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Prescribed-Time Mean-Square Nonlinear Stochastic Stabilization ⋆

Wuquan Li * **Miroslav Krstic** **

* *School of Mathematics and Statistics Science, Ludong University, Yantai 264025 China (e-mail: sea81@126.com).*
** *Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093 USA (e-mail:krstic@ucsd.edu).*

Abstract: We solve the prescribed-time mean-square stabilization problem, providing the first feedback solution to a stochastic null-controllability problem for strict-feedback nonlinear systems with stochastic disturbances. Our non-scaling backstepping design scheme's key novel design ingredient is that, rather than employing "blowing up" time-varying scaling of the backstepping coordinate transformation, we introduce, instead, a damping in the backstepping target systems which grows unbounded as time approaches the terminal time. With this approach, even for deterministic systems, a simpler controller results and the control effort is reduced compared to previous designs. We achieve prescribed-time stabilization in the mean-square sense.

Keywords: Stability of nonlinear systems, Time-varying systems, Lyapunov methods, Prescribed-time stabilization, Non-scaling design.

1. INTRODUCTION

After a spurt of activity in Lyapunov-based asymptotic control of stochastic nonlinear systems, see for instance Deng and Krstić (1997), Krstić and Deng (1998) and Pan and Basar (1999), much attention has been dedicated in recent years to stochastic nonlinear finite-time control. Specifically, Chen and Jiao (2010) and Yin et al. (2011) establish the Lyapunov criteria of stochastic finite-time stability; Yu et al. (2019) relaxes the constraint on the differential operator and gives a more general stochastic finite-time stability criteria. It should be noted that, the results in Chen and Jiao (2010), Yin et al. (2011) and Yu et al. (2019) achieve stochastic finite-time stabilization within some stochastic settling time, which typically depends on initial conditions and is often unknown (only almost surely finity can be ensured). However, the unknown and stochastic character of the settling time makes these results difficult to use in many real applications. In several real-world applications, discussed in Song et al. (2017) and Holloway and Krstic (2019a,b), stabilization is required within a known finite time to meet the control objectives, motivating the study of prescribed-time control.

In the prescribed-time control, the user can prescribe a known specific convergence time, irrespective of initial conditions. In this direction, Song et al. (2017) develops a scaling design method to solve the prescribed-time regulation problem of nonlinear systems in normal form, in which the system state is scaled by a time-varying function that grows unbounded towards the terminal time; Wang et al. (2019) presents the prescribed-time consensus design for networked first-order multi-agent systems; Holloway and Krstic (2019a) solves the prescribed-time estimation problem for linear systems in the observer canonical form. By leveraging the prescribed-time state feedback control in Song et al. (2017) and the prescribed-time observer in Holloway and Krstic (2019a), Holloway and Krstic (2019b) designs a prescribed-time output feedback controller for linear time-invariant systems in controllable canonical form; Krishnamurthy et al. (2019a,b) focus on the prescribed-time stabilization of nonlinear strict-feedback-like systems; Steeves et al. (2019a,b) study the prescribed-time output-feedback stabilization problems for reaction-diffusion equations. It should be emphasized that all the above-mentioned results on prescribed-time control are focused on deterministic systems. However, the perturbations and unmodelled dynamics in physical systems are often described by stochastic noise entering the model. Therefore, it is imperative to study the prescribed-time control of stochastic nonlinear systems.

Motivated by the above observations, we study the the prescribed-time mean-square stabilization for stochastic strict-feedback nonlinear systems. The contributions of this paper are two-fold:

(1) We present a new non-scaling design framework for stochastic nonlinear systems in this paper. Different from the scaling design in Song et al. (2017), Wang et al. (2019) and Holloway and Krstic (2019a,b) where the time-varying function is used to scale the states in all the transforma-

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tions, our approach does not use the scaling function in the coordinate transformations. To achieve prescribed-time stabilization, the time-varying scaling function is suitably used to design virtual controllers. In this way, a simpler controller can be designed since the computation burden for the derivative of the time-varying scaling function can be largely reduced with non-scaling transformations. Therefore, the control effort can be saved. This advantage is especially obvious when the system order is high. It should be emphasized that even for the deterministic nonlinear systems, the non-scaling design scheme proposed in this paper is new.

(2) Compared with the stochastic finite-time stability results, such as Chen and Jiao (2010), Yin et al. (2011) and Yu et al. (2019), where the settling time is stochastic, unknown and heavily relies on the initial conditions, the prescribed-time control developed in this paper has a clear advantage that the settling time is deterministic, known and irrespective of initial conditions, which allows the user to prescribe the convergence time a priori. Therefore, our control schemes are more practical in real applications.

2. PRELIMINARIES

Consider the following stochastic nonlinear system

$$dx = f(t,x,u)dt + g^T(t,x)d\omega, \quad \forall\, x_0 \in R^n, \tag{1}$$

where $x \in R^n$ and $u(t,x) \in R$ are the system state and control input. The functions $f : R^+ \times R^n \times R \to R^n$ and $g : R^+ \times R^n \to R^{m\times n}$ are piecewise continuous in t, locally bounded and locally Lipschitz continuous in x uniformly in $t \in R^+$. ω is an m-dimensional independent standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets).

We introduce the following scaling functions:

$$\mu_1(t) = \frac{T}{t_0 + T - t}, \tag{2}$$

$$\mu(t) = \left(\frac{T}{t_0 + T - t}\right)^m = \mu_1^m(t), \forall\, t \in [t_0, t_0 + T), \tag{3}$$

where $m \geq 2$ is a integer.

Obviously, $\mu(t)$ is a monotonically increasing function on $[t_0, t_0 + T)$ with $\mu(t_0) = 1$ and $\lim\limits_{t\to t_0+T} \mu(t) = +\infty$.

Next, we give the definition of prescribed-time mean-square stable.

Definition 1. For stochastic system (1) with $f(t,0,0) = 0$ and $g(t,0) = 0$, the equilibrium $x(t) = 0$ is prescribed-time mean-square stable if there exist positive constants k_i $(1 \leq i \leq 4)$ such that

$$E|x(t)|^2 \leq k_1|x(t_0)|^2(1 + \mu_1^{k_2}(t))e^{-k_3\mu_1^{k_4}(t)}, \quad \forall\, t \in [t_0, t_0 + T). \tag{4}$$

Remark 1. In Definition 1, by (2) we have $\lim\limits_{t\to t_0+T} E|x|^2 = 0$. Besides, denoting

$$\varrho(t) = k_1|x(t_0)|^2(1 + \mu_1^{k_2})e^{-k_3\mu_1^{k_4}}, \tag{5}$$

then we have

$$\begin{aligned}\frac{d\varrho}{dt} &= k_1\frac{k_2}{T}|x(t_0)|^2 e^{-k_3\mu_1^{k_4}}\mu_1^{k_2+1} \\ &\quad\cdot\left(1 - \frac{k_3k_4}{k_2}(\mu_1^{k_4-k_2} + \mu_1^{k_4})\right) \\ &\leq k_1\frac{k_2}{T}|x(t_0)|^2 e^{-k_3\mu_1^{k_4}}\mu_1^{k_2+1}\left(1 - \frac{k_3k_4}{k_2}\mu_1^{k_4}\right).\end{aligned} \tag{6}$$

It can be deduced from (6) that $E|x|^2$ is a strictly decreasing function in $[T^*, t_0 + T)$, where

$$T^* = \max\left\{t_0, t_0 + T - T(\frac{k_3k_4}{k_2})^{1/k_4}\right\}. \tag{7}$$

From (7), it is obvious that $t_0 \leq T^* < t_0 + T$.

For stochastic system (1), the following Lemma provides a basic tool for proving the existence of a solution and analyzing the prescribed-time mean-square stability, whose proof is omitted here.

Lemma 1. Consider the system (1). If there exist a nonnegative function $U(t,x) \in C^{1,2}(R^+ \times R^n; R^+)$, and positive constants c_0 and M_0 such that

$$\lim_{|x|\to+\infty} U(t,x) = +\infty, \tag{8}$$

$$\mathcal{L}U(t,x) \leq -c_0\mu U + \mu M_0, \quad \forall\, t \in [t_0, t_0 + T), \tag{9}$$

then the following conclusions hold:

(1) System (1) has an almost surely unique solution on $[t_0, t_0 + T)$ for any $x_0 \in R^n$.

(2) The function $U(t,x)$ satisfies

$$EU(t,x) \leq e^{-c_0\int_{t_0}^t \mu(s)ds}U(t_0, x_0) + \frac{M_0}{c_0}, \quad \forall\, t \in [t_0, t_0 + T). \tag{10}$$

3. PROBLEM FORMULATION

Consider a class of stochastic nonlinear systems described by

$$dx_i = (x_{i+1} + f_i(t,x))dt + g_i^T(t,x)d\omega,\ 1 \leq i \leq n-1, \tag{11}$$

$$dx_n = (u + f_n(t,x))dt + g_n^T(t,x)d\omega, \tag{12}$$

where $x = (x_1, \ldots, x_n)^T \in R^n$ and $u \in R$ are the system state and control input. The functions $f_i : R^+ \times R^n \to R$ and $g_i : R^+ \times R^n \to R^m$ are piecewise continuous in t, locally bounded and locally Lipschitz continuous in x uniformly in $t \in R^+$, $f_i(t,0) = 0$, $g_i(t,0) = 0$, $i = 1, \ldots, n$. ω is an m-dimensional independent standard Wiener process whose definition can be found in system (1).

To proceed further, we need the following assumption.

Assumption 1. For $i = 1, \cdots, n$, there exist positive constants c_{i1} and c_{i2} such that

$$|f_i(t,x)| \leq c_{i1}(|x_1| + \cdots + |x_i|), \tag{13}$$

$$|g_i(t,x)| \leq c_{i2}(|x_1| + \cdots + |x_i|). \tag{14}$$

In this paper, for system (11)-(12) with Assumption 1, we first develop a novel non-scaling design scheme, by which a new time-varying controller is designed; then we analyze the prescribed-time mean-square stability of the closed-loop system.

4. CONTROLLER DESIGN

In this section, we design a time-varying controller for system (11)-(12) step by step.

Step 1. Define $V_1 = \frac{1}{4}\xi_1^4$, $\xi_1 = x_1$, from (11), (13)-(14) we have

$$\begin{aligned}\mathcal{L}V_1(\xi_1) &= \xi_1^3 x_2 + \xi_1^3 f_1 + \frac{3}{2}\xi_1^2 |g_1|^2 \\ &\leq \xi_1^3(x_2 - x_2^*) + \xi_1^3 x_2^* + \xi_1^4\left(c_{11} + \frac{3}{2}c_{12}^2\right). \quad (15)\end{aligned}$$

Choosing

$$x_2^* = -\mu^{\delta_1}\left(c_1 + c_{11} + \frac{3}{2}c_{12}^2\right)\xi_1 \triangleq -\mu^{\delta_1}\alpha_1\xi_1, \quad (16)$$

which substitutes into (15) yields

$$\mathcal{L}V_1(\xi_1) \leq -c_1\mu^{\delta_1}\xi_1^4 + \xi_1^3(x_2 - x_2^*), \quad (17)$$

where $\delta_1 = 1$, $c_1 > 0$ is a design parameter and $\alpha_1 = c_1 + c_{11} + \frac{3}{2}c_{12}^2$.

Deductive Step. Assume that at step $k-1$, there are set of virtual controllers $x_2^*, \ldots, x_k^*$ defined by

$$x_2^* = -\mu^{\delta_1}\alpha_1\xi_1, \qquad \xi_1 = x_1, \quad (18)$$
$$x_3^* = -\mu^{\delta_2}\alpha_2\xi_2, \qquad \xi_2 = x_2 - x_2^*, \quad (19)$$
$$\vdots \qquad \qquad \vdots$$
$$x_k^* = -\mu^{\delta_{k-1}}\alpha_{k-1}\xi_{k-1}, \quad \xi_{k-1} = x_{k-1} - x_{k-1}^*, \quad (20)$$

such that

$$\begin{aligned}\mathcal{L}V_{k-1}(\bar{\xi}_{k-1}) \leq & -\sum_{i=1}^{k-1}(c_i - a_{k-1,i})\mu^{\delta_i}\xi_i^4 \\ & + \xi_{k-1}^3(x_k - x_k^*), \quad (21)\end{aligned}$$

where $\alpha_1, \ldots, \alpha_{k-1}$ are positive constants, $c_i > 0$ is a design parameter, $a_{k-1,1}, \ldots, a_{k-1,k-2}$ are arbitrary positive constants, $a_{k-1,k-1} = 0$, $\bar{\xi}_{k-1} = (\xi_1, \ldots, \xi_{k-1})^T$, $V_{k-1}(\bar{\xi}_{k-1}) = \frac{1}{4}\sum_{i=1}^{k-1}\xi_i^4$ and

$$\delta_1 = 1, \qquad \delta_i = 3 \cdot 5^{i-2}, \qquad 2 \leq i \leq k-1. \quad (22)$$

To complete the induction, at the kth step, we consider the ξ_k-system.

Define $\xi_k = x_k - x_k^*$, from (18)-(20) we obtain

$$\xi_k = x_k + \sum_{i=1}^{k-1}\beta_i(t)x_i, \quad (23)$$

$$\beta_i(t) = \prod_{j=i}^{k-1}\mu^{\delta_j}\alpha_j. \quad (24)$$

Noting that $\frac{\partial^2(\beta_i x_i)}{\partial x_i \partial x_j} = 0$, by (11), (23) and Itô's formula we get

$$\begin{aligned} d\xi_k = & \left(x_{k+1} + f_k + \sum_{i=1}^{k-1}\dot{\beta}_i x_i + \sum_{i=1}^{k-1}\beta_i(x_{i+1} + f_i)\right)dt \\ & + \left(g_k^T + \sum_{i=1}^{k-1}\beta_i g_i^T\right)d\omega. \quad (25)\end{aligned}$$

We choose the Lyapunov function

$$V_k(\bar{\xi}_k) = V_{k-1}(\bar{\xi}_{k-1}) + \frac{1}{4}\xi_k^4. \quad (26)$$

It follows from (21), (25)-(26) and Itô's formula that

$$\begin{aligned}\mathcal{L}V_k(\bar{\xi}_k) \leq & -\sum_{i=1}^{k-1}(c_i - a_{k-1,i})\mu^{\delta_i}\xi_i^4 + \xi_{k-1}^3\xi_k \\ & + \xi_k^3 x_{k+1} + \xi_k^3 f_k + \xi_k^3\sum_{i=1}^{k-1}\beta_i(x_{i+1} + f_i) \\ & + \xi_k^3\sum_{i=1}^{k-1}\dot{\beta}_i x_i + \frac{3}{2}\xi_k^2\left|g_k^T + \sum_{i=1}^{k-1}\beta_i g_i^T\right|^2. \quad (27)\end{aligned}$$

For $i = 1, \cdots, k$, by (13)-(14) and (18)-(20) we get

$$|f_i(t,x)| \leq \hat{c}_{i1}(\mu^{\delta_1}|\xi_1| + \cdots + \mu^{\delta_{i-1}}|\xi_{i-1}| + |\xi_i|), \quad (28)$$
$$|g_i(t,x)| \leq \hat{c}_{i2}(\mu^{\delta_1}|\xi_1| + \cdots + \mu^{\delta_{i-1}}|\xi_{i-1}| + |\xi_i|), \quad (29)$$

where $\hat{c}_{i1}$ and $\hat{c}_{i2}$ are positive constants.

By (22), (28) and Young's inequality in Krstić and Deng (1998) we have

$$\begin{aligned}\xi_{k-1}^3\xi_k \leq & \, a_{k,k-1,1}\mu^{\delta_{k-1}}\xi_{k-1}^4 \\ & + \frac{1}{4}\left(\frac{4}{3}a_{k,k-1,1}\right)^{-3}\xi_k^4, \quad (30)\end{aligned}$$

$$\begin{aligned}\xi_k^3 f_k \leq & \, \hat{c}_{k1}|\xi_k|^3(\mu^{\delta_1}|\xi_1| + \cdots + \mu^{\delta_{k-1}}|\xi_{k-1}| + |\xi_k|) \\ \leq & \sum_{i=1}^{k-1}a_{k,i,2}\mu^{\delta_i}\xi_i^4 + \mu^{\delta_{k-1}}\Big(\hat{c}_{k1} + \frac{3}{4}\hat{c}_{k1}^{4/3} \\ & \cdot\sum_{i=1}^{k-1}(4a_{k,i,2})^{-1/3}\Big)\xi_k^4, \quad (31)\end{aligned}$$

where $a_{k,k-1,1}$ and $a_{k,i,2}$ are arbitrary positive constants.

From (24) and the definition of δ_k we have

$$\beta_i = \mu^{\delta_i + \cdots + \delta_{k-1}}\prod_{j=i}^{k-1}\alpha_j, \quad (32)$$

$$|\dot{\beta}_i| \leq \left(\frac{m}{T}\sum_{j=i}^{k-1}\delta_j\right)\mu^{\delta_i + \cdots + \delta_{k-1} + 1}\prod_{j=i}^{k-1}\alpha_j. \quad (33)$$

By (18)-(20), (22), (32) and Young's inequality we obtain

$$\begin{aligned}\xi_k^3\sum_{i=1}^{k-1}\beta_i x_{i+1} \leq & |\xi_k|^3\sum_{i=1}^{k-1}\mu^{\delta_i + \cdots + \delta_{k-1}} \\ & \cdot\Big(\prod_{j=i}^{k-1}\alpha_j\Big)\left(|\xi_{i+1}| + \mu^{\delta_i}\alpha_i|\xi_i|\right)\end{aligned}$$

$$\leq \mu^{\delta_{k-1}}\alpha_{k-1}\xi_k^4 + \sum_{i=1}^{k-1}\mu^{2\delta_i+\delta_{i+1}+\cdots+\delta_{k-1}} \cdot(\alpha_i^2+1)\Big(\prod_{j=i-1}^{k-1}\alpha_j\Big)|\xi_i||\xi_k|^3$$
$$\leq \sum_{i=1}^{k-1} a_{k,i,3}\mu^{\delta_i}\xi_i^4 + \mu^{7\delta_{k-1}/3}\bigg(\alpha_{k-1} + \frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,3})^{-1/3}(\alpha_i^2+1)^{4/3} \cdot\Big(\prod_{j=i-1}^{k-1}\alpha_j\Big)^{4/3}\bigg)\xi_k^4, \tag{34}$$

where $a_{k,i,3}$ is an arbitrary positive constant and $\alpha_0 = 1$.

Similar to (34) we get

$$\xi_k^3\sum_{i=1}^{k-1}\beta_i f_i \leq \sum_{i=1}^{k-1} a_{k,i,4}\mu^{\delta_i}\xi_i^4 + \frac{3}{4}\mu^{2\delta_{k-1}}\bigg(\sum_{i=1}^{k-1}(4a_{k,i,4})^{-1/3} \cdot\Big(\bar{c}_{k-1,1}(k-1)\prod_{j=1}^{k-1}\alpha_j\Big)^{4/3}\bigg)\xi_k^4, \tag{35}$$

$$\xi_k^3\sum_{i=1}^{k-1}\dot{\beta}_i x_i \leq \sum_{i=1}^{k-1} a_{k,i,5}\mu^{\delta_i}\xi_i^4 + \frac{3}{4}\mu^{2\delta_{k-1}}\sum_{i=1}^{k-1}(4a_{k,i,5})^{-1/3} \cdot\Big(\prod_{j=i}^{k-1}\alpha_j\Big)^{4/3}\Big(\frac{2m}{T}\sum_{j=i}^{k-1}\delta_j\Big)^{4/3}\xi_k^4, \tag{36}$$

where $a_{k,i,4}$ and $a_{k,i,5}$ are arbitrary positive constants, $\xi_0 = 0$ and $\bar{c}_{k-1,1} = \max\{\hat{c}_{11}, \hat{c}_{21}, \ldots, \hat{c}_{k-1,1}\}$.

By (29) and (32) we obtain

$$\frac{3}{2}\xi_k^2\Big|g_k^T + \sum_{i=1}^{k-1}\beta_i g_i^T\Big|^2 \leq \frac{3}{2}k\hat{c}_{k2}^2\xi_k^4 + \frac{3}{2}k\sum_{i=1}^{k-1}\mu^{2\delta_i+\cdots+2\delta_{k-1}} \cdot\Big(\hat{c}_{k2} + \sum_{s=i}^{k-1}\prod_{j=s}^{k-1}\alpha_s\hat{c}_{s2}\Big)^2\xi_i^2\xi_k^2. \tag{37}$$

From (22) we have

$$3\delta_i + 4(\delta_{i+1}+\cdots+\delta_{k-1}) = \begin{cases} 5\delta_{k-1}, & \text{if } i = 1, \\ 5\delta_{k-1} - 2\delta_i, & \text{if } 2 \leq i \leq k-1. \end{cases} \tag{38}$$

By (38) and Young's inequality, (37) can be written as

$$\frac{3}{2}\xi_k^2\Big|g_k^T + \sum_{i=1}^{k-1}\beta_i g_i^T\Big|^2 \leq \sum_{i=1}^{k-1} a_{k,i,6}\mu^{\delta_i}\xi_i^4 + \mu^{5\delta_{k-1}}\bigg(\frac{3}{2}k\hat{c}_{k2}^2 + \frac{9}{16}k^2\cdot\sum_{i=1}^{k-1}a_{k,i,6}^{-1}\Big(\hat{c}_{k2} + \sum_{s=i}^{k-1}\prod_{j=s}^{k-1}\alpha_s\hat{c}_{s2}\Big)^4\bigg)\xi_k^4, \tag{39}$$

where $a_{k,i,6}$ is an arbitrary positive constant.

From (22), (30)-(31), (34)-(36) and (39) we can choose

$$\delta_k = \max\Big\{\delta_{k-1}, 2\delta_{k-1}, \frac{7}{3}\delta_{k-1}, 5\delta_{k-1}\Big\} = 3\cdot 5^{k-2}. \tag{40}$$

Next, we choose the virtual controller as

$$x_{k+1}^* = -\mu^{\delta_k}\xi_k\bigg(c_k + \hat{c}_{k1} + \alpha_{k-1} + \frac{3}{2}k\hat{c}_{k2}^2 + \frac{1}{4}(\frac{4}{3}a_{k,k-1,1})^{-3} + \frac{3}{4}\hat{c}_{k1}^{4/3}\sum_{i=1}^{k-1}(4a_{k,i,2})^{-1/3}$$
$$+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,3})^{-1/3}(\alpha_i^2+1)^{4/3}\Big(\prod_{j=i-1}^{k-1}\alpha_j\Big)^{4/3} + \frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,4})^{-1/3}\Big(\bar{c}_{k-1,1}(k-1)\prod_{j=1}^{k-1}\alpha_j\Big)^{4/3}$$
$$+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,5})^{-1/3}\Big(\prod_{j=i}^{k-1}\alpha_j\Big)^{4/3}\Big(\frac{2m}{T}\sum_{j=i}^{k-1}\delta_j\Big)^{4/3} + \frac{9}{16}k^2\sum_{i=1}^{k-1}a_{k,i,6}^{-1}\Big(\hat{c}_{k2} + \sum_{s=i}^{k-1}\prod_{j=s}^{k-1}\alpha_s\hat{c}_{s2}\Big)^4\bigg)$$
$$\triangleq -\mu^{\delta_k}\alpha_k\xi_k. \tag{41}$$

With (40), substituting (30)-(31), (34)-(36), (39) and (41) into (27) yields

$$\mathcal{L}V_k(\bar{\xi}_k) \leq -\sum_{i=1}^{k}(c_i - a_{k,i})\mu^{\delta_i}\xi_i^4 + \xi_k^3(x_{k+1} - x_{k+1}^*), \tag{42}$$

where $c_k > 0$ is a design parameter, $a_{k,k} = 0$ and

$$a_{k,i} = a_{k-1,i} + \sum_{j=2}^{6}a_{k,i,j}, \quad i = 1,\cdots,k-2, \tag{43}$$

$$a_{k,k-1} = a_{k-1,k-1} + \sum_{j=1}^{6}a_{k,i,j}. \tag{44}$$

Step n. Similar to (41)-(42), by choosing the actual control law

$$u = -\mu^{\delta_n}\alpha_n\xi_n, \tag{45}$$

we have

$$\mathcal{L}V_n(\bar{\xi}_n) \leq -\sum_{i=1}^{n}(c_i - a_{n,i})\mu^{\delta_i}\xi_i^4, \tag{46}$$

where $c_n > 0$ is a design parameter, $\delta_n = 3\cdot 5^{n-2}$, α_n is a positive constant, $a_{n,n} = 0$, $a_{n,1},\ldots,a_{n,n-1}$ are positive constants, $\xi_n = x_n - x_n^*$ and $V_n(\bar{\xi}_n) = \frac{1}{4}\sum_{i=1}^{n}\xi_i^4$.

Choosing the design parameters as

$$c_i > a_{n,i}, \quad i = 1,\cdots,n-1, \tag{47}$$
$$c_n > 0, \tag{48}$$

from (22) and (46)-(48) we have

$$\mathcal{L}V_n(\bar{\xi}_n) \leq -\frac{1}{4}c\sum_{i=1}^{n}\mu^{\delta_i}\xi_i^4 \leq -c\mu V_n, \tag{49}$$

where $c = 4\min_{1\leq i\leq n-1}\{c_i - a_{n,i}, c_n\}$.

Remark 2. From the design process, it can be observed that the order of μ in the controller is suitably constructed so that the negative term $-\mu^{\delta_i}\xi_i^4$ dominates the nonnegative terms produced by Itô's formula. For example, the order of μ in the the virtual controller (41) is carefully chosen as $\delta_k = 3\cdot 5^{k-2}$. On the one hand, if $\delta_k < 3\cdot 5^{k-2}$, from (37)-(39) we conclude that some nonlinear terms like $\mu^p\xi_1^4$ ($p>1$) appear, which cannot be dominated by $-\mu^{\delta_1}\xi_1^4$, losing the guarantee of stability. On the other hand, if $\delta_k > 3\cdot 5^{k-2}$, although the stochastic prescribed-time stability is achieved, the control effort will be larger. Therefore, a good choice of δ_k is nontrivial. In fact, it can be deduced from (27)-(40) that the minimum suitable value of δ_k is mainly decided by the Hessian term $\frac{3}{2}\xi_k^2\big|g_k^T + \sum_{i=1}^{k-1}\beta_i g_i^T\big|^2$ (more details are found in (37)-(40)).

Remark 3. In this section, we propose a new non-scaling backstepping design scheme for stochastic nonlinear system (11)-(12) to achieve prescribed-time mean-square stable. The merit of this design is not using the time-varying μ to scale the coordinate transformations $\xi_i = x_i - x_i^*$, $i = 1,\cdots,n$, and μ is suitably introduced into the virtual controller x_i^*. This approach is essentially different from the scaling method developed in Song et al. (2017), Wang et al. (2019) and Holloway and Krstic (2019a,b) where the scaled transformation $\xi_i = \mu^k(x_i - x_i^*)$ is used for the controller design at every step. The main advantage of our approach is that a simpler controller is designed and the computation burden arising from the derivative of μ is largely reduced with non-scaling transformations. Therefore, the control effort can be saved.

5. STABILITY ANALYSIS

In the following theorem, we give the main stability results on system (11)-(12).

Theorem 1. Consider the plant consisting of (11)-(12), (45) and (47)-(48). If Assumption 1 holds, then the following conclusions hold:

1) The plant has an almost surely unique solution on $[t_0, t_0+T)$;

2) The equilibrium at the origin of the plant is prescribed-time mean-square stable with $\lim_{t\to t_0+T} E|x|^2 = \lim_{t\to t_0+T} Eu^2 = 0$. Moreover, for $\forall\, t\in[t_0, t_0+T)$, the following estimates hold:

$$E|x|^2 \le \sqrt{n}\Big(n + \sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i}\Big) \cdot\left(x_1^4(t_0) + \sum_{k=2}^{n}\Big(x_k(t_0) + \sum_{i=1}^{k-1}\prod_{j=i}^{k-1}\alpha_j x_i(t_0)\Big)^4\right)^{1/2} \cdot e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)}, \tag{50}$$

$$Eu^2 \le \sqrt{n}\alpha_n^2\mu^{2\delta_n}\left(x_1^4(t_0) + \sum_{k=2}^{n}\Big(x_k(t_0) + \sum_{i=1}^{k-1}\prod_{j=i}^{k-1}\alpha_j x_i(t_0)\Big)^4\right)^{1/2} \cdot e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)}. \tag{51}$$

Proof. From (18)-(20) we have

$$x = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_1\mu^{\delta_1} & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_{n-1}\mu^{\delta_{n-1}} & 1 \end{bmatrix}\xi, \tag{52}$$

where $\xi = (\xi_1,\cdots,\xi_n)^T$.

By (52) we get

$$|x| \le \left(n + \sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i}\right)^{1/2}|\xi|, \tag{53}$$

which means that

$$|\xi| \ge \left(n + \sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i}\right)^{-1/2}|x|. \tag{54}$$

Noting that the plant satisfies the local Lipschitz condition and that $V_n(\bar{\xi}_n) = \frac{1}{4}\sum_{i=1}^{n}\xi_i^4$, by (49) and (54), the conditions (8) and (9) in Lemma 1 hold.

Therefore, by Lemma 1, conclusion 1) holds and

$$EV_n(t,x) \le e^{-c\int_{t_0}^{t}\mu(s)ds}V_n(t_0,x_0). \tag{55}$$

By (55) and Schwarz inequality we obtain

$$E|\xi|^2 \le \left\{E|\xi|^4\right\}^{1/2} \le 2\sqrt{n}e^{-\frac{c}{2}\int_{t_0}^{t}\mu(s)ds}V_n^{1/2}(t_0,x_0), \forall\, t\in[t_0,t_0+T). \tag{56}$$

By (3) and (56) we get

$$E|\xi|^2 \le 2\sqrt{n}e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)} \cdot V_n^{1/2}(t_0,x_0), \quad \forall\, t\in[t_0,t_0+T). \tag{57}$$

From (23)-(24), (53) and (57) we have

$$E|x|^2 \le \sqrt{n}\Big(n + \sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i}\Big) \cdot\left(x_1^4(t_0) + \sum_{k=2}^{n}\Big(x_k(t_0) + \sum_{i=1}^{k-1}\prod_{j=i}^{k-1}\alpha_j x_i(t_0)\Big)^4\right)^{1/2} \cdot e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)}, \forall\, t\in[t_0,t_0+T). \tag{58}$$

Noting that $c>0$, $T>0$ and $m\ge 2$, by (3) we obtain

$$\lim_{t\to t_0+T}\mu^k e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)} = 0 \tag{59}$$

holds for any $k\in R$.

By (58) and (59) we get

$$\lim_{t\to t_0+T} E|x|^2 = 0. \tag{60}$$

Similar to (58) and (60), it follows from (45) and (57) that (51) holds and

$$\lim_{t\to t_0+T} E|u|^2 = 0. \tag{61}$$

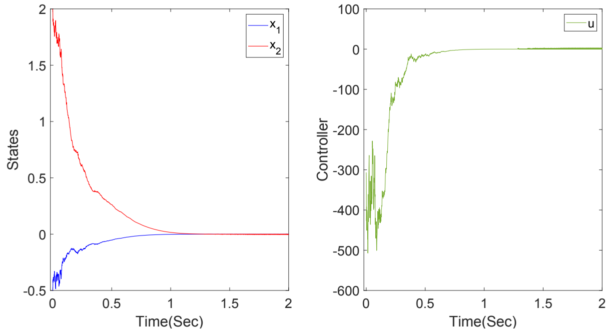

Fig. 1. The response of the closed-loop system (62)-(64).

6. A SIMULATION EXAMPLE

In the following example, we show that even when Assumption 1 is not satisfied, by following the controller design developed in this paper, we still solve the prescribed-time mean-square stabilization problem when the nonlinear terms are in some special forms.

Example 1. Consider the following system

$$dx_1 = x_2 dt + x_1 \sin x_1 d\omega, \tag{62}$$

$$dx_2 = (u + x_1 x_2)dt + x_1^{5/3} d\omega. \tag{63}$$

Obviously, the drift term $x_1 x_2$ and the diffusion term $x_1^{5/3}$ don't satisfy the linear growth condition required in Assumption 1.

By following the design procedure developed in Section 4, we design the control law as

$$\begin{aligned} u = &-\Big(c_1 + c_2 + \frac{33}{4} + \frac{3}{4}x_2^{4/3} + \frac{3}{4}\big(\frac{m}{T}\big)^{4/3} \\ &\cdot\Big(c_1 + \frac{3}{2}\Big)^{4/3} + \frac{3}{4}\Big(c_1 + \frac{3}{2}\Big)^{8/3} \\ &+ \frac{9}{4}\Big(c_1 + \frac{3}{2} + x_1^{2/3}\Big)^4\Big)\mu^3 \xi_2, \end{aligned} \tag{64}$$

where $c_1 > 0$ and $c_2 > 0$ are design parameters and $\tilde{c}_0 = 4\min\{c_1 - \frac{5}{4}, c_2\}$.

For simulation, we select $t_0 = 0$, $T = 2$, $m = 2$, the parameters $c_1 = \frac{3}{2}$, $c_2 = \frac{1}{2}$, and randomly set the initial conditions as $x_1(0) = -0.5$, $x_2(0) = 2$. Fig.1 gives the response of the closed-loop system (62)-(64). From Fig.1, we can find that $\lim_{t\to 2} E|x|^2 = \lim_{t\to 2} E|u| = 0$. In other words, the prescribed-time mean-square stabilization can be achieved. Therefore, the effectiveness of the controller design is demonstrated.

7. CONCLUDING REMARKS

In this paper we have addressed the prescribed-time mean-square stabilization design for stochastic strict-feedback nonlinear systems. By developing a new non-scaling backstepping design method, a new controller is designed to guarantee that the equilibrium at the origin of the closed-loop system is prescribed-time mean-square stable.

For the stochastic nonlinear systems, many open issues are worth investigating, such as generalizing the results in this paper to output-feedback control shown in Li et al. (2020).

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