Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Global Exponential Stability Criteria for Proportional Delay High-Order Neural Networks: A Hyper-Exponential Stability Technique

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Abstract: A new method is presented for stability analysis of proportional delay high-order neural networks. The network model is first transformed into a system with a constant time delay and unbounded time-varying coefficients, and then it is proven that the former is globally exponentially stable if and only if the laster is globally hyper-exponentially stable. The global hyper-exponential stability criteria of the laster are investigated by employing the generalized Halanay inequality and constructing a novel Lyapunov function that can avoid the computation of upper-right derivative. From which, the global exponential stability criteria of the former are derived. To illustrate the advantages of this proposed method, numerical simulation examples are given. Compared with the existing results, the contributions of this paper lie in: (i) An Lyapunov function different from ones in literature is constructed; (ii) The derived global exponential stability criteria possess simple forms, which are easy to verify; and (iii) The concept of hyper-exponential stability is proposed. The proposed method is also available to multi-proportional delay neural networks.

Keywords: proportional delay; global exponential stability; global hyper-exponential stability; high-order neural networks; Halanay inequality.

1. INTRODUCTION

Since the Hopfield-type neural networks have been proposed by Hopfield (1982), both the theoretical research and the application research of neural networks have entered a period of rapid development. The parallel computing power, associative storage functions, and self-organizing self-learning capabilities of neural networks make them widely applied in image processing, information engineering, robot control and other fields (see Berdnik et al. (2006); Kariniotakis et al. (1996); Wang et al. (2016); Shi et al. (2016); Wang et al. (2018); Velichko et al. (2019); Wang and Yang (2020); Hu et al. (2016); Zhang et al. (2019)).

In the operation of networks, the delay is inevitable (see Zhang et al. (2016a,b); Shen et al. (2020); Lin et al. (2013); Chen et al. (2019)). On the other hand, most of the above applications require the neural network to be stable, while time delay is usually a factor of breaking the stability. As a result, the studies on stability of various delayed neural networks have also made many advancements (see Pratap et al. (2019); Li et al. (2018); Zhang and Zeng (2019a,b); Shi et al. (2019, 2020); Li et al. (2018); Gao et al. (2019); Zhang et al. (2017)). Currently, from the perspective of whether it depends on time, the delays in literature can be divided into two kinds: time-varying delays and constant delays. In terms of its characteristics, proportional delay is proportional to time, so it is unbounded and time-varying. Up to now, there are few research on proportional delay neural networks. This is, of course, a large part of the reason because the ordinary method of dealing with bounded delays is difficult to apply directly to the case of unbounded time-varying delays. On the other hand, the development of proportional delay neural networks is also constrained, because the development of the proportional delay differential equations is relatively slow. This is also the reason why this paper studies proportional delay neural networks.

The existing results focus on mainly low-order neural networks, which has limited capacity. The high-order neural networks (HONNs) have the considered advantages and the excellent characteristics than the low-order ones, and hence they have been paid more and more attention (see Kariniotakis et al. (1996); Berdnik et al. (2006)). Although there are many achievements on stability for HONNs with time delay(s) (see Ren and Cao (2006); Liu et al. (2005); Lou and Cui (2007); Yu (2016); Xu and Li (2017); Zheng et al. (2015); Zhou (2015); Huang et al. (2019); Shen et al. (2020) and the references therein), few results are related to proportional delay HONNs (see Yu (2016); Xu and Li (2017); Zheng et al. (2015); Zhou (2015); Huang et al. (2019); Shen et al. (2020)). Yu (2016) derived several global exponential stability criteria of HONNs with neutral-type proportional delays. Xu and Li (2017) addressed the exponential stability of global network equilibrium. Zheng et al. (2015) obtained sufficient conditions for p-th stability of proportional delay HONNs. Zhou (2015) investigated sufficient conditions for global exponential periodicity and stability of generalized proportional delay HONNs. Huang et al. (2019) dealt with asymptotic stability of neutral proportional delay HONNs. Shen et al. (2020) presented asymptotic stability criteria for a kind of HONNs with one proportional delay.

Motivated by the previous discussions, the paper aims at presenting a new method to give global exponential stability criteria of proportional delay HONNs. Note that it is difficult to directly investigate global exponential stability criteria by applying the traditional method used in the case of bounded time-varying delays, since proportional delay is unbounded. Therefore, we will indirectly solve the problem through introducing an appropriate model transformation and the definition of hyper-exponential stability. First, the considered network model is transformed into a time-delay system which has unbounded time-varying coefficients, and then it is shown that the resulting time-delay system is globally hyper-exponentially stable if and only if the considered HONN is globally exponentially stable. Second, criteria for the global hyper-exponential stability of the resulting time-delay system (i.e., exponential stability of the HONN under consideration) are derived by picking a special Lyapunov function and using the generalized Halanay inequality. Compared with one in Zheng et al. (2015), the proposed method has less conservative and does not require the computation of matrix norms and matrix measures, which has been illustrated by numerical examples in Section 4.

Advantages of the proposed method lie in: (i) A new Lyapunov function is constructed, which computes only its derivative instead of upper-right derivative; (ii) The derived global exponential stability criteria are effective and easy to verify; and (iii) The concept of hyper-exponential stability is introduced. In addition, in terms of method promotion, the proposed method can also be applied to the analysis and design of proportional delay neural networks, even other problems involving proportional delay(s).

Notations: Let $\Re^{m\times n}$ be the set of all $m \times n$ matrices over the real number field $\Re$, and set $\Re^n = \Re^{n\times 1}$. The Kronecker product, $A \otimes B$, of $A := [a_{ij}] \in \Re^{m\times n}$ and $B \in \Re^{p\times q}$ are defined by $[a_{ij}B] \in \Re^{mp\times nq}$. The symbol $\|\cdot\|$ represents the Euclidean norm on $\Re^n$ or the spectral norm on $\Re^{m\times n}$. Denote by $\mathrm{diag}(\cdot)$ and $\mathrm{col}(\cdot)$ the diagonal matrix and the column matrix, respectively.

2. PROBLEM FORMULATION

Consider the following proportional delay HONN (see Zheng et al. (2015)):

$$\begin{aligned}\dot{u}_i(t) =& - d_i u_i(t) + \sum_{j=1}^{n}[a_{ij} f_j(u_j(t)) + b_{ij} g_j(u_j(qt))] \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk} g_j(u_j(qt)) g_k(u_k(qt)) + J_i, \\ &t \geq 1, i = 1, 2, \ldots, n, \end{aligned} \tag{1a}$$

$$u_i(s) = \psi_i(s), i = 1, 2, \ldots, n, s \in [q, 1], \tag{1b}$$

where $u_i(t)$ is the membrane potential of the i-th neuron at time t, $\psi_i(\cdot)$ represents initial function, $0 < q < 1$ and $d_i > 0$ are known constants, J_i represents external input, a_{ij}, b_{ij} and T_{ijk} are the synaptic connection weights, and $f_j(\cdot)$ and $g_j(\cdot)$ are the j-th neuron's activation functions that satisfy

$$\begin{aligned} &| f_i(v_1) - f_i(v_2) | \leq \alpha_i \mid v_1 - v_2 |, \\ &| g_i(v_1) | \leq \gamma_i, \\ &| g_i(v_1) - g_i(v_2) | \leq \beta_i \mid v_1 - v_2 | \end{aligned} \tag{2}$$

for any $v_1, v_2 \in \Re$ and $i = 1, 2, \ldots, n$, where α_i, β_i and γ_i are known positive constants independent from v_1 and v_2.

Since $qt = t - (1-q)t$, so $(1-q)t$ is called the unbounded transmission delay. Let

$$\begin{aligned} &u(t) = \mathrm{col}(u_1(t), u_2(t), \ldots, u_n(t)),\ A = [a_{ij}] \in \Re^{n\times n}, \\ &B = [b_{ij}] \in \Re^{n\times n},\ D = \mathrm{diag}(d_1, d_2, \ldots, d_n), \\ &\psi(s) = \mathrm{col}(\psi_1(s), \psi_2(s), \ldots, \psi_n(s)), \\ &f(u(t)) = \mathrm{col}(f_1(u_1(t)), f_2(u_2(t)), \ldots, f_n(u_n(t))), \\ &g(u(qt)) = \mathrm{col}(g_1(u_1(qt)), g_2(u_2(qt)), \ldots, g_n(u_n(qt))), \\ &G(u(qt)) = I_n \otimes g(u(qt)),\ T_i = [T_{ijk}] \in \Re^{n\times n}, \\ &T = \mathrm{col}(T_1, T_2, \ldots, T_n),\ J = \mathrm{col}(J_1, J_2, \ldots, J_n). \end{aligned}$$

Then HONN (1) can be expressed as:

$$\begin{aligned}\dot{u}(t) =& - Du(t) + Af(u(t)) + Bg(u(qt)) \\ &+ G^{\mathrm{T}}(u(qt))Tg(u(qt)) + J,\ t \geq 1, \end{aligned} \tag{3a}$$

$$u(s) = \psi(s), s \in [q, 1]. \tag{3b}$$

Set

$$\begin{cases} y(t) = u(\mathrm{e}^t),\ t \geq 0, \\ \varphi(s) = \psi(\mathrm{e}^s),\ s \in [-\tau, 0], \end{cases} \tag{4}$$

where $\tau = -\ln q > 0$. Then $u(q\mathrm{e}^t) = u(\mathrm{e}^{t-\tau}) = y(t - \tau),\ t \geq 0,$, and hence HONN (3) is transformed into:

$$\begin{aligned}\dot{y}(t) =& \mathrm{e}^t\{-Dy(t) + Af(y(t)) + Bg(y(t-\tau)) \\ &+ G^{\mathrm{T}}(y(t-\tau))Tg(y(t-\tau)) + J\},\ t \geq 0, \end{aligned} \tag{5a}$$

$$y(s) = \varphi(s),\ s \in [-\tau, 0]. \tag{5b}$$

Definition 1. Let u^* be an equilibrium point of HONN (3). If there exist two constants $M > 0$ and $\lambda > 0$ such that

$$\|u(t, \psi) - u^*\| \leq M \sup_{q\leq s\leq 1} \|\psi(s) - u^*\| \mathrm{e}^{-\lambda t}$$

for any $t \geq 1$ and $\psi \in \mathcal{C}([q, 1], \Re^n)$, where $u(t, \psi)$ is the solution of HONN (3), then u^* is globally exponentially stable.

Definition 2. Let y^* be an equilibrium point of system (5). If there exist two constants $M > 0$ and $\lambda > 0$ such that

$$\|y(t,\varphi) - y^*\| \leq M \sup_{-\tau\leq s\leq 0} \|\varphi(s) - y^*\| \mathrm{e}^{-\lambda \mathrm{e}^t}$$

for any $t \geq 0$ and $\varphi \in \mathcal{C}([-\tau,0],\Re^n)$, where $y(t,\varphi)$ is the solution of system (5), then y^* is globally hyper-exponentially stable.

Remark 1. HONN (3) is transformed into system (5) via (4). If the equilibrium y^* of system (5) is globally hyper-exponentially stable, then according to Definition 2, we can get that $\|y(t,\varphi)-y^*\| \leq M \sup_{-\tau\leq s\leq 0} \|\varphi(s)\|\mathrm{e}^{-\lambda \mathrm{e}^t}$ for any $t \geq 0$ and $\varphi \in \mathcal{C}([-\tau,0],\Re^n)$, and some $M > 0$ and $\lambda > 0$. Let $s = \mathrm{e}^t$, then $\|u(s,\psi) - u^*\| \leq M \sup_{q\leq\sigma\leq 1} \|\psi(\sigma)\|\mathrm{e}^{-\lambda s}$ for any $s \geq 1$ and $\psi \in \mathcal{C}([q,1],\Re^n)$. By Definition 1, HONN (3) is globally exponentially stable. Conversely, one can easily show that if HONN (3) is globally exponentially stable, then system (5) is globally hyper-exponentially stable. Therefore, the hyper-exponential stability of system (5) is equivalent to the exponential stability of HONN (3).

Remark 2. The stability criteria investigated by Ren and Cao (2006); Liu et al. (2005) can not be applied to system (5), because only systems with constant coefficients are involved in these literature, while the coefficients of system (5) are unbounded and time-varying.

Remark 3. The approach proposed by Shen et al. (2020) can establish only asymptotic stability criteria for HONN (1), but not exponential stability criteria; see Shen et al. (2020, Remark 1) for the details.

This paper aims at proposing a new method to investigate less conservative global exponential stability criteria for the equilibrium u^* of HONN (3) (i.e., hyper-exponential stability criteria for the equilibrium y^* of system (5)). To the end, we require the following lemma:

Lemma 1. Tian (2004) Suppose

$$\dot{u}(t) \leq \gamma(t) - \alpha(t)u(t) + \beta(t) \sup_{t-\tau\leq\sigma\leq t} u(\sigma), t \geq t_0,$$

where $\tau \geq 0$, $\gamma : [t_0,\infty) \to [0,\gamma^*]$, $\alpha : [t_0,\infty) \to [\alpha_0,\infty]$ and $\beta : [t_0,\infty) \to [0,\infty]$ are continuous functions satisfying $0 \leq \beta(t) \leq q\alpha(t)$ for any $t \geq t_0$ with constants $\gamma^* > 0$, $\alpha_0 > 0$ and $0 \leq q < 1$. Then

$$u(t) \leq \tfrac{\gamma^*}{(1-q)\alpha_0} + \sup_{t_0-\tau\leq\sigma\leq t_0} u(\sigma)\mathrm{e}^{-\mu^*(t-t_0)}, t \geq t_0,$$

where $\mu^* = \inf_{t\geq t_0} \{\mu(t) : \mu(t) - \alpha(t) + \beta(t)\mathrm{e}^{\tau\mu(t)} = 0\} > 0$.

3. GLOBAL EXPONENTIAL STABILITY CRITERIA

In this section we will investigate sufficient conditions under which the equilibrium of HONN (3) is globally exponentially stable, i.e., system (5) is globally hyper-exponentially stable.

Assume that $y^* \in \Re^n$ is an equilibrium of system (5), that is,

$$-Dy^* + Af(y^*) + Bg(y^*) + G^{\mathrm{T}}(y^*)Tg(y^*) + J = 0.$$

Let $x(t) = y(t) - y^*$, then it follows that

$$\begin{aligned}\dot{x}(t) =& \mathrm{e}^t\{-Dx(t) + A\eta(x(t)) + B\zeta(x(t-\tau)) \\ &+ G^{\mathrm{T}}(x(t-\tau) + y^*)T\zeta(x(t-\tau)) \\ &+ G^{\mathrm{T}}(y^*)\hat{T}\zeta(x(t-\tau))\},\end{aligned} \tag{6}$$

where $\eta(\cdot) = f(\cdot + y^*) - f(y^*)$, $\zeta(\cdot) = g(\cdot + y^*) - g(y^*)$, $\hat{T} = \mathrm{col}(T_1^{\mathrm{T}}, T_2^{\mathrm{T}}, \ldots, T_n^{\mathrm{T}})$. For convenience, we define

$$\Delta = \begin{bmatrix} \Gamma_1 + \bar{q}\alpha^2 I + P & PA & PB & P & P \\ A^{\mathrm{T}}P & -Q & 0 & 0 & 0 \\ B^{\mathrm{T}}P & 0 & -\underline{p}I + \Gamma_2 & 0 & 0 \\ P & 0 & 0 & -s_1 I & 0 \\ P & 0 & 0 & 0 & -s_2 I \end{bmatrix},$$

where $\Gamma_1 = -PD - DP + \beta P$, $\Gamma_2 = \tilde{\gamma}(s_1 T^{\mathrm{T}}T + s_2\hat{T}^{\mathrm{T}}\hat{T})$, $\alpha = \max_{1\leq i\leq n} \alpha_i$, $\beta = \max_{1\leq i\leq n} \beta_i$, $\tilde{\gamma} = \sum_{i=1}^{n} \gamma_i^2$.

Theorem 1. Assume that (2) is satisfied, if $0 < \beta < \sqrt{q}$ and there exist positive scalars $\underline{p}, \bar{q}, s_1$ and s_2, and positive definite matrices $P, Q \in \Re^{n\times n}$ such that $\Delta < 0$, $\underline{p}I \leq P$ and $Q \leq \bar{q}I$, then the equilibrium y^* of system (5) is globally hyper-exponentially stable, i.e., the equilibrium u^* of HONN (3) is globally exponentially stable.

Proof. Based on the Schur complement Lemma, it follows from $\Delta < 0$ that

$$\Delta_0 := \begin{bmatrix} \Gamma_1 + \bar{q}\alpha^2 I + P + \Gamma_3 & PA & PB \\ A^{\mathrm{T}}P & -Q & 0 \\ B^{\mathrm{T}}P & 0 & -\underline{p}I + \Gamma_2 \end{bmatrix} < 0 \tag{7}$$

with $\Gamma_3 = (s_1^{-1} + s_2^{-1})P^2$.

Based on $Q \leq \bar{q}I$ and (2), one can obtain by direct computation that

$$\eta^{\mathrm{T}}(x(t))Q\eta(x(t)) \leq \bar{q}\alpha^2 x^{\mathrm{T}}(t)x(t), \tag{8}$$

$$\zeta^{\mathrm{T}}(x(t-\tau))\zeta(x(t-\tau)) \leq \beta^2 x^{\mathrm{T}}(t-\tau)x(t-\tau), \tag{9}$$

$$G(x(t-\tau) + y^*)G^{\mathrm{T}}(x(t-\tau) + y^*) \leq \tilde{\gamma}I_n, \tag{10}$$

$$G(y^*)G^{\mathrm{T}}(y^*) \leq \tilde{\gamma}I_n. \tag{11}$$

Set

$$\Delta_{21} = \begin{bmatrix} P & P \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ \Delta_{22} = \begin{bmatrix} 0 & 0 & G^{\mathrm{T}}(x(t-\tau) + y^*)T \\ 0 & 0 & G^{\mathrm{T}}(y^*)\hat{T} \end{bmatrix}.$$

Then

$$\Delta_{21}\mathrm{diag}(s_1 I, s_2 I)^{-1}\Delta_{21}^{\mathrm{T}} = \mathrm{diag}(\Gamma_3, 0, 0) \tag{12}$$

and

$$\begin{aligned}&\Delta_{22}^{\mathrm{T}}\mathrm{diag}(s_1 I, s_2 I)\Delta_{22} \\ =&\ \mathrm{diag}(0, 0, s_2\hat{T}^{\mathrm{T}}G(y^*)G^{\mathrm{T}}(y^*)\hat{T} \\ &+s_1 T^{\mathrm{T}}G(y(t-\tau))G^{\mathrm{T}}(y(t-\tau))T) \\ \leq&\ \mathrm{diag}(0, 0, \Gamma_2).\end{aligned} \tag{13}$$

The combination of (10), (11) and (13) derives that

$$\Delta_{22}^{\mathrm{T}}\mathrm{diag}(s_1 I, s_2 I)\Delta_{22} \leq \mathrm{diag}(0, 0, \Gamma_2).$$

This, together with (7) and (12), implies that $\Delta_1 + \Delta_{21}\mathrm{diag}(s_1 I, s_2 I)^{-1}\Delta_{21}^{\mathrm{T}} + \Delta_{22}^{\mathrm{T}}\mathrm{diag}(s_1 I, s_2 I)\Delta_{22} < 0$, where

$$\Delta_1 = \begin{bmatrix} \Gamma_1 + \bar{q}\alpha^2 I + P & PA & PB \\ A^{\mathrm{T}}P & -Q & 0 \\ B^{\mathrm{T}}P & 0 & -\underline{p}I \end{bmatrix}.$$

Noting that

$$\begin{aligned}&\Delta_{21}\Delta_{22}+\Delta_{22}^{\mathrm{T}}\Delta_{21}^{\mathrm{T}}\\ \leq&\Delta_{21}\mathrm{diag}(s_1I,s_2I)^{-1}\Delta_{21}^{\mathrm{T}}+\Delta_{22}^{\mathrm{T}}\mathrm{diag}(s_1I,s_2I)\Delta_{22},\end{aligned}$$

we obtain

$$\Delta_2:=\Delta_1+\Delta_{21}\Delta_{22}+\Delta_{22}^{\mathrm{T}}\Delta_{21}^{\mathrm{T}}<0.$$

Pick the following Lyapunov function:

$$V(x(t))=\mathrm{e}^{\beta\mathrm{e}^t}x^{\mathrm{T}}(t)Px(t),\ t\geq 0.$$

Computing the time derivative of $V(x(t))$ along the trajectories of system (6), one can derive

$$\begin{aligned}\dot V(x(t))=&\mathrm{e}^{\beta\mathrm{e}^t}\mathrm{e}^tx^{\mathrm{T}}(t)P\{\beta x(t)-2Dx(t)+2A\eta(x(t))\\&+2B\zeta(x(t-\tau))+2G^{\mathrm{T}}(y^*)\hat T\zeta(x(t-\tau))\\&+2G^{\mathrm{T}}(x(t-\tau)+y^*)T\zeta(x(t-\tau))\}\\=&\mathrm{e}^{\beta\mathrm{e}^t}\mathrm{e}^t\{\xi^{\mathrm{T}}(t)\Delta_4\xi(t)+\eta^{\mathrm{T}}(x(t))Q\eta(x(t))\\&+\underline{p}\zeta^{\mathrm{T}}(x(t-\tau))\zeta(x(t-\tau))\},\end{aligned}\tag{14}$$

where

$$\xi(t)=\mathrm{col}(x(t),\eta(x(t),\zeta(x(t-\tau)),$$

$$\Delta_3=\begin{bmatrix}\Gamma_1&PA&\Omega\\A^{\mathrm{T}}P&0&0\\\Omega^{\mathrm{T}}&0&0\end{bmatrix},$$

$$\Delta_4=\Delta_3+\mathrm{diag}(0,-Q,-\underline{p}I),$$

$$\Omega=PB+PG^{\mathrm{T}}(x(t-\tau)+y^*)T+PG^{\mathrm{T}}(y^*)\hat T.$$

Applying (14), (8) and (9), we derive that

$$\begin{aligned}&\dot V(x(t))\\ \leq&\mathrm{e}^{\beta\mathrm{e}^t}\mathrm{e}^t\{\xi^{\mathrm{T}}(t)\Delta_5\xi(t)+\underline{p}\beta^2x^{\mathrm{T}}(t-\tau)x(t-\tau)\}.\end{aligned}$$

where $\Delta_5=\Delta_4+\mathrm{diag}(\alpha^2\overline{q}I,0,0)$. This, together with $\Delta_5=\Delta_2-\mathrm{diag}(P,0,0)$, implies that

$$\dot V(x(t))\leq\mathrm{e}^t\{-V(x(t))+\beta^2V(x(t-\tau))\}.$$

Using Lemma 1 to $t_0=0$, $\gamma^*\to 0$, $\alpha(t)=\mathrm{e}^t$, $\beta(t)=\beta^2\mathrm{e}^t$ and $\gamma(t)=0$, we get

$$\begin{aligned}\mathrm{e}^{\beta\mathrm{e}^t}x(t)^{\mathrm{T}}Px(t)=&V(x(t))\\ \leq&\sup_{-\tau\leq s\leq 0}V(x(s))\mathrm{e}^{-\mu^*t},t\geq 0,\end{aligned}$$

where $\mu^*=\inf_{t\geq 0}\{\mu(t):\mu(t)-\mathrm{e}^t+\beta^2\mathrm{e}^t\mathrm{e}^{\tau\mu(t)}=0\}>0$. Therefore,

$$\|x(t)\|_2\leq M\sup_{-\tau\leq s\leq 0}\|\varphi(s)\|\mathrm{e}^{-\frac{\beta}{2}\mathrm{e}^t},$$

where $M=\mathrm{e}^{\frac{\beta}{2}}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$. It can be obtained from Definition 2 that the zero equilibrium of system (6) (i.e., the equilibrium y^* of system (5)) is globally hyper-exponentially stable). This, together with Remark 1, implies that the equilibrium u^* of HONN (3) is globally exponentially stable. The proof is complete. □

Remark 4. When $T=0$, the HONN (1) simplifies into (see Zheng et al. (2015)):

$$\begin{aligned}\dot u_i(t)=&-d_iu_i(t)+\sum_{j=1}^n[a_{ij}f_j(u_j(t))+b_{ij}g_j(u_j(qt))]+J_i,\\&t\geq 1,i=1,2,\ldots,n.\end{aligned}\tag{15}$$

It should be emphasized that Theorem 1 is still applicable to system (15).

For the low-order neural networks with a pair of proportional delays, which can be described as (see Zheng et al. (2015)):

$$\begin{aligned}\dot u_i(t)=&-d_iu_i(t)+\sum_{j=1}^n[a_{ij}f_j(u_j(t))+b_{ij}g_j(u_j(q_1t))\\&+c_{ij}h_j(u_j(q_2t))]+J_i,\ t\geq 1,i=1,2,\ldots,n,\end{aligned}\tag{16}$$

where $0<q_i<1$, $i=1,2$, the activation functions f_j and g_j satisfy (2), and there exists $l_i>0$ such that

$$|\ h_i(u)-h_i(v)\ |\leq l_i\ |\ u-v\ |,\forall u,v\in\Re,i=1,2,\ldots,n.\tag{17}$$

Similar to that mentioned above, the delays of neural network (16) are $(1-q_i)t$, $i=1,2$, which are unbounded proportional delays. Furthermore, system (16) can be transformed into the following form:

$$\begin{aligned}\dot y(t)=&\mathrm{e}^t\{-Dy(t)+Af(y(t))+Bg(y(t-\tau_1))\\&+Ch(y(t-\tau_2))+J\},t\geq 1,\end{aligned}\tag{18}$$

where $y(t)=u(\mathrm{e}^t)$, $\tau_1=-\ln q_1>0$ and $\tau_2=-\ln q_2>0$. Set $l=\max_{1\leq i\leq n}l_i$, $q=\min\{q_1,q_2\}$ and $\tau=\max\{\tau_1,\tau_2\}$. Similar to Theorem 1, we can give directly the following conclusion.

Theorem 2. Assume that (2) and (17) are satisfied. If $\beta^2+l^2<q$ and there exist positive scalars $\underline{p}$ and $\overline{q}$, and positive definite matrices $P,Q\in\Re^{n\times n}$ such that $Q\leq\overline{q}I$, $\underline{p}I\leq P$ and

$$\tilde\Delta:=\begin{bmatrix}\Gamma_1+P+\overline{q}\alpha^2I&PA&PB&PC\\A^{\mathrm{T}}P&-Q&0&0\\B^{\mathrm{T}}P&0&-\underline{p}I&0\\C^{\mathrm{T}}P&0&0&-\underline{p}I\end{bmatrix}<0,$$

then the equilibrium y^* of system (18) is globally hyper-exponentially stable, or equivalently, the equilibrium u^* of system (16) is globally exponentially stable.

Remark 5. One can apply this proposed method to establish global exponential stability criteria for multi-proportional delay HONNs. We do not list the corresponding results which may be complicated.

4. NUMERICAL EXAMPLES

This section illustrates the validity of theoretical results obtained above by employing two numerical examples.

Example 1. Consider HONN (1) with $n=2$, $q=0.5$, $d_1=6$, $d_2=2$, $a_{11}=3$, $a_{12}=0.14$, $a_{21}=0.2$, $a_{22}=0.31$, $b_{11}=0.09$, $b_{12}=0.25$, $b_{21}=0.21$, $b_{22}=0.45$, $T_{111}=0.05$, $T_{112}=0.14$, $T_{121}=-0.06$, $T_{122}=0.05$, $T_{211}=0.29$, $T_{212}=0.1$, $T_{221}=0.23$, $T_{222}=0.14$, $J_1=1.5$, $J_2=2$, $f_1(x)=2g_1(x)=\tanh(0.6x)$, $f_2(x)=2g_2(x)=\tanh(0.8x)$.

It is clear that (2) is satisfied by setting $\alpha_1=0.6$, $\alpha_2=0.8$, $\beta_1=0.3$, $\beta_2=0.4$ and $\gamma_1=\gamma_2=0.5$. Hence $\alpha=0.8$, $\beta=0.4$ and $\tilde\gamma=0.5$. Furthermore, $T=\mathrm{col}(T_1,T_2)$ and $\hat T=\mathrm{col}(T_1^{\mathrm{T}},T_2^{\mathrm{T}})$ with $T_1=\begin{bmatrix}0.05&0.14\\-0.06&0.05\end{bmatrix}$ and $T_2=\begin{bmatrix}0.29&0.1\\0.23&0.14\end{bmatrix}$. A feasible solution of the LMIs in Theorem 1 can be obtained as follows:

$$P=\begin{bmatrix}0.7643&-0.0686\\-0.0686&0.7783\end{bmatrix},\ Q=\begin{bmatrix}0.9641&0.0555\\0.0555&0.7619\end{bmatrix},$$

$$s_1 = 1.8269, s_2 = 1.8818, \underline{p} = 0.6424, \overline{q} = 1.1522.$$

Due to Theorem 1, the equilibrium (0.4156, 1.2967) of the HONN under consideration is globally exponentially stable, which is illustrated by the phase trajectories in Figure 1 and the state response diagrams in Figure 2.

However, for the example, the LMIs in Zheng et al. (2015, Theorem 1) is not feasible. So, the obtain global exponential stability criterion given in Theorem 1 is less conservative than one in Zheng et al. (2015, Theorem 1). And it is worth emphasizing that the criterion in Zheng et al. (2015, Theorem 1) needs to calculate matrix norms and matrix measures, while the criterion given in Theorem 1 is not involving the computation of matrix norms and matrix measures.

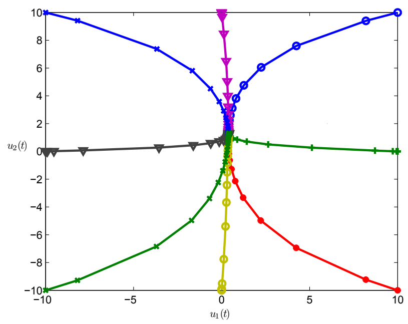

Fig. 1. Phrase trajectories of HONN in Example 1

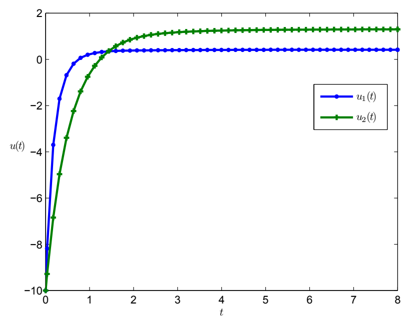

Fig. 2. State responses of HONN in Example 1

Example 2. Consider the first-order neural network (16) with $n = 2$, $q_1 = 0.5, q_2 = 0.8$, $d_1 = 4, d_2 = 5$, $a_{11} = 1$, $a_{12} = -2$, $a_{21} = 0$, $a_{22} = 1$, $b_{11} = 1$, $b_{12} = 0$, $b_{21} = 1$, $b_{22} = -2$, $c_{11} = 0$, $c_{12} = 1$, $c_{21} = -1$, $c_{22} = -0.5$, $J_1 = 2$, $J_2 = 3$, $f_1(x) = 2g_1(x) = 2h_1(x) = \tanh(0.5x)$, $f_2(x) = 2g_2(x) = 2h_2(x) = \tanh(0.6x)$.

Setting $\alpha_1 = 0.5$, $\beta_1 = l_1 = 0.25$, $\alpha_2 = 0.6$, $\beta_2 = l_2 = 0.3$ and $\gamma_1 = \gamma_2 = 0.5$, we obtain $\alpha = 0.6$, $\beta = l = 0.3$ and $\tilde{\gamma} = 0.5$. So, the premise condition of Theorem 2 is satisfied. A feasible solution of the LMIs in Theorem 2 is obtained as follows:

$$P = \begin{bmatrix} 0.5130 & 0.0052 \\ 0.0052 & 0.4511 \end{bmatrix}, \; Q = \begin{bmatrix} 1.0402 & -0.1356 \\ -0.1356 & 1.3198 \end{bmatrix},$$

$$\underline{p} = 0.4131, \; \overline{q} = 1.8928.$$

Therefore, by Theorem 2, the equilibrium (0.4531, 0.6049) of the considered neural network is globally exponentially stable, which is illustrated by the phase trajectories in Figure 3 and the state response diagrams in Figure 4.

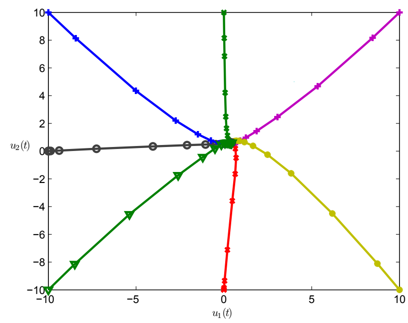

Fig. 3. Phrase trajectories of HONN in Example 2

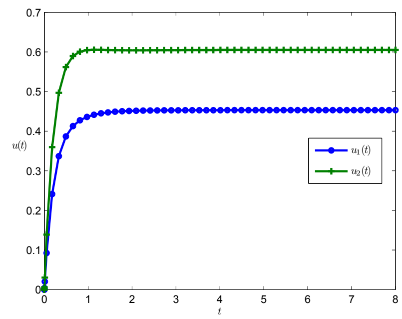

Fig. 4. State responses of HONN in Example 2

5. CONCLUSION

In this paper, the novel global exponential stability criteria of HONNs with proportional delay has been investigated. By applying the hyper-exponential stability technique, the global exponential stability criteria are derived by picking a special Lyapunov function and using the generalized Halanay inequality. Furthermore, the proposed method can be applied to the analysis and design of proportional delay neural networks, even other problems involving proportional delay.

ACKNOWLEDGEMENTS

This work was partially supported by the National Natural Science Foundation of China (Grant nos. 61703148, 61673098 and 61873306), the Natural Science Foundation of Heilongjiang Province (Grant no LH2019F030), the Outstanding Youth Fund of Heilongjiang University (Grant no JCL201903), and the Heilongjiang University Innovation Fund for Graduates (Grant no YJSCX2019-074HLJU).

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