegative, $V_x > 0$ for all $\hat{x} \neq 0$ and $V_k > 0$ for all $\hat{\xi} \neq 0$, so we have $V > 0$ for all $(\hat{x} \neq 0, \hat{\xi} \neq 0)$. Since $\hat{x} \equiv 0 \Rightarrow \hat{y} \equiv 0$ and $(\hat{y} \equiv 0, \hat{\xi} \equiv 0) \Rightarrow \hat{u} \equiv 0 \Rightarrow \hat{q} \equiv 0$, we also have $V(0, 0) = 0$.

Evaluating the derivate of V along the incremental model (30) gives:

$$\begin{aligned} \dot{V} = &-\hat{x}^T(\nabla f(x) - \nabla f(x^*)) + \hat{y}^T \hat{u} - \hat{y}^T \hat{q} \\ &-\hat{u}^T \hat{y} - \hat{y}^T \hat{y} - \hat{u}^T \hat{q} - \hat{y}^T \hat{q} + \hat{q}^T \hat{u} - \hat{q}^T \hat{q} \end{aligned} \tag{32a}$$

$$= -\hat{x}^T(\nabla f(x) - \nabla f(x^*)) - (\hat{y} + \hat{q})^T(\hat{y} + \hat{q}) \tag{32b}$$

$$\leq -\varepsilon V_x - (\hat{y} + \hat{q})^T(\hat{y} + \hat{q}) \leq 0. \tag{32c}$$

Observe that the first inequality is due to the exponential shifted passivity of the map $\hat{u} \to (\hat{y} + \hat{q})$ and the shifted passivity of the map $-(\hat{y} + \hat{q}) \to \hat{u}$.

To establish asymptotic stability we follow similar route as in the proof of Proposition 4. Let $\mathcal{M} = \{(\hat{x}, \hat{\xi}) \in \mathbb{R}^n \times \mathbb{R}^m | V = 0\}$. Observe that V_x vanishes only at $\hat{x} = 0$. Also, the second term in (32c) vanishes when $\hat{y} = -\hat{q}$. But as $\hat{x} \equiv 0 \Rightarrow \hat{y} \equiv 0$ and $(\hat{y} \equiv 0, \hat{\xi} \equiv 0) \Rightarrow \hat{u} \equiv 0 \Rightarrow \hat{q} \equiv 0$, the largest invariant set in $\mathcal{M}$ in the equilibrium point (x^*, ξ^*). Asymptotic stability follows from the LaSalle invariance principle. Since V is a global Lyapunov function, we conclude global asymptotic stability of the equilibrium point.

4. NUMERICAL EXAMPLE

To illustrate the utility of the proposed approach to optimization problem arising in model predictive control (MPC) (Maeder et al., 2009), we consider a continuous-time system with the following dynamics:

$$\dot{x}_p(t) = (a - 1)x_p(t) + x_u(t) + d(t),\ a < 1 \tag{33}$$

$$y_p(t) = x_p(t) \tag{34}$$

where $x_p \in \mathbb{R}$ is the plant state, $x_u \in \mathbb{R}$ is the control input which must satisfy the constraint $x_u \geq 0$ and $d \in \mathbb{R}$ is an input disturbance. Suppose the output y_p is to track a zero reference with zero offset at steady state. To achieve this, we employ linear offset-free model predictive control (Maeder et al., 2009). First, we discretized the plant using forward difference approximation and a sampling time of 1 second to obtain

$$x_p[k + 1] = ax_p[k] + x_u[k] + d[k],\ a < 1 \tag{35}$$

$$y_p[k] = x_p[k] \tag{36}$$

and we adopt the following unity prediction horizon model predictive control:

$$\min_{x_u[0]} (x_u[0] - x_u^t)^2 + (x_p[1] - x_p^t)^2 \tag{37a}$$

$$\text{subject to} \quad x_u[0] \geq 0, \tag{37b}$$

$$x_p[1] = a x_p[0] + x_u[0] + d[0], \tag{37c}$$

$$\begin{bmatrix} a-1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_p^t \\ x_u^t \end{bmatrix} = \begin{bmatrix} -d[0] \\ 0 \end{bmatrix}, \tag{37d}$$

$$x_p[0] = \hat{x}_p[k], \ d[0] = \hat{d}[k], \tag{37e}$$

where x_p^t and x_u^t are the steady-state targets enforced by (37d), and $\hat{x}_p[k]$ and $\hat{d}[k]$ are the observed values at the current time instant. The MPC can be reduced to the following quadratic program (QP) problem that has to be solved at every time instant:

$$\min_{x_u[0]} \ 2x_u[0]^2 + 4\left(d[0] + \frac{1}{2} a x_p(0)\right) x_u[0] + \text{constant terms} \tag{38a}$$

$$\text{subject to} \quad x_u[0] \geq 0 \tag{38b}$$

The QP has a closed-form solution

$$x_u^*[k] = \begin{cases} -\hat{d}[k] - \frac{1}{2} a \hat{x}_p[k] & -\hat{d}[k] - \frac{1}{2} a \hat{x}_p[k] \geq 0 \\ 0 & -\hat{d}[k] - \frac{1}{2} a \hat{x}_p[k] < 0 \end{cases} \tag{39}$$

Using the proposed continuous-time solver, we obtain the controller dynamics without antiwindup augmentation as:

$$\begin{bmatrix} \dot{x}_u \\ \dot{\xi} \end{bmatrix} = \begin{cases} \begin{bmatrix} -5 & 1 \\ -k_i & 0 \end{bmatrix} \begin{bmatrix} x_u \\ \xi \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} (\hat{d}(t) + \frac{1}{2} a \hat{x}_p(t)) \\ \text{for } \sigma(\xi - x_u) = \xi - x_u, \\ \begin{bmatrix} -4 & 0 \\ -k_i & 0 \end{bmatrix} \begin{bmatrix} x_u \\ \xi \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} (\hat{d}(t) + \frac{1}{2} a \hat{x}_p(t)) \\ \text{for } \sigma(\xi - x_u) = 0, \end{cases} \tag{40}$$

and with antiwindup augmentation as:

$$\begin{bmatrix} \dot{x}_u \\ \dot{\xi} \end{bmatrix} = \begin{cases} \begin{bmatrix} -5 & 1 \\ -k_i & 0 \end{bmatrix} \begin{bmatrix} x_u \\ \xi \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} (\hat{d}(t) + \frac{1}{2} a \hat{x}_p(t)) \\ \text{for } \sigma(\xi - x_u) = \xi - x_u, \\ \begin{bmatrix} -4 & 0 \\ 0 & -k_i \end{bmatrix} \begin{bmatrix} x_u \\ \xi \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} (\hat{d}(t) + \frac{1}{2} a \hat{x}_p(t)) \\ \text{for } \sigma(\xi - x_u) = 0. \end{cases} \tag{41}$$

Note that (40) and (41) correspond to (22) and (28) respectively and that the equilibrium conditions of (41) solve the QP problem (38) for fixed plant and disturbance estimates. As seen in (41), the effect of the antiwindup augmentation is to decouple the x_u and the ξ dynamics whenever $\sigma(\xi - x_u) = 0$ holds.

For zero offset tracking, we construct a continuous-time observer of the form:

$$\dot{\hat{x}}_p(t) = (a-1)\hat{x}_p(t) + x_u(t) + \hat{d}(t) + L_x(y_p(t) - \hat{x}_p(t))$$

$$\dot{\hat{d}}(t) = L_d(y_p(t) - \hat{x}_p(t))$$

where we have incorporated a constant input disturbance model. The observer gains L_x and L_d are chosen as:

$$\begin{bmatrix} L_x \\ L_d \end{bmatrix} = \begin{bmatrix} \frac{a}{2} \\ 1 \end{bmatrix} \tag{43}$$

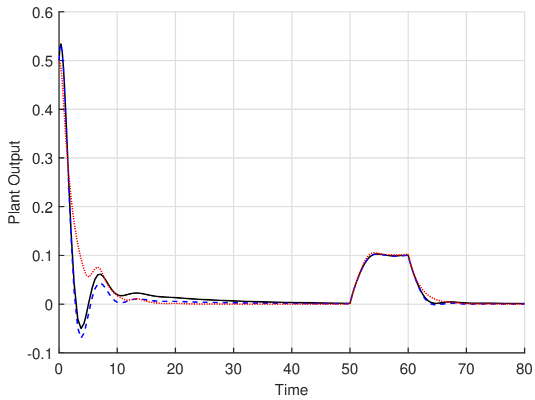

Fig. 3. Output $[x_p(t)]$: solver with antiwindup (solid-black), solver without antiwindup (Dashed-blue), and solver with closed-form solution (Dotted-red)

to ensure that the observer is stable and also to ensure that the observer dynamics are faster than that of the plant. The resulting controller dynamics with antiwindup

$$\begin{bmatrix} \dot{\hat{x}}_p \\ \dot{\hat{d}} \\ \dot{x}_u \\ \dot{\xi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{a}{2} - 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -2a & -4 & -5 & 1 \\ 0 & 0 & -k_i & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ \hat{d} \\ x_u \\ \xi \end{bmatrix} + \begin{bmatrix} \frac{a}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} y_p \\ \text{for } \sigma(\xi - x_u) = \xi - x_u \\ \begin{bmatrix} \frac{a}{2} - 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -2a & -4 & -4 & 0 \\ 0 & 0 & 0 & -k_i \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ \hat{d} \\ x_u \\ \xi \end{bmatrix} + \begin{bmatrix} \frac{a}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} y_p \\ \text{for } \sigma(\xi - x_u) = 0 \end{cases} \tag{44}$$

shows that integral action is restored for the case when $\sigma(\xi - x_u) = 0$. Note that we already establish global asymptotically stability for the solver dynamics for fixed input but here we allow for time varying $\hat{d}(t)$ and $\hat{x}_p(t)$. Figs. 3 through 5 show respectively the output, the input and the Lagrangian responses for the plant when coupled with three different solver dynamics: the primal-dual dynamics with and without antiwindup compensation, and the closed form solver connected to an observer (used as a benchmark). A step disturbance is introduced at simulation time $t = 50$ to demonstrate the zero-offset tracking property of the different schemes as well as the effectiveness of the observer. As seen, both dynamics performed fairly well as compared to the benchmark.

REFERENCES

Adegbege, A.A. (2019). Incrementally passive primal-dual dynamics for real-time optimization. In *American Control Conference*, 1767–1772.

Adegbege, A.A. (2020). Nonlinear control of gradient dynamics with shifted passivity. European Control Conference (To appear).

Arrow, K.J., Hurwicz, L., and Uzawa, H. (1958). *Studies in Linear and Non-linear Programming.* Stanford University Press, California.

Bertsekas, D.P. (2014). *Constrained optimization and Lagrange multiplier methods.* Academic press.

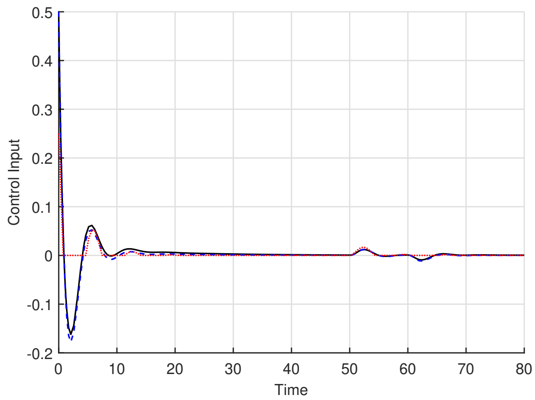

Fig. 4. Input $[x_u(t)]$: solver with antiwindup (solid-black), solver without antiwindup (Dashed-blue), and solver with closed-form solution (Dotted-red)

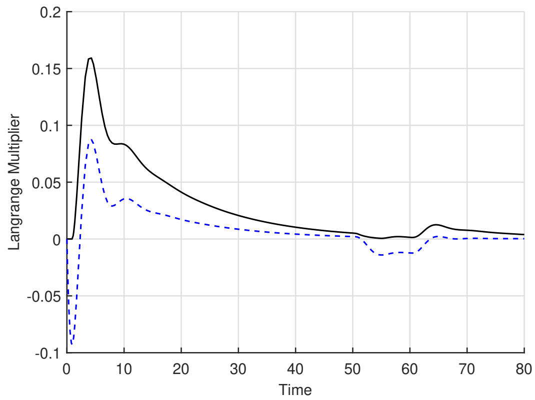

Fig. 5. Lagrange $[\xi(t)]$: solver with antiwindup (solid-black) and solver without antiwindup (Dashed-blue)

Boyd, S. and Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press, New York.

Cherukuri, A., Mallada, E., and Cortés, J. (2016). Asymptotic convergence of constrained primal–dual dynamics. *Systems & Control Letters*, 87, 10–15.

Desoer, C. and Vidyasagar, M. (1975). *Feedback Systems: Input-Output Properties*. Academic Press, Inc., Orlando, FL, USA.

Ding, D. and Jovanović, M.R. (2019). Global exponential stability of primal-dual gradient flow dynamics based on the proximal augmented lagrangian. In *American Control Conference*, 3414–3419.

Feijer, D. and Paganini, F. (2010). Stability of primal-dual gradient dynamics and applications to network optimization. *Automatica*, 46(12), 1974 – 1981.

Gomes da Silva Jr, J., Oliveira, M., Coutinho, D., and Tarbouriech, S. (2014). Static anti-windup design for a class of nonlinear systems. *International Journal of Robust and Nonlinear Control*, 24(5), 793–810.

Ishizaki, T., Ueda, A., and Imura, J. (2016). Convex gradient controller design for incrementally passive systems with quadratic storage functions. In *IEEE 55th Conference on Decision and Control*, 1721–1726.

Jayawardhana, B., Ortega, R., Garcia-Canseco, E., and Castanos, F. (2007). Passivity of nonlinear incremental systems: Application to pi stabilization of nonlinear rlc circuits. *Systems & control letters*, 56(9-10), 618–622.

Jokic, A., Lazar, M., and van den Bosch, P.P.J. (2009). On constrained steady-state regulation: Dynamic kkt controllers. *IEEE Transactions on Automatic Control*, 54(9), 2250–2254.

Khalil, H.K. (2002). *Nonlinear Systems*. Prentice-Hall, Inc., Upper Saddle River, NJ.

Kosaraju, K.C., Chinde, V., Pasumarthy, R., Kelkar, A., and Singh, N.M. (2018). Stability analysis of constrained optimization dynamics via passivity techniques. *IEEE Control Systems Letters*, 2(1), 91–96.

Kose, T. (1956). Solutions of saddle value problems by differential equations. *Econometrica*, 24(1), 59–70.

Levenson, R.M. and Adegbege, A.A. (2016). Analog circuit for real-time optimization of constrained control. In *American Control Conference*, 6947–6952.

Ma, X. and Elia, N. (2015). Convergence analysis for the primal-dual gradient dynamics associated with optimal power flow problems. In *European Control Conference*, 1261–1266.

Maeder, U., Borrelli, F., and Morari, M. (2009). Linear offset-free model predictive control. *Automatica*, 45(10), 2214–2222.

Monshizadeh, N., Monshizadeh, P., Ortega, R., and van der Schaft, A. (2019). Conditions on shifted passivity of port-hamiltonian systems. *Systems & Control Letters*, 123, 55–61.

Mulder, E.F., Kothare, M.V., and Morari, M. (2001). Multivariable anti-windup controller synthesis using linear matrix inequalities. *Automatica*, 37(9), 1407 – 1416.

Nguyen, H.D., Vu, T.L., Turitsyn, K., and Slotine, J.J. (2018). Contraction and robustness of continuous time primal-dual dynamics. *IEEE control systems letters*, 2(4), 755–760.

Nicotra, M.M., Liao-McPherson, D., and Kolmanovsky, I.V. (2018). Embedding constrained model predictive control in a continuous-time dynamic feedback. *IEEE Transactions on Automatic Control*, 64(5), 1932–1946.

Qu, G. and Li, N. (2019). On the exponential stability of primal-dual gradient dynamics. *IEEE Control Systems Letters*, 3(1), 43–48.

Simpson-Porco, J.W. (2018). Equilibrium-independent dissipativity with quadratic supply rates. *IEEE Transactions on Automatic Control*, 64(4), 1440–1455.

Skibik, T. and Adegbege, A.A. (2018). An architecture for analog vlsi implementation of embedded model predictive control. In *American Control Conference*, 4676–4681.

Stegink, T., Persis, C.D., and der Schaft, A.V. (2015). Port-hamiltonian formulation of the gradient method applied to smart grids. *IFAC-PapersOnLine*, 48(13), 13 – 18.

Van der Schaft, A. (2000). *L2-Gain and Passivity Techniques in Nonlinear Control*. Communications and Control Engineering. Springer Verlag, London.

Vichik, S., Arcak, M., and Borrelli, F. (2016). Stability of an analog optimization circuit for quadratic programming. *Systems & Control Letters*, 88, 68–74.

Wassim, M.H. and Chellaboina, V. (20008). *Nonlinear Dynamical Systems and Control, a Lyapunov-based approach*. Princeton University Press, New Jersey.

Yoshida, K., Inoue, M., and Hatanaka, T. (2019). Instant mpc for linear systems and dissipativity-based stability analysis. *IEEE Control Systems Letters*.