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Filterless Least-Squares Based Adaptive Stochastic Continuous-Time Nonlinear Control ⋆

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Abstract: In continuous-time system identification and adaptive control, the least-squares parameter estimation algorithm has always been used with regressor filtering, in order to avoid using time-derivatives of the measured state. Filtering adds to the dynamic order of the identifier and affects its performance. We solve the problem of filterless least-squares-based adaptive control for stochastic strict-feedback nonlinear systems with an unknown parameter in the drift term. The novel ingredient in our least-squares identification is that the update law for the parameter estimate is not a simple integrator but it also incorporates a feedthrough effect, namely, the parameter estimator is of relative degree zero (rather than one) relative to the update function. The feedthrough in the update law is a carefully designed nonlinear function, which incorporates the integration with respect to state (and not time) of the regressor function, the purpose of which is to eliminate the need for time-filtering of the regressor. Our backstepping design of the control law compensates the adverse effect of the noise (the Hessian nonlinear term, involving the diffusion nonlinearity, in the Lyapunov analysis) on the least-squares estimator. Such a controller also enables a construction of an single overall Lyapunov function, quadratic in the parameter error and quartic in the transformed state, to guarantee that the equilibrium at the origin of the closed-loop system is globally stable in probability and the states are regulated to zero almost surely.

Keywords: Adaptive control, Stability of nonlinear systems, Lyapunov methods, Stochastic nonlinear systems, Filterless least-squares.

1. INTRODUCTION

Least-squares is an appealing method to identify unknown parameters since it can adjust adaptive rates online so that all parametric estimates converge with approximately the same speed resulting in performance and robustness advantages, see for instance Berghuis et al. (1995), Krstic (2009) and Nguyen (2013).

The least-squares based stochastic adaptive control problem is first solved in Astrom and Wittenmark (1973) for single-input single-output systems perturbed by filtered white noise. For discrete-time stochastic systems, Kumar and Moore (1982) presents a weighted least-squares approach in parameter estimation where the weightings are selected according to a stability measure and guided by a global convergence theory; Chen and Guo (1986, 1991) study the convergence rate of least-squares identification for autoregressive moving average with exogenous input (ARMAX) models. For continuous-time stochastic systems, Zhang and Caines (1996) uses least-squares based switching control strategies to reduce the computational load; Gao and Pasik-Duncan (1997) establishes the stability of first-order linear systems by using the weighted least-squares algorithm without involving any excitations; Duncan et al. (1999) modifies the weighted least-squares algorithm by using a random regularization to ensure that the family of estimated models are uniformly controllable and observable. It should be emphasized that all the above-mentioned results on the least-squares based adaptive control of stochastic systems, are all focused on linear systems. However, as demonstrated by Khalil (2002), most physical systems have nonlinear terms of one sort or another. Therefore, it is imperative to study the least-squares identification and adaptive control of stochastic nonlinear systems.

Motivated by the above observations, we study the least-squares identification and adaptive control of stochastic strict-feedback nonlinear system with an unknown parameter in the drift term. Specifically, we consider a class of stochastic nonlinear systems described by

$$dx_i = x_{i+1}dt, \qquad i = 1, \cdots, n-1, \tag{1}$$

$$dx_n = (u + \varphi_1^T(x)\theta)dt + \varphi_2^T(x)d\omega, \tag{2}$$

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where $x = (x_1, \ldots, x_n)^T \in R^n$ and $u \in R$ are the system state and control input. The functions $\varphi_1 : R^n \to R^p$ and $\varphi_2 : R^n \to R^r$ are locally Lipschitz continuous in x, $\varphi_1(0) = 0$, $\varphi_2(0) = 0$. $\theta \in R^p$ is an unknown parameter. ω is an r-dimensional independent standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets).

We provide in this paper three novel contributions:

(1) We propose a new least-squares identification scheme in this paper. On the one hand, although our identification scheme is motivated by Krstic (2009), the stochastic character of the studied systems makes the least-squares estimation scheme in Krstic (2009) inapplicable in this paper. The main reason is that the estimator in Krstic (2009) will produce uncontrollable Hessian terms by using Itô's formula. To overcome this difficulty, we propose a new least-squares estimator by introducing weighted terms equipped with design parameters. By suitably selecting these estimator parameters, the controller can be endowed with the ability to deal with these Hessian terms; On the other hand, the least-squares estimator in this paper is also essentially different from that in stochastic continuous-time linear system Gao and Pasik-Duncan (1997) and Duncan et al. (1999) where the differential dx of the system state x is required to be measurable in the parameter estimation procedure. In this paper, the information of dx is not used in the estimators.

(2) We design a new adaptive controller. Specifically, the controller designed in this paper not only can deal with the Hessian terms produced by the least-squares estimator and the system itself to guarantee that the equilibrium at the origin of the closed-loop system is globally stable in probability, but also have the ability to make the least-squares estimator converge.

(3) In this paper, by introducing weighted terms with design parameters, we guarantee the estimator to have the property of convergence. This approach is different from that in Duncan et al. (1999), which uses a slowly increasing function as the weighted term in the estimator, and from that in Gao (1996), which uses excitation conditions to ensure convergence.

The remainder of this paper is organized as follows. In Section 2, we first construct a least-squares estimator for system (1)-(2). Then we design an adaptive controller in Section 3, which follows in Section 4 with the stability analysis of the closed-loop system. We finally end our paper with some concluding remarks in Section 5.

2. PARAMETER ESTIMATION

Define

$$h(x_1, \cdots, x_n) = \Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^2 \varphi_1(x_1, \cdots, x_n), \quad (3)$$

where $k_1, \cdots, k_{n-1}$ are parameters to be designed later.

For system (1)-(2), we design a parameter estimator as

$$\dot{\alpha} = -\Gamma h \varphi_1^T \alpha - \Gamma \sum_{i=1}^{n-1} x_{i+1} \int_0^{x_n} \frac{\partial h(x_1, \cdots, x_{n-1}, \sigma)}{\partial x_i} d\sigma - \frac{1}{2}\Gamma \frac{\partial h}{\partial x_n} |\varphi_2|^2 - \Gamma h u, \quad (4)$$

$$\dot{\Gamma} = -\Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^2 \Gamma \varphi_1 \varphi_1^T \Gamma, \Gamma(0) = \Gamma^T(0) > 0, \quad (5)$$

$$\hat{\theta} = \alpha + \Gamma \int_0^{x_n} h(x_1, \cdots, x_{n-1}, \sigma) d\sigma. \quad (6)$$

From (4) and (6) we have

$$\dot{\alpha} = \Gamma h \varphi_1^T \Gamma \int_0^{x_n} h(x_1, \cdots, x_{n-1}, \sigma) d\sigma - \Gamma \sum_{i=1}^{n-1} x_{i+1} \int_0^{x_n} \frac{\partial h(x_1, \cdots, x_{n-1}, \sigma)}{\partial x_i} d\sigma - \frac{1}{2}\Gamma \frac{\partial h}{\partial x_n} |\varphi_2|^2 - \Gamma h (u + \varphi_1^T \hat{\theta}), \quad (7)$$

by which and (5)-(6) we obtain

$$d\hat{\theta} = \Gamma h \varphi_1^T (\theta - \hat{\theta}) dt + \Gamma h \varphi_2^T dw. \quad (8)$$

Denote $\tilde{\theta} = \theta - \hat{\theta}$, from (8) we have

$$d\tilde{\theta} = -\Gamma h \varphi_1^T \tilde{\theta} dt - \Gamma h \varphi_2^T dw. \quad (9)$$

From (5) we have

$$\frac{d\Gamma^{-1}(t)}{dt} = \Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^2 \varphi_1 \varphi_1^T, \ \Gamma^{-1}(0) > 0, \quad (10)$$

which means that $\Gamma^{-1}(t)$ is a positive definite matrix and $\Gamma^{-1}(t) \geq \Gamma^{-1}(0)$.

Defining $V_{\tilde{\theta}} = \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, from (3), (5) and (9)-(10) we get

$$\begin{aligned} \mathcal{L}V_{\tilde{\theta}} &= -2\Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^2 \tilde{\theta}^T \varphi_1 \varphi_1^T \tilde{\theta} + \tilde{\theta}^T (\dot{\Gamma^{-1}}) \tilde{\theta} \\ &\quad + \Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^4 \mathrm{Tr}\{\varphi_2 \varphi_1^T \Gamma \varphi_1 \varphi_2^T\} \\ &\leq -\Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^2 \tilde{\theta}^T \varphi_1 \varphi_1^T \tilde{\theta} \\ &\quad + \Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^4 \mathrm{Tr}\{\varphi_2 \varphi_1^T \Gamma(0) \varphi_1 \varphi_2^T\}. \end{aligned} \quad (11)$$

Remark 1. A novel least-squares estimator is proposed in (3)-(6), in which the time derivatives from the parametric model is absorbed into the parameter estimator and thus the need for filtering is removed. Although this kind of estimator design is motivated by Krstic (2009) for deterministic systems, the estimator in (3)-(6) is substantially different from that in Krstic (2009). In fact, the estimator in Krstic (2009) is inapplicable for stochastic system (1)-(2). Specifically, when the term $\left(x_n + \sum_{j=1}^{n-1} k_j x_j\right)^2$ is deleted from $\dot{\Gamma}$ in (5) and is removed from $h(x_1, \cdots, x_n)$ in (3), (4), (6), the estimator in this paper immediately reduces to the parameter estimator used in Krstic (2009). In this case, (11) becomes

$$\mathcal{L}V_{\tilde{\theta}} \leq -\tilde{\theta}^T \varphi_1 \varphi_1^T \tilde{\theta} + \mathrm{Tr}\{\varphi_2 \varphi_1^T \Gamma(0) \varphi_1 \varphi_2^T\}. \quad (12)$$

The Hessian term $Tr\{\varphi_2 \varphi_1^T \Gamma(0) \varphi_1 \varphi_2^T\}$ in (12) can neither be damped by the negative term $-\tilde{\theta}^T \varphi_1 \varphi_1^T \tilde{\theta}$ nor be dealt with by the controller due to the lack of weighted functions. To solve this difficult problem, we first introduce the term $\left(x_n + \sum_{j=1}^{n-1} k_j x_j\right)^2$ into the estimator (3)-(6), then suitably choose the design parameters $k_1, \cdots, k_{n-1}$ by using the controller gains $\alpha_1, \cdots, \alpha_{n-1}$ designed in the next subsection. In this way, the Hessian term $\mathrm{Tr}\{\varphi_2 \varphi_1^T \Gamma(0) \varphi_1 \varphi_2^T\}$ can be dealt with by the controller. More details are in the next two subsections.

3. ADAPTIVE CONTROL DESIGN

In this subsection, we design an adaptive controller for system (1)-(2) step by step.

Step 1. Define $V_1 = \frac{1}{4}\xi_1^4$, $\xi_1 = x_1$, then we have

$$\mathcal{L}V_1 = \xi_1^3(x_2 - x_2^*) + \xi_1^3 x_2^*. \quad (13)$$

Choosing

$$x_2^* = -c_1 \xi_1 \triangleq -\xi_1 \alpha_1, \quad (14)$$

which substitutes into (13) yields

$$\mathcal{L}V_1 = -c_1 \xi_1^4 + \xi_1^3 (x_2 - x_2^*), \quad (15)$$

where $c_1 > 0$ is a design parameter and $\alpha_1 = c_1$.

Deductive Step. Assume that at step k, there are a proper and positive definite Lyapunov function $V_k(\bar{\xi}_k)$ and a set of virtual controllers $x_{s+1}^* = -\xi_s \alpha_s$, $\xi_s = x_s - x_s^*$, $s = 2, \cdots, k$, such that

$$\mathcal{L}V_k \leq -\sum_{i=1}^{k} (c_i - a_{ik}) \xi_i^4 + \xi_k^3 (x_{k+1} - x_{k+1}^*), \quad (16)$$

where $a_{k,k} = 0$, α_s and $a_{s-1,k}$ are positive constants, $s = 2, \cdots, k$.

Let $\xi_{k+1} = x_{k+1} - x_{k+1}^*$. Noting that $\xi_{k+1} = x_{k+1} + \sum_{s=1}^{k} \alpha_k \cdots \alpha_s x_s$, we have

$$d\xi_{k+1} = \left(x_{k+2} + \sum_{s=1}^{k} \alpha_k \cdots \alpha_s x_{s+1} \right) dt. \quad (17)$$

At step $k+1$, choosing $V_{k+1} = V_k + \frac{1}{4}\xi_{k+1}^4$ and noting $x_{s+1} = \xi_{s+1} - \xi_s \alpha_s$, $s = 1, \cdots, k$, from (16)-(17) we obtain

$$\begin{aligned} \mathcal{L}V_{k+1} \leq & -\sum_{i=1}^{k} (c_i - a_{ik}) \xi_i^4 + \xi_k^3 \xi_{k+1} + \xi_{k+1}^3 x_{k+2} \\ & + \xi_{k+1}^3 \sum_{s=1}^{k} \alpha_k \cdots \alpha_s (\xi_{s+1} - \xi_s \alpha_s). \end{aligned} \quad (18)$$

By Young's inequality (eq.(2.253) in Krstic et al. (1995)) we get

$$\xi_k^3 \xi_{k+1} \leq a_{k,k+1,1} \xi_k^4 + \frac{27}{256} a_{k,k+1,1}^{-3} \xi_{k+1}^4, \quad (19)$$

$$\begin{aligned} \sum_{s=1}^{k} \alpha_k \cdots \alpha_s \xi_{s+1} \xi_{k+1}^3 \leq & \sum_{s=1}^{k-1} a_{s+1,k+1,2} \xi_{s+1}^4 \\ & + \left(\alpha_k + \sum_{s=1}^{k-1} \frac{3}{4} (4a_{s+1,k+1,2})^{-\frac{1}{3}} (\alpha_k \cdots \alpha_s)^{\frac{4}{3}} \right) \xi_{k+1}^4, \end{aligned} \quad (20)$$

$$\begin{aligned} -\sum_{s=1}^{k} \alpha_k \cdots \alpha_{s+1} \alpha_s^2 \xi_s \xi_{k+1}^3 \leq & \sum_{s=1}^{k} a_{s,k+1,3} \xi_s^4 \\ & + \sum_{s=1}^{k} \frac{3}{4} (4a_{s,k+1,3})^{-\frac{1}{3}} (\alpha_k \cdots \alpha_{s+1} \alpha_s^2)^{\frac{4}{3}} \xi_{k+1}^4, \end{aligned} \quad (21)$$

where $a_{k,k+1,1}$, $a_{2,k+1,2}$, $\cdots$, $a_{k+1,k+1,2}$, $a_{1,k+1,3}$, $\cdots$, $a_{k,k+1,3}$, are arbitrary positive constants.

Denoting

$$a_{1,k+1} = a_{1k} + a_{1,k+1,3}, \quad (22)$$

$$a_{i,k+1} = a_{ik} + a_{i,k+1,2} + a_{i,k+1,3}, i = 2, \ldots, k-1, \quad (23)$$

$$a_{k,k+1} = a_{kk} + a_{k,k+1,1} + a_{k,k+1,2} + a_{k,k+1,3}, \quad (24)$$

and substituting (19)-(24) into (18) yields

$$\begin{aligned} \mathcal{L}V_{k+1} \leq & -\sum_{i=1}^{k} (c_i - a_{i,k+1}) \xi_i^4 + \xi_{k+1}^3 (x_{k+2} - x_{k+2}^*) \\ & + \xi_{k+1}^3 x_{k+2}^* + \Big(\alpha_k + \frac{27}{256} a_{k,k+1,1}^{-3} \\ & + \sum_{s=1}^{k-1} \frac{3}{4} (4a_{s+1,k+1,2})^{-\frac{1}{3}} (\alpha_k \cdots \alpha_s)^{\frac{4}{3}} \\ & + \sum_{s=1}^{k} \frac{3}{4} (4a_{s,k+1,3})^{-\frac{1}{3}} (\alpha_k \cdots \alpha_{s+1} \alpha_s^2)^{\frac{4}{3}} \Big) \xi_{k+1}^4. \end{aligned} \quad (25)$$

Choosing the virtual controller

$$\begin{aligned} x_{k+2}^* = & -\xi_{k+1} \Big(c_{k+1} + \alpha_k + \frac{27}{256} a_{k,k+1,1}^{-3} \\ & + \sum_{s=1}^{k-1} \frac{3}{4} (4a_{s+1,k+1,2})^{-\frac{1}{3}} (\alpha_k \cdots \alpha_s)^{\frac{4}{3}} \\ & + \sum_{s=1}^{k} \frac{3}{4} (4a_{s,k+1,3})^{-\frac{1}{3}} (\alpha_k \cdots \alpha_{s+1} \alpha_s^2)^{\frac{4}{3}} \Big) \\ \triangleq & -\xi_{k+1} \alpha_{k+1}, \end{aligned} \quad (26)$$

with which (25) can be rewritten as

$$\mathcal{L}V_{k+1} \leq -\sum_{i=1}^{k+1} (c_i - a_{i,k+1}) \xi_i^4 + \xi_{k+1}^3 (x_{k+2} - x_{k+2}^*), \quad (27)$$

where $c_{k+1} > 0$ is a design parameter and $a_{k+1,k+1} = 0$.

Step n. Defining $\xi_n = x_n + \xi_{n-1} \alpha_{n-1}$, by (1), (2) and (26) we have

$$
\begin{aligned}
d\xi_n =& \Big(u + \varphi_1^T(x)\theta + \sum_{i=1}^{n-1} \alpha_{n-1}\cdots\alpha_i x_{i+1}\Big)dt \\
&+ \varphi_2^T(x)d\omega.
\end{aligned} \tag{28}
$$

Choosing $V_n = \frac{1}{4}\sum_{i=1}^{n} \xi_i^4$, from (28) we obtain

$$
\begin{aligned}
\mathcal{L}V_n \le& -\sum_{i=1}^{n-1}(c_i - a_{i,n-1})\xi_i^4 + \xi_{n-1}^3\xi_n \\
&+ \xi_n^3\Big(u + \varphi_1^T(x)\theta + \sum_{i=1}^{n-1} \alpha_{n-1}\cdots\alpha_i x_{i+1}\Big) \\
&+ \frac{3}{2}\xi_n^2|\varphi_2|^2.
\end{aligned} \tag{29}
$$

By Young's inequality we have

$$
\xi_{n-1}^3\xi_n \le a_{n-1,n,1}\xi_{n-1}^4 + \frac{1}{4}\left(\frac{4}{3}a_{n-1,n,1}\right)^{-3}\xi_n^4, \tag{30}
$$

where $a_{n-1,n,1}$ is any positive constant.

Since $\varphi_2(0) = 0$, $\varphi_2(x)$ can be written as $\varphi_2 = x^T\varphi_{20}(x)$, then we get

$$
\begin{aligned}
&\frac{3}{2}\xi_n^2|\varphi_2|^2 \\
&\le \frac{3}{2}\xi_n^2(x_1^2 + \cdots + x_n^2)|\varphi_{20}|^2 \\
&\le 3\xi_n^2(\xi_1^2 + \xi_2^2 + \xi_1^2\alpha_1^2 + \cdots + \xi_n^2 + \xi_{n-1}^2\alpha_{n-1}^2)|\varphi_{20}|^2 \\
&\le \xi_n^2(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)\varphi_{21} \\
&\le \sum_{i=1}^{n-1} a_{i,n,2}\xi_i^4 + \Big(\sum_{i=1}^{n-1}\frac{1}{4a_{i,n,2}}\varphi_{21}^2 + \varphi_{21}\Big)\xi_n^4,
\end{aligned} \tag{31}
$$

where $a_{i,n,2}$ is any positive constant and $\varphi_{21}(x)$ is a nonnegative function.

Substituting (30)-(31) into (29) we get

$$
\begin{aligned}
\mathcal{L}V_n \le& -\sum_{i=1}^{n-1}(c_i - a_{i,n})\xi_i^4 \\
&+ \xi_n^3\left(u + \varphi_1^T(x)\theta + \sum_{i=1}^{n-1}\alpha_{n-1}\cdots\alpha_i x_{i+1}\right) \\
&+ \left[\varphi_{21} + \frac{1}{4}\left(\frac{4}{3}a_{n-1,n,1}\right)^{-3} + \sum_{i=1}^{n-1}\frac{1}{4a_{i,n,2}}\varphi_{21}^2\right]\xi_n^4,
\end{aligned} \tag{32}
$$

where

$$
a_{i,n} = a_{i,n-1} + a_{i,n,2}, \quad i = 1, \ldots, n-2, \tag{33}
$$

$$
a_{n-1,n} = a_{n-1,n-1} + a_{n-1,n,1} + a_{n-1,n,2}. \tag{34}
$$

Choosing the actual controller as

$$
\begin{aligned}
u =& -\xi_n\Big(2 + \varphi_{21} + \frac{1}{4}(\frac{4}{3}a_{n-1,n})^{-3} \\
&+ \sum_{i=1}^{n-1}\frac{1}{4a_{i,n,2}}\varphi_{21}^2 + 2\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\}\Big) \\
&- \varphi_1^T\hat{\theta} - \sum_{i=1}^{n-1}\alpha_{n-1}\cdots\alpha_i x_{i+1},
\end{aligned} \tag{35}
$$

and substituting (35) into (32) yields

$$
\begin{aligned}
\mathcal{L}V_n \le& -\sum_{i=1}^{n-1}(c_i - a_{i,n})\xi_i^4 - 2\xi_n^4 + \xi_n^3\varphi_1^T\tilde{\theta} \\
&- 2\xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\}.
\end{aligned} \tag{36}
$$

4. STABILITY ANALYSIS

The first theorem below gives the stability results of the plant (1)-(6).

Theorem 1. Consider the closed-loop system consisting of the plant (1)-(6) and the controller (35). If the parameters k_i in (3) and c_i in (26) are selected as

$$
k_i = \prod_{j=1}^{n-i}\alpha_{n-j}, \quad i = 1, \cdots, n-1, \tag{37}
$$

$$
c_i = a_{i,n} + 1, \quad i = 1, \cdots, n-1, \tag{38}
$$

then the following conclusions hold:

1) The closed-loop system has an almost surely unique solution on $[0, +\infty)$;

2) The equilibrium $x = 0$, $\tilde{\theta} = 0$ is globally stable in probability;

3) $\lim_{t\to+\infty} x(t) = 0$ a.s..

Proof. From the controller design process developed in the last subsection we have

$$
\xi_n = x_n + \sum_{i=1}^{n-1} x_i\left(\prod_{j=1}^{n-i}\alpha_{n-j}\right). \tag{39}
$$

By (37) and (39), (11) can be rewritten as

$$
\mathcal{L}V_{\tilde{\theta}} \le -\xi_n^2\tilde{\theta}^T\varphi_1\varphi_1^T\tilde{\theta} + \xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\}. \tag{40}
$$

Substituting (38) into (36) we have

$$
\begin{aligned}
\mathcal{L}V_n \le& -\sum_{i=1}^{n}\xi_i^4 - \xi_n^4 + \xi_n^3\varphi_1^T\tilde{\theta} \\
&- 2\xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\}.
\end{aligned} \tag{41}
$$

Choosing $V = V_n + V_{\tilde{\theta}}$, from (40-(41) we get

$$
\begin{aligned}
\mathcal{L}V \le& -\sum_{i=1}^{n}\xi_i^4 - \xi_n^4 + \xi_n^3\varphi_1^T\tilde{\theta} - (\xi_n\varphi_1^T\tilde{\theta})^2 \\
&- \xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\} \\
=& -\sum_{i=1}^{n}\xi_i^4 - \frac{3}{4}(\xi_n\varphi_1^T\tilde{\theta})^2 - \xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\} \\
&- \left(\xi_n^2 - \frac{1}{2}\xi_n\varphi_1^T\tilde{\theta}\right)^2.
\end{aligned} \tag{42}
$$

By (42) and Theorem 2.1 in Deng et al. (2001), that conclusions 1)-2) hold and

$$
\lim_{t\to+\infty}\sum_{i=1}^{n}\xi_i^4 = 0 \quad a.s.. \tag{43}
$$

By (43) and the definitions of ξ_i we have

$$\lim_{t\to+\infty} x(t) = 0 \quad a.s., \tag{44}$$

which means that conclusion 3) holds.

The following lemma provides a basic property of the least-squares identification algorithm (3)-(6), which plays an essential role in the verification of convergence of the estimator. Due to the page limit, we omit its proof here.

Lemma 1. For the parameter estimator (3)-(6) with (37), using the controller (35) satisfying (38), we have $\int_0^{+\infty}(\xi_n\varphi_1^T\tilde{\theta})^2 ds < +\infty$ a.s..

Based on Lemma 1, we get the main results on the convergence of least-squares identification scheme (3)-(6).

Theorem 2. Using the controller (35) with (38), the parameter estimator (3)-(6) and (37) has a convergence property, i.e., $\hat{\theta}(t)$ converges almost surely to a finite vector-valued random variable θ_0.

Proof. From (3), (5), (9) and (37) we have

$$\dot{\Gamma} = -\xi_n^2\Gamma\varphi_1\varphi_1^T\Gamma, \quad \Gamma(0) = \Gamma^T(0) > 0, \tag{45}$$

$$d\tilde{\theta} = -\Gamma\xi_n^2\varphi_1\varphi_1^T\tilde{\theta}dt - \Gamma\xi_n^2\varphi_1\varphi_2^T dw. \tag{46}$$

By (46) we obtain

$$\tilde{\theta}(t) = \tilde{\theta}(0) - \int_0^t \Gamma\xi_n^2\varphi_1\varphi_1^T\tilde{\theta}ds - M_2(t), \tag{47}$$

where

$$\mathcal{M}_2(t) = \int_0^t \Gamma\xi_n^2\varphi_1\varphi_2^T dw. \tag{48}$$

From (45) we get

$$\int_0^t \xi_n^2\Gamma\varphi_1\varphi_1^T\Gamma ds = \Gamma(0) - \Gamma(t) \leq \Gamma(0), \tag{49}$$

which means that

$$\int_0^t |\xi_n\Gamma\varphi_1|^2 ds \leq Tr\{\Gamma(0)\}. \tag{50}$$

It follows from (50) and Lemma 1 that

$$\begin{aligned}&\left|\int_0^{+\infty}\Gamma\xi_n^2\varphi_1\varphi_1^T\tilde{\theta}ds\right| \\ &\leq \int_0^{+\infty}|\Gamma\xi_n^2\varphi_1\varphi_1^T\tilde{\theta}|ds \\ &\leq \left(\int_0^{+\infty}|\xi_n\Gamma\varphi_1|^2 ds\right)^{1/2}\left(\int_0^{+\infty}(\xi_n\varphi_1^T\tilde{\theta})^2 ds\right)^{1/2} \\ &< +\infty \quad a.s..\end{aligned} \tag{51}$$

For any $r \geq 0$, defining the stopping time

$$\sigma_r = \inf\{t \geq 0 : |\tilde{\theta}| + |x| \geq r\}. \tag{52}$$

Obviously, $\sigma_r \to +\infty$ almost surely as $r \to +\infty$.

From (42) and (52) we have

$$\begin{aligned}&E\left\{\int_0^{t\wedge\sigma_r}\xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\}ds\right\} \\ &\leq V(0) - EV(t\wedge\sigma_r) \leq V(0).\end{aligned} \tag{53}$$

For (53), firstly set $r \to +\infty$ then set $t \to +\infty$, using Fatou's lemma, we get

$$E\left\{\int_0^{+\infty}\xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\}ds\right\} \leq V(0). \tag{54}$$

On the other hand, noting that $\Gamma^{-1}(0) > 0$ we obtain

$$\begin{aligned}&\int_0^t \xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(t)\varphi_1\varphi_2^T\}ds \\ &= \int_0^t \xi_n^4|\varphi_2|^2\mathrm{Tr}\{\varphi_1^T\Gamma(t)\varphi_1\}ds \\ &= \int_0^t \xi_n^4|\varphi_2|^2\mathrm{Tr}\{\varphi_1^T\Gamma(t)\Gamma^{-1}(t)\Gamma(t)\varphi_1\}ds \\ &\geq \lambda_{\min}\{\Gamma^{-1}(0)\}\int_0^t \xi_n^4|\varphi_2|^2|\Gamma(t)\varphi_1|^2 ds.\end{aligned} \tag{55}$$

From (5) we know $0 < \Gamma(t) \leq \Gamma(0)$, then we have

$$\begin{aligned}&\int_0^t \xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(t)\varphi_1\varphi_2^T\}ds \\ &\leq \int_0^{+\infty}\xi_n^4\mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\}ds.\end{aligned} \tag{56}$$

By (54)-(56) we get

$$\begin{aligned}&E\left\{\int_0^t \xi_n^4|\varphi_2|^2|\Gamma(t)\varphi_1|^2 ds\right\} \\ &\leq \frac{1}{\lambda_{\min}\{\Gamma^{-1}(0)\}}V(0) < +\infty.\end{aligned} \tag{57}$$

By (57) and Fubini's theorem, we have

$$\begin{aligned}&E\left\{\int_0^t |\Gamma(t)\xi_n^2\varphi_1\varphi_2^T|^2 ds\right\} \\ &= \int_0^t E\left\{|\Gamma(t)\xi_n^2\varphi_1\varphi_2^T|^2\right\} ds \\ &\leq \int_0^t E\left\{\xi_n^4|\varphi_2|^2|\Gamma(t)\varphi_1|^2\right\} ds\end{aligned}$$

$$= E\left\{\int_0^t \xi_n^4|\varphi_2|^2|\Gamma(t)\varphi_1|^2 ds\right\}$$
$$< +\infty. \tag{58}$$

Therefore, $\mathcal{M}_2(t)$ defined in (48) is a continuous martingale.

From (57)-(58) we get

$$\sup_{t\geq 0} E|\mathcal{M}_2(t)|^2 \leq \frac{1}{\lambda_{\min}\{\Gamma^{-1}(0)\}}V(0) < +\infty. \tag{59}$$

By Doob's martingale convergence theorem, we obtain

$$\lim_{t\to+\infty} \mathcal{M}_2(t) = \mathcal{M}_{01}, \tag{60}$$

where $\mathcal{M}_{01}$ is a a random variable satisfying $E|\mathcal{M}_{01}|^2 < +\infty$.

By (47), (51) and (60) we get

$$\lim_{t\to+\infty} \hat{\theta}(t) = \theta_0 \ \ \text{a}.s., \tag{61}$$

where θ_0 is a vector-valued random variable.

Remark 2. In this paper, by introducing the term $\Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^2$ into the estimator (3)-(6), we develop a new least-squares identification scheme. The weighted term $\Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^2$ has the following advantages:

(1) From the proof of Theorem 1, we find that this weighted term can make the Hessian term $\Big(x_n + \sum_{j=1}^{n-1} k_j x_j\Big)^4 \cdot \mathrm{Tr}\{\varphi_2\varphi_1^T\Gamma(0)\varphi_1\varphi_2^T\}$ in (11) be easily dominated by the controller (35).

(2) From the proof of Theorem 2, it is obviously that this weighted term plays an important role in the convergence of the estimator (3)-(6). In fact, as demonstrated by Proposition 2 in Nassiri-Toussi and Ren (1994), even for stochastic linear systems, if there is no any weighted term in the least-squares algorithms, the boundedness of $\hat{\theta}$ can hardly be guaranteed, much less the convergence.

5. CONCLUDING REMARKS

In this paper we have studied the least-squares identification and adaptive control for stochastic strict-feedback nonlinear systems with an unknown parameter in the drift term. The designed controller guarantees that the equilibrium at the origin of the closed-loop system is globally stable in probability and the states are regulated to zero almost surely. By suitably selecting the estimator parameters, we prove that the proposed least-squares estimator is convergent.

For the least-squares based adaptive control of stochastic nonlinear systems, many important issues are still open and worth investigating, such as the adaptive controls for more general stochastic systems shown in Li et al. (2020).

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