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Adaptive Prescribed-Time Control for Uncertain Nonlinear Systems with Non-affine Actuator Failures ⋆

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Abstract: In this paper, we present a novel prescribed-time fault-tolerant control method for a class of nonlinear systems with time-varying unmodeled actuator faults. Non-affine actuator failures and uncertain control direction can be addressed in a universal control framework, where any prior information about faults is not required in control design. We show that, with the proposed control scheme, the system trajectory can converge to a user-defined residual-set within prescribed settling time. The requirements on pre-assigned rapidity and accuracy can be simultaneously satisfied, leading to the settling time and convergence set only determined by fewer user-defined parameters rather than approximation errors, which is fundamentally different from conventional finite/fixed-time control. Simulation and experiment results are provided to validate the effectiveness of the proposed controller.

Keywords: adaptive control, prescribed-time stability, unmodeled actuator fault, fault-tolerant control, nonlinear systems.

1. INTRODUCTION

Compared with asymptotic control, finite-time control provides a new control framework to obtain an estimable settling time. Because of more superior performance (i.e., convergence speed, robustness, and accuracy) than traditional asymptotic control, corresponding definitions Bhat and Bernstein (2000); Yu et al. (2005); Shen and Xia (2008); Zheng et al. (2011); Hu and Jiang (2018) and applications Wang et al. (2018); Van et al. (2017); Li and Wang (2013); Yang et al. (2015) have been extensively studied in the field of finite-time control. However, the settling time is dependent on system initial values, which indicates that the settling time cannot be obtained once the initial values are unknown or sensitive to external noise. To deal with this issue, as an extended concept, fixed-time stability Polyakov (2012) was proposed to address the dependence of settling time on initial conditions, and thus triggered increasing subsequent researches Polyakov et al. (2015); Chen and Li (2018); Wang et al. (2019); Yang et al. (2017); Zuo et al. (2018); Ríos et al. (2017). By adjusting design parameters, the system trajectory under fixed-time control is driven to converge to a given set within predefined time subject to control parameter constraints. It is noted that, however, the settling time follows the restriction of design parameters, and thus cannot be user-defined arbitrarily though it is independent on initial conditions. In addition, the so-called finite-time or fixed-time is often the estimated bound of the actual settling time, which is larger or far larger than the actual one. Featured by user-defined settling time, prescribed-time stability Sánchez-Torres et al. (2015); Fraguela et al. (2012) was proposed recently for nonlinear system stabilization to realize that the system trajectory was ensured to strictly converge to zero within prescribed time Song et al. (2017); Wang et al. (2019); Song et al. (2019), which achieved a higher accuracy compared to finite/fixed-time control. However, in terms of energy consumption, the convergence accuracy is required to satisfy the predefined requirement, rather than the excessive precision in practice. Limited by sensor accuracy and actuator output, user-defined convergence accuracy is more practical so that the aforementioned methods would obtain limited performance. To the best knowledge of the authors, there is no work aiming at dealing with such issue.

Moreover, a premise that all actuators work properly is widely utilized in the fore-mentioned literature. However, external disturbances and systems uncertainties are inevitable for practical control systems such that actuators are vulnerable to various faults. A classic fault-tolerant control method is to estimate the boundary of partial

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loss of actuator effectiveness via universal approximation approaches, such as fuzzy-logic systems (FLSs). The unknown disturbance of state and input was discussed in Li and Yang (2016), where the lumped uncertainty was approximated by FLSs. With Takagi-Sugeno fuzzy models, an actuator-fault compensation control strategy Jiang et al. (2010) was developed for attitude control systems. Modeled as both loss of effectiveness and lock-in-place, actuator faults existing in large-scale and stochastic nonlinear systems were compensated in Tong et al. (2014a,b), respectively. However, there exist the following two issues in the mentioned control methods. (1) These adaptive control schemes will become ineffective when dealing with unmodeled nonlinear faults augmented by states and control input. (2) In spite of bounded states realized in the aforementioned approaches, the convergence set is dependent on the approximation error. The above analysis indicates that the fault-tolerant control with respect to a more general actuator-fault form is more challenging, and there has not been no work discussing prescribed-time convergence in the framework of fault-tolerant control.

Motivated by the above discussion, we present a new prescribed-time control method for a class of nonlinear systems with unknown disturbances, uncertain control direction and non-affine actuator faults. We show that, with the developed control scheme, the prescribed-time convergence of the tracking error into a user-defined residual-set is guaranteed.

The remainder of this paper is organized as follows. Sec. 2 introduces the definition of *practically prescribed-time stability* and the corresponding property. Sec. 3 gives the prescribed-time fault-tolerant control strategy. In Sec. 4 simulation and experiment results show the effectiveness of the proposed controller, followed by the conclusion in Sec. 5.

2. PRELIMINARIES

2.1 Time-varying piece-wise function

To achieve prescribed-time stabilization, we introduce a time-varying scaling piece-wise function as follows

$$\varsigma(t) = \begin{cases} \exp\left(\alpha\left(t_0 + T - t\right)\right) - 1, & t \in [t_0, t_0 + T) \\ 1 - \tanh\left(\alpha\left(t - t_0 - T\right)\right), & t \in [t_0 + T, +\infty) \end{cases} \quad (1)$$

where $\exp(\cdot)$ and $\tanh(\cdot)$ represent the exponential and hyperbolic tangent function, respectively. T denotes the prescribed settling time such that $T \geq T_s > 0$, in which T_s represents the time consuming of signal transmission and processor computing. α is a user-defined positive constant. t_0 stands for the initial time. Note that $\varsigma(t)$ is piece-wise monotonically decreasing to zero. In addition, the time derivative of $\varsigma(t)$ can be obtained as below

$$\dot{\varsigma}(t) = \begin{cases} -\alpha\left(\varsigma(t) + 1\right), & t \in [t_0, t_0 + T) \\ \alpha\left[\left(1 - \varsigma(t)\right)^2 - 1\right], & t \in [t_0 + T, +\infty) \end{cases} \quad (2)$$

which indicates that $\dot{\varsigma}^{\dagger} \triangleq |\dot{\varsigma}(t)|$ is continuous at $t = t_0 + T$, and then the smooth property of $\varsigma(t)$ is available.

2.2 Practically prescribed-time stability

Definition 1. (Sánchez-Torres et al. (2015); Fraguela et al. (2012)) Consider a class of nonlinear systems

$$\dot{x}(t) = f(x(t), t) \quad (3)$$

where $x(t) \in U_0 \subset \mathbb{R}^n$ is the system state and $f: U_0 \times \mathbb{R}_+ \to \mathbb{R}^n$ is a continuous-differential function. The origin of (3) is *prescribed-time stable* (PTS) if it is *fixed-time stable* and the settling time T is artificially designed such that $T_s \leq T \leq T_{\max} < +\infty$.

Definition 2. (Wang et al. (2020)) The origin of (3) is called to be *practically prescribed-time stable* (PPTS) if $\|x(t)\| \leq \varepsilon$ for $t \geq t_0 + T$, where ε and T are pre-specified, and $T_s \leq T \leq T_{\max} < +\infty$.

Lemma 1. (Wang et al. (2020)) If there exists a positive-definite continuous-differential function $V(x(t), t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ and scalars $b > 0, c \geq 0$, $0 \leq \eta < +\infty$ such that

$$\dot{V} \leq -bV - 2\frac{\dot{\varsigma}^{\dagger}}{\varsigma}V + \frac{\eta}{\varsigma} + c, \quad (4)$$

then the trajectory of the system (3) is PPTS with $D = \{x \,|V(x) \leq \eta/\alpha\}$ holding for $t \geq t_0 + T$.

Remark 1. The prescribed settling time and convergence domain are simultaneously available in *Lemma 1*, which extends the concept of PTS. In addition, with $\eta = 0, c = 0$, we have $D = \{x \,|V(x) \leq 0\}$, leading to $x = 0$ for $t \in [t_0 + T, +\infty)$. Thus, the origin of system (3) is PTS. Therefore, PTS is a special case of PPTS, which is more universal to carry out stability analysis and control synthesis.

Remark 2. For the finite-time stabilization issue of systems with unknown disturbance, *practically finite-time stability* Shen and Xia (2008) and *finite-time uniformly ultimately boundedness* Hu and Jiang (2018) were proposed to give a parameter-dependent convergence set. However, the convergence domain is dependent on the upper-bound of disturbances. By contrast, the residual set in *Lemma 1* can be uniform, only dependent on two user-defined parameters rather than upper-bound of disturbances.

3. PRACTICALLY PRESCRIBED-TIME CONTROLLER DESIGN

We consider a class of nonlinear systems with parametric and non-parametric uncertainties, and unmodeled actuator faults

$$\begin{aligned} &\dot{x}_i(t) = B_i(\bar{x}_i(t), t)x_{i+1}(t) + \theta_i^{\mathrm{T}}(t)f_i(t) + h_i(\bar{x}_i(t), t), \\ &i = 1, 2, ..., n-1, \\ &\dot{x}_n(t) = d_n B_n(\bar{x}_n(t), t)\varphi(u(t), \bar{x}_n(t)) \\ &\qquad + \theta_n^{\mathrm{T}}(t)f_n(t) + h_n(\bar{x}_n(t), t), \end{aligned} \quad (5)$$

where $x_i(t) \in \mathbb{R}^m$ is the state vector; $B_i(\bar{x}_i(t), t) \in \mathbb{R}^{m \times m}$ is the unknown gain function with $\bar{x}_i(t) = \left[x_1(t)^{\mathrm{T}}, ..., x_i(t)^{\mathrm{T}}\right]^{\mathrm{T}}$; $\theta_i^{\mathrm{T}}(t) \in \mathbb{R}^{m \times p}$ is the unknown time-varying function while $f_i(t) \in \mathbb{R}^{p \times 1}$ is known nonlinear function vector; $h_i(\bar{x}_i(t), t)$ is the unknown non-parametric disturbance, and $\varphi(u(t), \bar{x}_n(t)) : \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}^m$ is the control input with unmodeled actuator faults. $d_n \in \{-1, 1\}$ denotes the unknown control direction.

Assumption 1 (Jin (2017)). There exist upper and lower bounds for the unknown gain function, that is $\underline{\gamma}_i \leq \|B_i(\bar{x}_i(t), t)\| \leq \bar{\gamma}_i$, $i = 1, 2, ..., n$, where $\underline{\gamma}_i$ and $\bar{\gamma}_i$ are unknown constants.

Assumption 2 (Jin (2019)). The unknown functions $\theta_i(t)$ and $h_i(\bar{x}_i(t), t)$ are bounded such that $\|\theta_i(t)\| \leq \bar{\theta}_i$,

$\|h_i(\bar{x}_i(t), t)\| \le \bar{h}_i$, $i = 1, 2, ..., n$, where $\bar{\theta}_i$ and $\bar{h}_i$ are unknown constants.

Assumption 3(Zhang and Guay (2003)). $\frac{\partial \varphi(u, \bar{x}_n)}{\partial u} \neq 0$ and there exist unknown positive constants such that $\underline{\pi} \le \|\partial \varphi(u, \bar{x}_n)/\partial u\| \le \bar{\pi}$ and $\underline{\pi}^{\star} \le \|\varphi(u, \bar{x}_n)\| \le \bar{\pi}^{\star}$.

Remark 3: Since the measured states are subject to specific physical range, *Assumptions* 1-2 are rational in accord with practical applications. The conventional assumption that unknown gain functions are symmetric is relaxed in (5). Different from conventional actuator faults discussed in Jiang et al. (2010); Tong et al. (2014a,b); Jin (2017, 2019), the nonlinear and non-affine fault with respect to control input and states is considered in this paper, which is a more general form including saturation and dead-zone nonlinearity Hu et al. (2008).

Define the desired state variable as $x_{id}(t) \in \mathbb{R}^m$, which is bounded, continuous, and differentiable up to the n-th order, and the tracking error is represented by $e_i(t) = x_i(t) - x_{id}(t)$. The control objective can be stated as follows. For the nonlinear system (5), design the control scheme to achieve that the tracking error converges into a prescribed range within prescribed time.

We present the design procedure by back-stepping approach, which consists of n steps. For brevity, the argument presented in (t) and $(\bar{x}_i(t), t)$ will be omitted later.

Step 1: Define $z_1 = e_1$ and $z_2 = x_2 - \rho_1$, where ρ_1 is the virtual stabilizing function to be designed later. The time derivative of z_1 gives

$$\dot{z}_1 = B_1 (z_2 + \rho_1) + \theta_1^{\mathrm{T}} f_1 + h_1 - \dot{x}_{1d}. \quad (6)$$

Define $V_{z_1} = \frac{1}{2} z_1^{\mathrm{T}} z_1$. Taking the time-derivative of V_{z_1} and substituting (6) into it yield

$$\dot{V}_{z_1} = z_1^{\mathrm{T}} B_1 z_2 + z_1^{\mathrm{T}} B_1 \rho_1 + z_1^{\mathrm{T}} \theta_1^{\mathrm{T}} f_1 + z_1^{\mathrm{T}} h_1 - z_1^{\mathrm{T}} \dot{x}_{1d}. \quad (7)$$

Note that in (7), we have

$$z_1^{\mathrm{T}} \theta_1^{\mathrm{T}} f_1 \le \|z_1\| \bar{\theta}_1 \|f_1\| < \varepsilon_1 \bar{\theta}_1 + \bar{\theta}_1 \frac{z_1^{\mathrm{T}} z_1 f_1^{\mathrm{T}} f_1}{\sqrt{z_1^{\mathrm{T}} z_1 f_1^{\mathrm{T}} f_1 + \varepsilon_1^2}}, \quad (8)$$

$$z_1^{\mathrm{T}} h_1 \le \|z_1\| \bar{h}_1 < \varepsilon_1 \bar{h}_1 + \bar{h}_1 \frac{z_1^{\mathrm{T}} z_1}{\sqrt{z_1^{\mathrm{T}} z_1 + \varepsilon_1^2}}, \quad (9)$$

where $\varepsilon_i > 0$ is a small positive constant for $i = 1, 2, ..., n$.

Substituting (8)-(9) to (7) yields

$$\dot{V}_{z_1} \le z_1^{\mathrm{T}} B_1 z_2 + z_1^{\mathrm{T}} B_1 \rho_1 + z_1^{\mathrm{T}} \Theta_1 \xi_1 - z_1^{\mathrm{T}} \dot{x}_{1d} + \varepsilon_1 (\bar{\theta}_1 + \bar{h}_1), \quad (10)$$

where $\Theta_1 = [\bar{\theta}_1 I_m, \bar{h}_1 I_m]$, $\xi_1 = \left[\frac{z_1^{\mathrm{T}} f_1^{\mathrm{T}} f_1}{\sqrt{z_1^{\mathrm{T}} z_1 f_1^{\mathrm{T}} f_1 + \varepsilon_1^2}}, \frac{z_1^{\mathrm{T}}}{\sqrt{z_1^{\mathrm{T}} z_1 + \varepsilon_1^2}} \right]^{\mathrm{T}}$, I_m is the m-dimensional identity matrix.

For the third term on the right side of (10), one has

$$z_1^{\mathrm{T}} \Theta_1 \xi_1 < \varepsilon_1 \bar{\Theta}_1 + \bar{\Theta}_1 \frac{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1}{\sqrt{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1 + \varepsilon_1^2}}, \quad (11)$$

where $\|\Theta_1\| < \bar{\Theta}_1$ with $\bar{\Theta}_1$ being an unknown constant.

Define the Lyapunov function candidate as

$$V_1 = V_{z_1} + \frac{1}{2\sigma_{\bar{\Theta}_1}} \tilde{\bar{\Theta}}_1^2 + \frac{\underline{\gamma}_1}{2\sigma_{\beta_1}} \tilde{\beta}_1^2, \quad (12)$$

where $\tilde{\bar{\Theta}}_i = \bar{\Theta}_i - \hat{\bar{\Theta}}_i$, $\tilde{\beta}_i = \beta_i - \hat{\beta}_i$, and $1/\underline{\gamma}_i = \beta_i$ for $i = 1, 2, ..., n-1$. $\hat{\bar{\Theta}}_i$ and $\hat{\beta}_i$ are the estimation value of $\bar{\Theta}_i$ and β_i, respectively. $\sigma_{\bar{\Theta}_i}$ and σ_{β_i} are designed parameters. Taking derivative of (12) with respect to time and introducing (10)-(11) into it yield

$$\begin{aligned} \dot{V}_1 \le\ & z_1^{\mathrm{T}} B_1 z_2 + z_1^{\mathrm{T}} B_1 \rho_1 + \bar{\Theta}_1 \frac{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1}{\sqrt{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1 + \varepsilon_1^2}} \\ & - z_1^{\mathrm{T}} \dot{x}_{1d} + \varepsilon_1 (\bar{\theta}_1 + \bar{h}_1 + \bar{\Theta}_1) + \frac{1}{\sigma_{\bar{\Theta}_1}} \tilde{\bar{\Theta}}_1 \dot{\tilde{\bar{\Theta}}}_1 + \frac{\underline{\gamma}_1}{\sigma_{\beta_1}} \tilde{\beta}_1 \dot{\tilde{\beta}}_1. \end{aligned} \quad (13)$$

Design the virtual stabilizing function as

$$\rho_1 = -\frac{\hat{\beta}_1^2 z_1 \bar{\rho}_1^{\mathrm{T}} \bar{\rho}_1}{\sqrt{\hat{\beta}_1^2 z_1^{\mathrm{T}} z_1 \bar{\rho}_1^{\mathrm{T}} \bar{\rho}_1 + \varepsilon_1^2}} \quad (14)$$

with

$$\begin{aligned} \bar{\rho}_1 =\ & \hat{\bar{\Theta}}_1 \frac{z_1 \xi_1^{\mathrm{T}} \xi_1}{\sqrt{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1 + \varepsilon_1^2}} - \dot{x}_{1d} \\ & + \left(k_1 + \frac{\varsigma^{\dagger}}{\varsigma}\right) z_1 - \frac{\eta}{\varsigma} \frac{z_1}{\|z_1\|^2 + \varepsilon_1^2} \end{aligned} \quad (15)$$

where k_1 and η are positive tunable parameters.

Note that the following inequality holds

$$z_1^{\mathrm{T}} B_1 \rho_1 \le -\underline{\gamma}_1 \frac{\hat{\beta}_1^2 z_1^{\mathrm{T}} z_1 \bar{\rho}_1^{\mathrm{T}} \bar{\rho}_1}{\sqrt{\hat{\beta}_1^2 z_1^{\mathrm{T}} z_1 \bar{\rho}_1^{\mathrm{T}} \bar{\rho}_1 + \varepsilon_1^2}} \le \varepsilon_1 \underline{\gamma}_1 - \underline{\gamma}_1 z_1^{\mathrm{T}} \hat{\beta}_1 \bar{\rho}_1. \quad (16)$$

Therefore, with (13)-(16), we have

$$\begin{aligned} \dot{V}_1 \le\ & z_1^{\mathrm{T}} B_1 z_2 - \underline{\gamma}_1 z_1^{\mathrm{T}} \hat{\beta}_1 \bar{\rho}_1 + \bar{\Theta}_1 \frac{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1}{\sqrt{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1 + \varepsilon_1^2}} \\ & - z_1^{\mathrm{T}} \dot{x}_{1d} - \frac{1}{\sigma_{\bar{\Theta}_1}} \tilde{\bar{\Theta}}_1 \dot{\hat{\bar{\Theta}}}_1 - \frac{\underline{\gamma}_1}{\sigma_{\beta_1}} \tilde{\beta}_1 \dot{\hat{\beta}}_1 + c_1 \end{aligned} \quad (17)$$

where $c_1 = \varepsilon_1 \left(\bar{\theta}_1 + \bar{h}_1 + \bar{\Theta}_1 + \underline{\gamma}_1 \right)$.

Construct the adaptive laws as below

$$\dot{\hat{\bar{\Theta}}}_1 = \sigma_{\bar{\Theta}_1} \left[\frac{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1}{\sqrt{z_1^{\mathrm{T}} z_1 \xi_1^{\mathrm{T}} \xi_1 + \varepsilon_1^2}} - \hat{\bar{\Theta}}_1 \right], \quad (18)$$

$$\dot{\hat{\beta}}_1 = \sigma_{\beta_1} \left[z_1^{\mathrm{T}} \bar{\rho}_1 - \hat{\beta}_1 \right]. \quad (19)$$

Then, substituting (15) into (17) and applying (18)-(19) yield

$$\begin{aligned} \dot{V}_1 \le\ & z_1^{\mathrm{T}} B_1 z_2 - \left(k_1 + \frac{\varsigma^{\dagger}}{\varsigma}\right) z_1^{\mathrm{T}} z_1 + \frac{\eta}{\varsigma} \\ & + \tilde{\bar{\Theta}}_1 \hat{\bar{\Theta}}_1 + \underline{\gamma}_1 \tilde{\beta}_1 \hat{\beta}_1 + c_1 \\ =\ & z_1^{\mathrm{T}} B_1 z_2 - 2\left(k_1 + \frac{\varsigma^{\dagger}}{\varsigma}\right) V_1 + \frac{\eta}{\varsigma} + k_1 \left(\frac{1}{\sigma_{\bar{\Theta}_1}} \tilde{\bar{\Theta}}_1^2 \right. \\ & \left. + \frac{\underline{\gamma}_1}{\sigma_{\beta_1}} \tilde{\beta}_1^2 \right) + \frac{\varsigma^{\dagger}}{\varsigma} \left(\frac{1}{\sigma_{\bar{\Theta}_1}} \tilde{\bar{\Theta}}_1^2 + \frac{\underline{\gamma}_1}{\sigma_{\beta_1}} \tilde{\beta}_1^2 \right) \\ & + \tilde{\bar{\Theta}}_1 \hat{\bar{\Theta}}_1 + \underline{\gamma}_1 \tilde{\beta}_1 \hat{\beta}_1 + c_1. \end{aligned} \quad (20)$$

Note that the following inequality holds

$$\begin{aligned}\tilde{\bar{\Theta}}_1\hat{\bar{\Theta}}_1 &= \tilde{\bar{\Theta}}_1(-\tilde{\bar{\Theta}}_1 + \bar{\Theta}_1)\\ &\leq -\tilde{\bar{\Theta}}_1^2 + \frac{1}{2}\tilde{\bar{\Theta}}_1^2 + \frac{1}{2}\bar{\Theta}_1^2\\ &= -\frac{1}{2}\tilde{\bar{\Theta}}_1^2 + \frac{1}{2}\bar{\Theta}_1^2.\end{aligned} \tag{21}$$

Let $\sigma_{\bar{\Theta}_1} = \sigma_{\beta_1} = 2k_1 + 2\alpha\left(\frac{1}{\varsigma(t)} + 1\right)$, then we have from (21) that

$$\begin{aligned}&\frac{k_1}{\sigma_{\bar{\Theta}_1}}\tilde{\bar{\Theta}}_1^2 + \frac{\dot{\varsigma}^\dagger}{\varsigma}\frac{1}{\sigma_{\bar{\Theta}_1}}\tilde{\bar{\Theta}}_1^2 + \tilde{\bar{\Theta}}_1\hat{\bar{\Theta}}_1\\ &\leq \frac{\tilde{\bar{\Theta}}_1^2}{\sigma_{\bar{\Theta}_1}}\left(k_1 + \frac{\alpha(1+\varsigma(t))}{\varsigma(t)}\right) - \frac{1}{2}\tilde{\bar{\Theta}}_1^2 + \frac{1}{2}\bar{\Theta}_1^2\\ &= \frac{1}{2}\bar{\Theta}_1^2.\end{aligned} \tag{22}$$

And in the same manner with (21)-(22), we have

$$k_1\frac{\underline{\gamma}_1}{\sigma_{\beta_1}}\tilde{\beta}_1^2 + \frac{\dot{\varsigma}^\dagger}{\varsigma}\frac{\underline{\gamma}_1}{\sigma_{\beta_1}}\tilde{\beta}_1^2 + \underline{\gamma}_1\tilde{\beta}_1\hat{\beta}_1 \leq \frac{\beta_1}{2}. \tag{23}$$

Let $\bar{c}_1 = c_1 + \frac{1}{2}\bar{\Theta}_1^2 + \frac{\beta_1}{2}$, by employing (20)-(23), and then one can obtain

$$\dot{V}_1 \leq z_1^{\mathrm{T}}B_1z_2 - 2k_1V_1 - 2\frac{\dot{\varsigma}^\dagger}{\varsigma}V_1 + \frac{\eta}{\varsigma} + \bar{c}_1. \tag{24}$$

Step $i(i = 2, 3, ..., n-1)$: Define $z_{i+1} = x_{i+1} - \rho_i$, where ρ_i is the virtual stabilizing function to be designed later. The time derivative of z_i gives

$$\dot{z}_i = B_iz_{i+1} + B_i\rho_i + \theta_i^{\mathrm{T}}f_i + h_i - \dot{\rho}_{i-1} \tag{25}$$

where

$$\dot{\rho}_{i-1} = \sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial x_j}\left(B_jx_{j+1} + \theta_j^{\mathrm{T}}f_j + h_j\right) + \varpi_i \tag{26}$$

with

$$\begin{aligned}\varpi_i = &\sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial\hat{\beta}_j}\dot{\hat{\beta}}_j + \sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial\hat{\bar{\Theta}}_j}\dot{\hat{\bar{\Theta}}}_j\\ &+ \sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial x_{1d}^{(j-1)}}x_{1d}^{(j)} + \sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial\varsigma^{(j-1)}}\varsigma^{(j)}.\end{aligned} \tag{27}$$

Define $V_{zi} = \frac{1}{2}z_i^{\mathrm{T}}z_i$. Taking the derivative of V_{zi} and combining it with (25) give

$$\begin{aligned}\dot{V}_{zi} =& z_i^{\mathrm{T}}B_iz_{i+1} + z_i^{\mathrm{T}}B_i\rho_i + z_i^{\mathrm{T}}\theta_i^{\mathrm{T}}f_i + z_i^{\mathrm{T}}h_i - z_i^{\mathrm{T}}\varpi_i\\ &- z_i^{\mathrm{T}}\sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial x_j}\left(B_jx_{j+1} + \theta_j^{\mathrm{T}}f_j + h_j\right).\end{aligned} \tag{28}$$

Similarly with (8)-(9), we have

$$z_i^{\mathrm{T}}\theta_i^{\mathrm{T}}f_i \leq \|z_i\|\,\bar{\theta}_i\,\|f_i\| < \varepsilon_i\bar{\theta}_i + \bar{\theta}_i\frac{z_i^{\mathrm{T}}z_if_i^{\mathrm{T}}f_i}{\sqrt{z_i^{\mathrm{T}}z_if_i^{\mathrm{T}}f_i + \varepsilon_i^2}} \tag{29}$$

$$z_i^{\mathrm{T}}h_i \leq \|z_i\|\,\bar{h}_i < \bar{h}_i\varepsilon_i + \bar{h}_i\frac{z_i^{\mathrm{T}}z_i}{\sqrt{z_i^{\mathrm{T}}z_i + \varepsilon_i^2}} \tag{30}$$

and

$$\begin{aligned}&- z_i^{\mathrm{T}}\sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial x_j}B_jx_{j+1}\\ &\leq \sum_{j=1}^{i-1}\bar{\gamma}_j\frac{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_j}\right\|^2\|x_{j+1}\|^2}{\sqrt{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_j}\right\|^2\|x_{j+1}\|^2 + \varepsilon_i^2}} + \sum_{j=1}^{i-1}\varepsilon_j\bar{\gamma}_j,\end{aligned} \tag{31}$$

$$\begin{aligned}&- z_i^{\mathrm{T}}\sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial x_j}\theta_j^{\mathrm{T}}f_j\\ &\leq \sum_{j=1}^{i-1}\bar{\theta}_j\frac{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_j}\right\|^2\|f_j\|^2}{\sqrt{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_j}\right\|^2\|f_j\|^2 + \varepsilon_i^2}} + \sum_{j=1}^{i-1}\varepsilon_j\bar{\theta}_j,\end{aligned} \tag{32}$$

$$\begin{aligned}&- z_i^{\mathrm{T}}\sum_{j=1}^{i-1}\frac{\partial\rho_{i-1}}{\partial x_j}h_j\\ &\leq \sum_{j=1}^{i-1}\bar{h}_j\frac{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_j}\right\|^2}{\sqrt{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_j}\right\|^2 + \varepsilon_i^2}} + \sum_{j=1}^{i-1}\varepsilon_j\bar{h}_j.\end{aligned} \tag{33}$$

Similarly with (10)-(11), we have

$$\begin{aligned}\dot{V}_{zi} \leq& z_i^{\mathrm{T}}B_iz_{i+1} + z_i^{\mathrm{T}}B_i\rho_i + z_i^{\mathrm{T}}\Theta_i\xi_i - z_i^{\mathrm{T}}\varpi_i\\ &+ \sum_{j=1}^{i}\varepsilon_j\left(\bar{\theta}_j + \bar{h}_j\right) + \sum_{j=1}^{i-1}\varepsilon_j\bar{\gamma}_j\end{aligned} \tag{34}$$

where $\Theta_i = \left[\bar{\theta}_1I_m, \bar{h}_1I_m, \bar{\gamma}_1I_m, ..., \bar{\theta}_iI_m, \bar{h}_iI_m, \bar{\gamma}_iI_m\right]$, and

$$\xi_i = \left[\frac{z_i^{\mathrm{T}}\left\|\frac{\partial\rho_{i-1}}{\partial x_1}\right\|^2\|f_1\|^2}{\sqrt{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_1}\right\|^2\|f_1\|^2 + \varepsilon_i^2}}, \frac{z_i^{\mathrm{T}}\left\|\frac{\partial\rho_{i-1}}{\partial x_1}\right\|^2}{\sqrt{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_1}\right\|^2 + \varepsilon_i^2}},\right.$$
$$\frac{z_i^{\mathrm{T}}\left\|\frac{\partial\rho_{i-1}}{\partial x_1}\right\|^2\|x_2\|^2}{\sqrt{\|z_i\|^2\left\|\frac{\partial\rho_{i-1}}{\partial x_1}\right\|^2\|x_2\|^2 + \varepsilon_i^2}}, ..., \frac{z_i^{\mathrm{T}}f_i^{\mathrm{T}}f_i}{\sqrt{z_i^{\mathrm{T}}z_if_i^{\mathrm{T}}f_i + \varepsilon_i^2}},$$
$$\left.\frac{z_i^{\mathrm{T}}}{\sqrt{z_i^{\mathrm{T}}z_i + \varepsilon_i^2}}, 0, ..., 0\right]^{\mathrm{T}}.$$

For the third term on the right side of (34), we have

$$z_i^{\mathrm{T}}\Theta_i\xi_i < \varepsilon_i\bar{\Theta}_i + \bar{\Theta}_i\frac{z_i^{\mathrm{T}}z_i\xi_i^{\mathrm{T}}\xi_i}{\sqrt{z_i^{\mathrm{T}}z_i\xi_i^{\mathrm{T}}\xi_i + \varepsilon_i^2}} \tag{35}$$

where $\|\Theta_i\| < \bar{\Theta}_i$, and $\bar{\Theta}_i$ is an unknown constant. Define the Lyapunov function candidate for *Step i*

$$V_i = V_{zi} + \frac{1}{2\sigma_{\bar{\Theta}_i}}\tilde{\bar{\Theta}}_i^2 + \frac{\underline{\gamma}_i}{2\sigma_{\beta_i}}\tilde{\beta}_i^2. \tag{36}$$

Then, the time-derivative of (36) becomes

$$\begin{aligned}\dot{V}_i \leq& z_i^{\mathrm{T}}B_iz_{i+1} + z_i^{\mathrm{T}}B_i\rho_i - z_i^{\mathrm{T}}\varpi_i\\ &+ \bar{\Theta}_i\frac{z_i^{\mathrm{T}}z_i\xi_i^{\mathrm{T}}\xi_i}{\sqrt{z_i^{\mathrm{T}}z_i\xi_i^{\mathrm{T}}\xi_i + \varepsilon_i^2}} + \sum_{j=1}^{i}\varepsilon_j\left(\bar{\theta}_j + \bar{h}_j\right)\\ &+ \sum_{j=1}^{i-1}\varepsilon_j\bar{\gamma}_j + \varepsilon_i\bar{\Theta}_i + \frac{1}{\sigma_{\bar{\Theta}_i}}\tilde{\bar{\Theta}}_i\dot{\hat{\bar{\Theta}}}_i + \frac{\underline{\gamma}_i}{\sigma_{\beta_i}}\tilde{\beta}_i\dot{\hat{\beta}}_i.\end{aligned} \tag{37}$$

Design the virtual stabilizing function as

$$\rho_i = -\frac{\hat{\beta}_i^2 z_i \bar{\rho}_i^{\mathrm{T}} \bar{\rho}_i}{\sqrt{\hat{\beta}_i^2 z_i^{\mathrm{T}} z_i \bar{\rho}_i^{\mathrm{T}} \bar{\rho}_i + \varepsilon_i^2}} \tag{38}$$

with

$$\bar{\rho}_i = \hat{\bar{\Theta}}_i \frac{z_i \xi_i^{\mathrm{T}} \xi_i}{\sqrt{z_i^{\mathrm{T}} z_i \xi_i^{\mathrm{T}} \xi_i + \varepsilon_i^2}} + \left(k_i + \frac{\dot{\varsigma}^{\dagger}}{\varsigma}\right) z_i - \varpi_i \tag{39}$$

where k_i is a positive tunable parameter for $i = 1, 2, ..., n$.

Note that the following inequality holds

$$z_i^{\mathrm{T}} B_i \rho_i \leq -\underline{\gamma}_i \frac{\hat{\beta}_i^2 z_i^{\mathrm{T}} z_i \bar{\rho}_i^{\mathrm{T}} \bar{\rho}_i}{\sqrt{\hat{\beta}_i^2 z_i^{\mathrm{T}} z_i \bar{\rho}_i^{\mathrm{T}} \bar{\rho}_i + \varepsilon_i^2}} \leq \varepsilon_i \underline{\gamma}_i - \underline{\gamma}_i z_i^{\mathrm{T}} \hat{\beta}_i \bar{\rho}_i. \tag{40}$$

From (37)-(40), one can obtain

$$\begin{aligned}\dot{V}_i \leq{}& z_i^{\mathrm{T}} B_i z_{i+1} - \underline{\gamma}_i z_i^{\mathrm{T}} \hat{\beta}_i \bar{\rho}_i - z_i^{\mathrm{T}} \varpi_i + \bar{\Theta}_i \frac{z_i^{\mathrm{T}} z_i \xi_i^{\mathrm{T}} \xi_i}{\sqrt{z_i^{\mathrm{T}} z_i \xi_i^{\mathrm{T}} \xi_i + \varepsilon_i^2}} \\ &+ c_i - \frac{1}{\sigma_{\bar{\Theta}_i}} \tilde{\bar{\Theta}}_i \dot{\hat{\bar{\Theta}}}_i - \frac{\underline{\gamma}_i}{\sigma_{\beta_i}} \tilde{\beta}_i \dot{\hat{\beta}}_i\end{aligned} \tag{41}$$

where $c_i = \sum_{j=1}^{i} \varepsilon_j \left(\bar{\theta}_j + \bar{h}_j + \bar{\gamma}_j\right) + \varepsilon_i \bar{\Theta}_i$.

Construct the adaptive laws as

$$\dot{\hat{\bar{\Theta}}}_i = \sigma_{\bar{\Theta}_i} \left[\frac{z_i^{\mathrm{T}} z_i \xi_i^{\mathrm{T}} \xi_i}{\sqrt{z_i^{\mathrm{T}} z_i \xi_i^{\mathrm{T}} \xi_i + \varepsilon_i^2}} - \hat{\bar{\Theta}}_i\right], \tag{42}$$

$$\dot{\hat{\beta}}_i = \sigma_{\beta_i} \left[z_i^{\mathrm{T}} \bar{\rho}_i - \hat{\beta}_i\right]. \tag{43}$$

By (42)-(43) and further simplification, (41) becomes

$$\begin{aligned}\dot{V}_i \leq{}& z_i^{\mathrm{T}} B_i z_{i+1} - 2\left(k_i + 2\frac{\dot{\varsigma}^{\dagger}}{\varsigma}\right) V_i + k_i \left(\frac{1}{\sigma_{\bar{\Theta}_i}} \tilde{\bar{\Theta}}_i^2 + \frac{\underline{\gamma}_i}{\sigma_{\beta_i}} \tilde{\beta}_i^2\right) \\ &+ \frac{\dot{\varsigma}}{\varsigma} \left(\frac{1}{\sigma_{\bar{\Theta}_i}} \tilde{\bar{\Theta}}_i^2 + \frac{\underline{\gamma}_i}{\sigma_{\beta_i}} \tilde{\beta}_i^2\right) + \tilde{\bar{\Theta}}_i \hat{\bar{\Theta}}_i + \underline{\gamma}_i \tilde{\beta}_i \hat{\beta}_i + c_i.\end{aligned} \tag{44}$$

Let $\sigma_{\bar{\Theta}_i} = \sigma_{\beta_i} = 2k_i + 2\alpha\left(\frac{1}{\varsigma(t)} + 1\right)$ for $i = 2, ..., n$, similarly with (22)-(23), then we have

$$\dot{V}_i \leq z_i^{\mathrm{T}} B_i z_{i+1} - 2k_i V_i - 2\frac{\dot{\varsigma}^{\dagger}}{\varsigma} V_i + \bar{c}_i \tag{45}$$

where $\bar{c}_i = c_i + \frac{1}{2}\bar{\Theta}_i^2 + \frac{\beta_i}{2}$.

Step n: Note that $z_n = x_n - \rho_{n-1}$, and the time derivative of z_n can be expressed as

$$\begin{aligned}\dot{z}_n ={}& \bar{B}_n \varphi(u, \bar{x}_n) + \theta_n^{\mathrm{T}} f_n + h_n \\ &- \sum_{j=1}^{n-1} \frac{\partial \rho_{i-1}}{\partial x_j} \left(B_j x_{j+1} + \theta_j^{\mathrm{T}} f_j + h_j\right) - \varpi_n\end{aligned} \tag{46}$$

where $\varpi_n = \sum_{j=1}^{n-1} \frac{\partial \rho_{i-1}}{\partial \hat{\beta}_j} \dot{\hat{\beta}}_j + \sum_{j=1}^{n-1} \frac{\partial \rho_{i-1}}{\partial \hat{\bar{\Theta}}_j} \dot{\hat{\bar{\Theta}}}_j + \sum_{j=1}^{n-1} \frac{\partial \rho_{i-1}}{\partial x_{1d}^{(j-1)}} x_{1d}^{(j)} + \sum_{j=1}^{n-1} \frac{\partial \rho_{i-1}}{\partial \varsigma^{(j-1)}} \varsigma^{(j)}$; $\bar{B}_n = d_n B_n$. Define $V_{zn} = \frac{1}{2} z_n^{\mathrm{T}} z_n$. Since $\partial\varphi(u, \bar{x}_n)/\partial u \neq 0$, we can derive $\bar{B}_n \frac{\partial \varphi(u, \bar{x}_n)}{\partial u} \neq 0$. Then by the differential mid-value theorem, there exist u_1 and u_2 such that

$$\varphi(u, \bar{x}_n) = \varphi(u_1, \bar{x}_n) + \varkappa(u_2, \bar{x}_n)(u - u_1), \tag{47}$$

where $\varkappa(u_2, \bar{x}_n) = \frac{\partial \varphi(u, \bar{x}_n)}{\partial u}|_{u=u_2}$; u_1 is the control input in some operating point of interest; $u_2 = col\{u_{2i}\} \in \mathbb{R}^m$, in which $u_{2i} \in \left[min\{u_{1i}, u_i\}, max\{u_{1i}, u_i\}\right]$. Then substituting (46)-(47) into $\dot{V}_{zn}$ and further simplification yield

$$\begin{aligned}\dot{V}_{zn} \leq{}& z_n^{\mathrm{T}} \bar{B}_n \varkappa(u_2, \bar{x}_n) u + z_n^{\mathrm{T}} \bar{B}_n (\varphi(u_1, \bar{x}_n) - \varkappa(u_2, \bar{x}_n) u_1) \\ &+ z_n^{\mathrm{T}} \Theta_n \xi_n - z_n^{\mathrm{T}} \varpi_n + \sum_{j=1}^{n} \varepsilon_j \left(\bar{\theta}_j + \bar{h}_j\right) + \sum_{j=1}^{n-1} \varepsilon_j \bar{\gamma}_j.\end{aligned} \tag{48}$$

From the *Assumption* 3, it follows that $z_n^{\mathrm{T}} \bar{B}_n \varphi(u_1, \bar{x}_n) \leq \frac{1}{2} z_n^{\mathrm{T}} z_n + \frac{1}{2} \left\|\bar{B}_n\right\|^2 \|\varphi(u_1, \bar{x}_n)\|^2 \leq \frac{1}{2} z_n^{\mathrm{T}} z_n + \frac{1}{2} \bar{\gamma}_n^2 (\bar{\pi}^\star)^2$, which means that (48) is equivalent to

$$\begin{aligned}\dot{V}_{zn} \leq{}& z_n^{\mathrm{T}} \bar{B}_n \varkappa(u_2, \bar{x}_n) u - z_n^{\mathrm{T}} \bar{B}_n \varkappa(u_2, \bar{x}_n) u_1 \\ &+ \frac{1}{2} z_n^{\mathrm{T}} z_n + z_n^{\mathrm{T}} \Theta_n \xi_n - z_n^{\mathrm{T}} \varpi_n \\ &+ \sum_{j=1}^{n} \varepsilon_j \left(\bar{\theta}_j + \bar{h}_j\right) + \sum_{j=1}^{n-1} \varepsilon_j \bar{\gamma}_j + \frac{1}{2} \bar{\gamma}_n^2 (\bar{\pi}^\star)^2\end{aligned} \tag{49}$$

where $\Theta_n = \left[\bar{\theta}_1 I_m, \bar{h}_1 I_m, \bar{\gamma}_1 I_m, ..., \bar{\theta}_n I_m, \bar{h}_n I_m, \bar{\gamma}_n I_m\right]$, and

$$\begin{aligned}\xi_n = \Bigg[& \frac{z_n^{\mathrm{T}} \left\|\frac{\partial \rho_{n-1}}{\partial x_1}\right\|^2 \|f_1\|^2}{\sqrt{\|z_n\|^2 \left\|\frac{\partial \rho_{n-1}}{\partial x_1}\right\|^2 \|f_1\|^2 + \varepsilon_n^2}}, \\ & \frac{z_n^{\mathrm{T}} \left\|\frac{\partial \rho_{n-1}}{\partial x_1}\right\|^2}{\sqrt{\|z_n\|^2 \left\|\frac{\partial \rho_{n-1}}{\partial x_1}\right\|^2 + \varepsilon_n^2}}, \frac{z_n^{\mathrm{T}} \left\|\frac{\partial \rho_{n-1}}{\partial x_1}\right\|^2 \|x_2\|^2}{\sqrt{\|z_n\|^2 \left\|\frac{\partial \rho_{n-1}}{\partial x_1}\right\|^2 \|x_2\|^2 + \varepsilon_n^2}}, ..., \\ & \frac{z_n^{\mathrm{T}} f_n^{\mathrm{T}} f_n}{\sqrt{z_n^{\mathrm{T}} z_n f_n^{\mathrm{T}} f_n + \varepsilon_n^2}}, \frac{z_n^{\mathrm{T}}}{\sqrt{z_n^{\mathrm{T}} z_n + \varepsilon_n^2}}\Bigg]^{\mathrm{T}}.\end{aligned} \tag{50}$$

For the fourth term on the right side of (49), we have

$$z_n^{\mathrm{T}} \Theta_n \xi_n < \varepsilon_n \bar{\Theta}_n + \bar{\Theta}_n \frac{z_n^{\mathrm{T}} z_n \xi_n^{\mathrm{T}} \xi_n}{\sqrt{z_n^{\mathrm{T}} z_n \xi_n^{\mathrm{T}} \xi_n + \varepsilon_n^2}} \tag{51}$$

where $\|\Theta_i\| < \bar{\Theta}_i$, and $\bar{\Theta}_i$ is an unknown constant. Define the Lyapunov function candidate for *Step n* as follows

$$V_n = V_{zn} + \frac{1}{2\sigma_{\bar{\Theta}_n}} \tilde{\bar{\Theta}}_n^2 + \frac{\underline{\gamma}_n \underline{\pi}}{2\sigma_{\beta_n}} \tilde{\beta}_n^2. \tag{52}$$

Design the controller as

$$u = -\frac{\hat{\beta}_n^2 z_n \bar{u}^{\mathrm{T}} \bar{u}}{\sqrt{\hat{\beta}_n^2 z_n^{\mathrm{T}} z_n \bar{u}^{\mathrm{T}} \bar{u} + \varepsilon_n^2}} + u_1 \tag{53}$$

where $\beta_n = 1/\underline{\pi}\underline{\gamma}_n$, and

$$\bar{u} = \hat{\bar{\Theta}}_n \frac{z_n \xi_n^{\mathrm{T}} \xi_n}{\sqrt{z_n^{\mathrm{T}} z_n \xi_n^{\mathrm{T}} \xi_n + \varepsilon_n^2}} + \left(k_n + \frac{\dot{\varsigma}^{\dagger}}{\varsigma} + \frac{1}{2}\right) z_n - \varpi_n. \tag{54}$$

Taking the time derivative of V_n and substituting (49)-(54) into it yield

$$\begin{aligned}\dot{V}_n \leq{}& -\underline{\gamma}_n \underline{\pi} z_n^{\mathrm{T}} \hat{\beta}_n \bar{u} + \bar{\Theta}_n \frac{z_n^{\mathrm{T}} z_n \xi_n^{\mathrm{T}} \xi_n}{\sqrt{z_n^{\mathrm{T}} z_n \xi_n^{\mathrm{T}} \xi_n + \varepsilon_n^2}} - z_n^{\mathrm{T}} \varpi_n \\ &- \frac{1}{\sigma_{\bar{\Theta}_n}} \tilde{\bar{\Theta}}_n \dot{\hat{\bar{\Theta}}}_n - \frac{\underline{\gamma}_n \underline{\pi}}{\sigma_{\beta_n}} \tilde{\beta}_n \dot{\hat{\beta}}_n + c_n\end{aligned} \tag{55}$$

where $c_n = \sum_{j=1}^{n} \varepsilon_j \left(\bar{\theta}_j + \bar{h}_j\right) + \sum_{j=1}^{n-1} \varepsilon_j \bar{\gamma}_j + \varepsilon_n(\bar{\Theta}_n + \bar{\gamma}_n \bar{\pi}) + \frac{1}{2}\bar{\gamma}_n^2 (\bar{\pi}^\star)^2$. Design the adaptive laws

$$\dot{\hat{\Theta}}_n = \sigma_{\Theta_i} \left[\frac{z_n^{\mathrm{T}} z_n \xi_n^{\mathrm{T}} \xi_n}{\sqrt{z_n^{\mathrm{T}} z_n \xi_n^{\mathrm{T}} \xi_n + \varepsilon_n^2}} - \hat{\Theta}_i \right], \tag{56}$$

$$\dot{\hat{\beta}}_n = \sigma_{\beta_n} \left[z_n^{\mathrm{T}} \bar{u} - \hat{\beta}_n \right]. \tag{57}$$

Combining (56)-(57) into (55) and further simplification yield

$$\dot{V}_n \leq -z_n^{\mathrm{T}} \left(k_n + \frac{\dot{\varsigma}^\dagger}{\varsigma} \right) z_n + \dot{\hat{\Theta}}_i \tilde{\Theta}_n + \underline{\gamma}_n \underline{\pi} \hat{\beta}_n \tilde{\beta}_n + c_n. \tag{58}$$

By the similar scaling manner with (22)-(23), we have

$$\dot{V}_n \leq -2k_n V_n - 2\frac{\dot{\varsigma}^\dagger}{\varsigma} V_n + \bar{c}_n \tag{59}$$

where $\bar{c}_n = c_n + \frac{1}{2}\bar{\Theta}_n^2 + \frac{\beta_n}{2}$. The above back-stepping design leads to the theorem as follows.

Theorem 2. For the system with parametric and non-parametric uncertainties (5), if *Assumptions* 1-3 hold, the controller (53) and adaptive laws (18)-(19), (42)-(43), (56)-(57) are adopted, the system (5) is PPTS with the settling time T, and the state trajectory is convergent to a compact set $D = \{x_1 \,|V_1 \leq \eta/\alpha\}$.

Proof. Making use of (59) and *Lemma 1*, we show that z_n will converge to zero at $t = t_0 + T$, which implies the prescribed-time performance of z_{i-1}. In an inductive manner, we can obtain from (24) that $\dot{V}_1 \leq -2k_1 V_1 - 2\frac{\dot{\varsigma}^\dagger}{\varsigma} V_1 + \frac{\eta}{\varsigma} + \bar{c}_1$, which means $V_1 \leq \frac{\eta}{\alpha}$ for $t \geq t_0 + T$ according to the PPTS property, as stated in *Lemma 1*. Note that $V_{z_1} = \frac{1}{2} z_1^{\mathrm{T}} z_1 \leq V_1 \leq \frac{\eta}{\alpha}$, then we have $\|e_1\| \leq \sqrt{\frac{2\eta}{\alpha}}$ for $t \geq t_0 + T$. Thus, the convergence domain follows $D = \{x_1 \,|V_1 \leq \eta/\alpha\}$ since the settling time reaches T, and the system trajectory will be kept in the prescribed convergence domain. The proof is therefore completed.

Remark 4: Compared to conventional fault-tolerant methods based on additive and multiplicative fault model Jiang et al. (2010); Tong et al. (2014a,b); Jin (2017, 2019), any prior information and dynamics model about actuator failures are not required in the control design, which forms a universal fault-tolerant control scheme to address non-affine nonlinear actuator faults. Furthermore, the convergence set only dependent on two predefined parameters is realized via employing the scaling piece-wise function in the framework of PPTS, which is substantially different from that determined by estimation errors of faults.

4. SIMULATION AND EXPERIMENT

4.1 Simulation on wing rock motion with faults

To validate the effectiveness of the proposed prescribed-time control scheme, the model Song et al. (2017) of wing rock motion for airplanes flying at high angle of attack is considered in the presence of time-varying and high-frequency uncertainty, and random external disturbance

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \theta_1 + (\theta_2 + \Delta\theta_2)\, x_1 + (\theta_3 + \Delta\theta_3)\, x_2 \\ &\quad + (\theta_4 + \Delta\theta_4)\, |x_1|\, x_2 + \theta_5\, |x_2|\, x_2 + d\varphi(u(t), \bar{x}_n(t)) + \theta_6 \end{aligned} \tag{60}$$

where x_1 and x_2 stand for the roll angle and rate, respectively. $\theta_k (k = 1, 2..., 5)$ denote the nominal wind-tunnel parameters, while $\Delta\theta_k (k = 2, 3, 4)$ represent the time-varying uncertainty. The nominal part is set as $\theta_1 = \theta_2 = \theta_3 = 1$, $\theta_4 = 2$, $\theta_5 = 3$, and θ_6 is the random disturbance subject to Gaussian distribution $N(2, 1)$. In addition, the uncertain part is set as below: $\Delta\theta_2 = \cos(t) - 1$, $\Delta\theta_3 = 2\sin(t) - 1$, $\Delta\theta_4 = 10\sin(100t)$, $\varphi(u) = (1 + 0.1\sin(0.2t))u$, $d = 1$. The control parameters are selected as $\alpha = 2$, $k_1 = 20$, and $k_2 = 20$. $u_1 = u(0) = 0$. The control objective is to track the reference signal $x_{1d} = 0$rad, $x_{2d} = 0$rad/s within prescribed settling time and residual set. We consider the following three cases with different initial conditions to show the prescribed-time property of the proposed control scheme.

Case 1: $x_1(t_0) = 1$rad, $x_{2d} = -1$rad/s;

Case 2: $x_1(t_0) = -0.95$rad, $x_{2d} = 0.1$rad/s;

Case 3: $x_1(t_0) = 0.5$rad, $x_{2d} = -0.2$rad/s;

where $t_0 = 0$s. The prescribed settling time and convergence domain are set as $T = 3$s and $\eta = 0.5$, respectively.

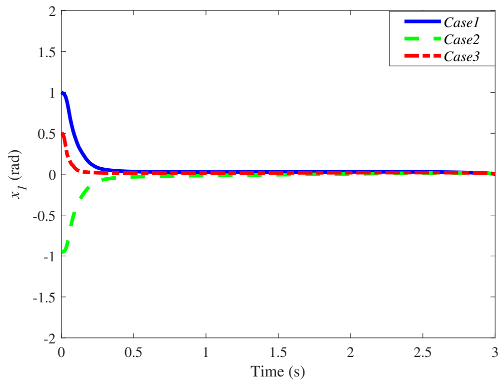

Fig. 1. The response of x_1 under different initial conditions.

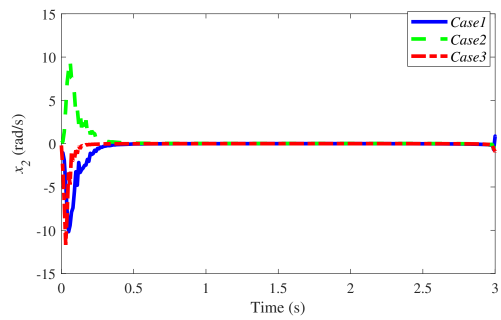

Fig. 2. The response of x_2 under different initial conditions.

Figs. 1-2 show the response of the roll angle driven by the prescribed-time control scheme (53) with $T = 3$s. The prescribed-time convergence performance with respect to different initial conditions is guaranteed. Besides the prescribed-time property, convergence accuracy can be

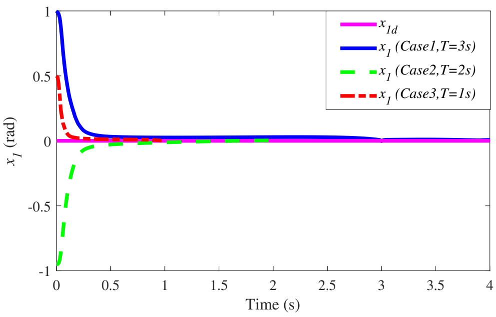

Fig. 3. The response of x_1 under different prescribed time.

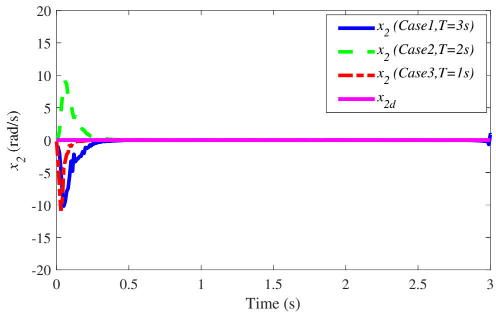

Fig. 4. The response of x_2 under different prescribed time.

guaranteed (i.e., $|x_1| \leq 2.5 \times 10^{-3}$, $|x_2| \leq 5.6 \times 10^{-3}$, $t > 3$s) even in the presence of high-frequency disturbance, uncertainties and actuator faults. To further demonstrate the prescribed-time property, we consider the case in which the settling time is set as $T = 1, 2, 3$s with different initial conditions, respectively, as illustrated in Figs. 3-4. As seen in Fig. 3, the settling time can be uniformly pre-specified with the guaranteed prescribed convergence domain, which is illustrated in Table 1.

4.2 Experiment on three-DOF tele-robot system

The tele-robot system for the experiment, consisted of two three-DOF manipulators, is developed (see Fig. 5), where each degree of freedom is actuated by DYNAMIXEL MX-106R powered by a 12V DC battery. RS485 standard is used for asynchronous serial communication (Baud rate up to 4.5Mbps). The encoder, mounted to the shaft of motor side, is AS5045 Rotary Sensor, which is a 12-bit rotary position sensor for absolute angular measurement and with a PWM output over a full turn of 360^o. An additive fault between 0 and 15Nm is applied in the right-side robot. The tracking response of master-slave tele-robot system is depicted in Fig. 6, where q_{m1} and q_{s1} represent the joint position of the first joint of the master (right side) and slave (left side), respectively. Limited by the page length, only the response of the first joint position is illustrated. It shows that, under the proposed controller, the PPTS performance and robustness against random actuator faults are ensured within prescribed settling time, namely $T = 3s$.

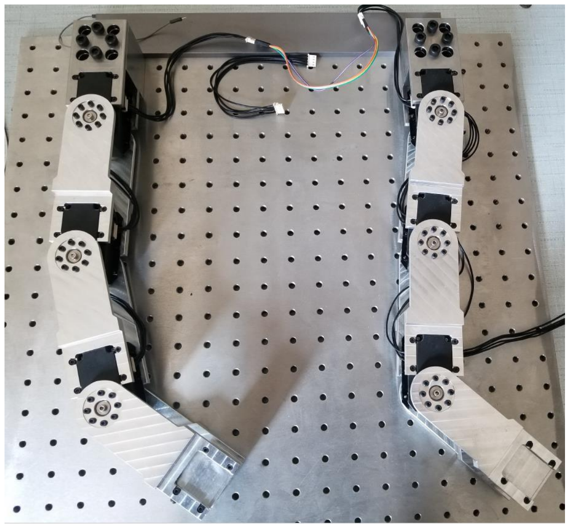

Fig. 5. The tele-robot plant.

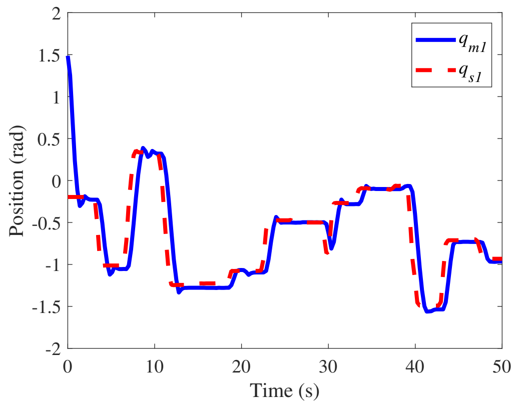

Fig. 6. Tracking response of the first joint.

5. CONCLUSION

In this paper, the prescribed-time tracking problem of a class of nonlinear systems subject to external disturbances and actuator faults is investigated. An adaptive fault-tolerant control scheme is developed to achieve *practically prescribed-time* convergence, featured by user-defined settling time and residual-set. Unknown non-affine faults, system uncertainties, external disturbances, and control direction can be addressed in a unified control framework. Detailed results have been presented to show the superior performance of the proposed method.

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Table 1. Control performance comparison.

Initial condition	Prescribed time (s)	Steady-state $x_1(rad)$	Steady-state $x_2(rad/s)$
Case 1	$T = 3$	2.5×10^{-3}	5.6×10^{-3}
Case 2	$T = 2$	1.6×10^{-3}	3.6×10^{-3}
Case 3	$T = 1$	2.2×10^{-3}	8.8×10^{-3}

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