Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Rapidly Oscillating Systems on the Plane: Weak Resonances and Asymptotic Control

Sergey Belikov*, and Ruslan Belikov**

**SPM Labs LLC, Tempe, AZ, USA*
***NASA AMES Research Center, Mountain View, CA, USA*

Abstract: Rapidly Oscillating Systems (ROS) is a class of perturbed dynamical systems with a small parameter that contains oscillating part with the period proportional to the parameter. These include a lot of classical perturbed systems that found wide applications in celestial mechanics, electronics, mechatronics, nanotechnology, etc. However, many important systems of the class cannot be treated classically. In recent years these non-classical ROS found important applications in control of bipedal walk, rapidly varying media, frequency demodulation, Atomic Force Microscopy, etc. A characteristic specific of the non-classical ROS is that oscillations depend not only on time, but also on phase variables. This allows to manipulate the oscillation functions and to use them for asymptotic control. The purpose of this paper is to illustrate non-classical ROS methods and their properties on simple, but representative, 2D examples without technical details of real applications. In some cases these also provide a new technique to the classical systems.

Keywords: oscillation control, averaging, asymptotic methods.

1. INTRODUCTION

The importance of perturbation techniques for modelling and control system design is described in [Bellman, R. (1966)]. In particular, Van der Pol equation is analysed in Chapter 2 of the book (p. 59-60) and in [Sanders, J.A., Verhulst, F., and Murdock, J. (2007): pp. 22-24]. This equation describes the multivibrator, and was one of the first successfully analysed systems in the field of electronic oscillators with megacycle frequency. Van der Pol proposed an empirical method of analysis, now called averaging. The small parameter ε was proportional to the period of the oscillator. This method was significantly extended, rigorously proved [Bogoliubov and Mitropolsky (1961)] and widely used in numerous applications by the name KBM (Krylov-Bogoliubov-Mitropolsky). In fact, the early averaging method is "going back to the founders of celestial mechanics and widely usable in all those areas of application, where a slow evolution has to be separated from fast oscillations" [Arnold (1988)]. In celestial mechanics the small parameter ε is usually the ratio of the mass of the planet to that of the Sun. Recently the averaging technique found important applications in Atomic Force Microscopy (AFM) [Belikov, S., and Magonov, S. (2017)], [Belikov, S. et al (2016)]. AFM simulation based on asymptotic dynamics is reported in [Belikov S., and Magonov, S. (2019)]. In AFM the small parameter ε is proportional to the inverse quality factor of the mode of the cantilever oscillation. Since the classic book [Bogoliubov, N., and Mitropolsky, Yu. (1961)] more modern survey on averaging has been published [Sanders, J.A., Verhulst, F., and Murdock, J. (2007)].

Multi-Resonant AFM is an important example of averaging in systems with several frequencies, where the so called *resonance hypersurfaces* [Arnold (1988), p. 154] may prevent the possibility of averaging. In this paper we define the weak resonances that do not prevent the averaging and illustrate them on classical and non-classical examples.

The first example of non-classical systems, now called ROS, which does not satisfy KBM conditions and cannot be averaged by this method was presented in [Sari, T. (1983)]. The Sari's equation is $dy/dt=\varepsilon\cdot\sin(t\cdot y)$. In contrast to classical KBM system, the oscillation function depends not only on time t, but on combination of t and phase variable y. This example was generalized in [Belikov, S. (1989)] where theorems of non-classical (non-KBM) averaging have been formulated for some classes of perturbed systems. Finally, non-classical ROS was introduced in [Belikov, S., and Belikov, R. (1996)] followed by additional applications [Belikov, S., and Belikov, R. (1999)], [Belikov, R., and Belikov, S. (2001)].

This paper is the second in a sequence. The first was [Belikov, S., and Belikov, R. (1997)]. We illustrate non-classical ROS methods and their properties on simple, but representative, 2D examples without technical details of real applications. In the first part [Belikov, S., and Belikov, R. (1997)] so called *rapid bifurcations* have been described and demonstrated. Rapid bifurcations by the name of *canard*s have been introduced in [Benoit, E., et al (1981)] for 2D singularly perturbed ODE, surveyed in [Zvonkin, A.K., and Shubin, M.A. (1984)], and described in monographs [Albeverio, S., et al. (1986)] and [O'Malley R.E. (2016)]. The extension to singularly perturbed systems of arbitrary dimension with a single slow variable was reported in [Belikov, S., and Samborskii, S. (1989)] and described with detailed proofs in [Belikov, S., and Samborskii, S. (1991)].

"Canard" rapid bifurcation for ROS was demonstrated in [Belikov, S., and Belikov, R. (1997)]. The name *rapid* is justified by the fact that canards exist in exponentially small (order of $\exp(-1/\varepsilon)$) interval of the parameter –the related theorem is called "A canard's life is short" [O'Malley R.E. (2016): p. 171].

Simulation of a rapid bifurcation in ROS is demonstrated in [Belikov, S., and Belikov, R. (1997)]. Although, the pictures of ROS phase portraits during the rapid bifurcations, in contrast to singularly perturbed systems, do not remind flying ducks, the phenomenon is similar and a life of ROS rapid bifurcation is also short.

Section 2 of this paper provides the necessary background and theorems formulated for 2D systems that are simpler than in general case [Belikov, S., and Belikov, R. (1996)] –this background is partly intersects with [Belikov, S., and Belikov, R. (1997)].

Section 3 formulates the *weak resonance* condition for ROS. Although formulated in terms of non-classical ROS, it is applicable to simplify analysis and provide additional insight to some classical systems.

Section 4 illustrates the unique property, especially valuable for control science and engineering, called *asymptotic control* [Belikov, S., and Belikov, R. (1996)]. One can manipulate the oscillation functions that depend on both fast and slow variables at "fast" or "micro" level to control the "slow" or "macro" level dynamics. A motivating example is a walking machine (or a living creature) that manipulates the step dynamics (micro level) to control the walking trajectory [Belikov, S., and Belikov, R. (1996)]. In Section 4 we demonstrate a simple asymptotic control that can be designed using classical optimal control methods [Boltyanskii, V.G. (1971)]. The purpose of the section is to illustrate the technique of the asymptotic control without technical details of a particular application and relate it to the classical example [Boltyanskii, V.G. (1971): p. 204].

2. BACKGROUND

Let us remind that one of the basic results of classical averaging is the following theorem (rigorous formulation and proof can be found in [Sanders, J.A., Verhulst, F., and Murdock, J. (2007): pp. 74-75]).

Theorem of KBM Averaging: *Consider the initial value problem*

$$\dot{x} = \varepsilon f(x,t), \quad x(0) = a \tag{1}$$

with $f: R^n \times R \to R^n$ *and*

$$\dot{z} = \varepsilon \bar{f}(z), \quad z(0) = a \tag{2}$$

where

$$\bar{f}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x,t)dt \tag{3}$$

Suppose

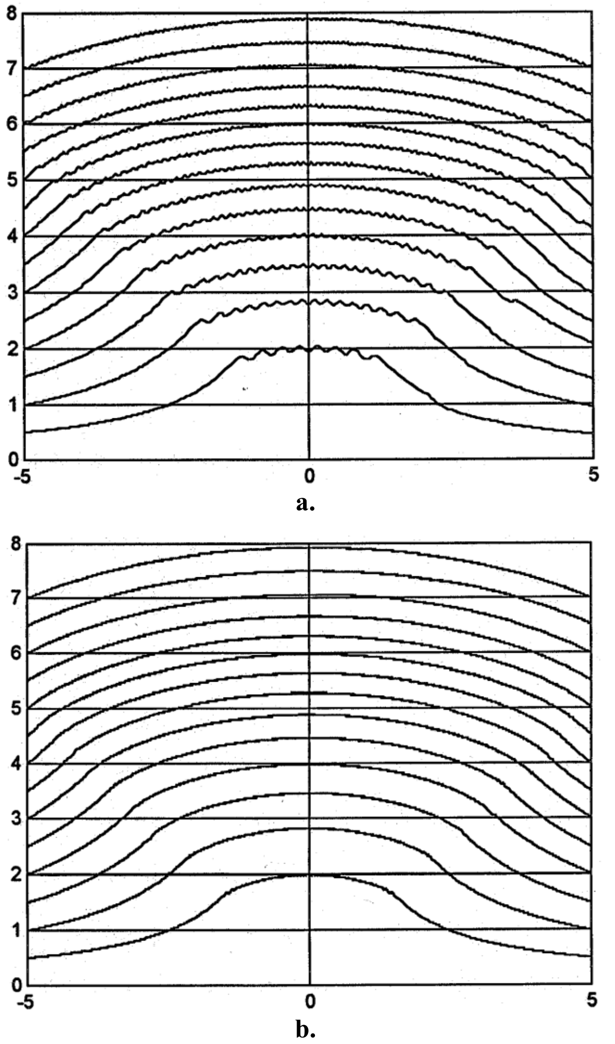


Fig. 1. Trajectories $x(\tau)$ of Eq. (7) with $\varepsilon = 0.2\pi$ (a.) and $\varepsilon = 0.1$ (b.)

1. *$f(x,t)$ is a KBM-vector field, i.e. the limit (3) exists and f satisfies some technical conditions* [Sanders, J.A., Verhulst, F., and Murdock, J. (2007): p. 69], [Bogoliubov, N., and Mitropolsky, Yu. (1961)].

2. *t belongs to the interval of time scale* $1/\varepsilon$. *Then*

$$\lim_{\varepsilon\to 0}\|x(t) - z(t)\| = 0 \tag{4}$$

Introducing slow time $\tau=\varepsilon t$, Eq.(1)-(2) can be written as

$$dx/d\tau = f(x, \varepsilon^{-1}\tau), \quad x(0) = a \tag{5}$$

$$dz/d\tau = \bar{f}(z), \quad z(0) = a \tag{6}$$

and τ belongs to the interval of time scale 1.

Unfortunately, in many important applications $f(x,t)$ is not a KBM vector field. A simple example is the following.

Example 1. Let us consider the following system of type (1):

$$\dot{x} = \varepsilon \sin\left(2\pi \cdot x \cdot t\right) \Rightarrow dx/d\tau = \sin\left(2\pi \cdot x\varepsilon^{-1}\tau\right) \quad (7)$$

Indeed, if f in (7) was a KBM field the averaged system (2) would be dz/dt≡0, while simulation of (7) in Fig. 1 demonstrates it is not the case –trajectories are far from constants.

This observation and demands of applications motivates considering averaging of the systems of the form

$$dx/d\tau = f\left(\tau, x, \varepsilon^{-1}\varphi(\tau, x)\right), \quad x(0) = a \quad (8)$$

and looking for ε-independent averaged system of the form

$$dz/d\tau = \bar{f}\left(\tau, z; \varphi()\right), \quad z(0) = a \quad (9)$$

where $\bar{f}\left(\tau, z; \varphi()\right)$ is an operator of $\varphi(\ )$. This approach provides a very important bonus for control systems –the systems can be controlled by $\varphi(\ )$ that becomes a feedback.

Unfortunately not all systems of the class (8) can be averaged, and [Belikov, S., and Belikov, R. (1996)] introduces the class of *Rapidly Oscillating Systems* (*ROS*), proves theorem of averaging, and demonstrates applications.

We consider the following system of ODEs:

$$\begin{aligned} &dx_k/dt = f_k(t, x, y) \\ &dy^i_{l_i}/dt = g^i_{l_i}(t, x, y)\Phi^i\left(t, x, y, \varepsilon^{-1}\varphi^i\left(t, x, y^i\right)\right) \end{aligned} \quad (10)$$

$x = [x_1, \ldots, x_n]$, $y^i = [y^i_1, \ldots, y^i_{m_i}]^T$; k=1,...,n; l_i=1,...,m_i; i=1,...,N; and Φ^i is periodic on the fourth argument with the period 1. System (10) is called (n, N) ROS with n non-(rapidly)-oscillating (or "macro") blocks and N rapidly oscillating (or "micro") blocks.

Here we reproduce the theorems for two-dimensional ROS, i.e. $n+N=2$, ($m_i \le 1$). In this case (10) is reduced either (1,1) or (0,2) ROS with either one or two rapidly oscillating blocks.

2.1 Averaging of (1,1) *ROS*

The (1,1) ROS is the following

$$\dot{x} = f(t, x, y), \quad \dot{y} = \Phi\left(t, x, y, \varepsilon^{-1}\varphi(t, x, y)\right) \quad (11)$$

where $\Phi(t, x, y, \eta)$ is periodic on η with period one. Let us define function

$$u(t, x, y) = \frac{\varphi'_t(t, x, y) + \varphi'_x(t, x, y) f(t, x, y)}{\varphi'_y(t, x, y)} \quad (12)$$

and

***Non-Resonance Condition for* (1,1) *ROS*:** *numerator and denominator of* (12) *do not vianish simultaneously.*

With technical conditions formulated in [Belikov, S., and Belikov, R. (1996)], including the non-resonance condition

The averaging of* (1,1) *ROS described by* (11) *is

$$\begin{aligned} &\dot{x} = f(t, x, y) \\ &\dot{y} = \left[\int_0^1 \left[u(t, x, y) + \Phi(t, x, y, \eta)\right]^{-1} d\eta\right]^{-1} - u(t, x, y) \end{aligned} \quad (13)$$

Example 2. Eq. (7) can be presented in the following (1,1) ROS form

$$\dot{x} = 1, \quad \dot{y} = \sin\left(2\pi \cdot \varepsilon^{-1} xy\right) \quad (14)$$

and applying formulas (12)-(13), the averaging is

$$\begin{aligned} &\dot{x} = 1, \quad \dot{y} = \begin{cases} \operatorname{sgn} u \cdot \sqrt{u^2 - 1} - u, \text{ if } |u| > 1 \\ -u, \quad \text{if } |u| \le 1 \end{cases} \\ &u = y/x \end{aligned} \quad (15)$$

Remark. To prove (15), use [Dwight, H.B. (1961): 436.00].

Example 3. Let us consider a pendulum perturbed by a rapidly oscillating torque that depends on both velocity and time

$$\dot{x} = y, \quad \dot{y} = -x + a \cdot \operatorname{sgn}\left(\sin\left(\varepsilon^{-1}(t + y)\right)\right) \quad (16)$$

where the parameter a is proportional to both the strength of the torque as well as inverse moment of inertia. Applying formulas (12)-(13), the averaging is

$$\dot{x} = y, \quad \dot{y} = -x - a^2(1 - x)^{-1} \quad (17)$$

that is a Hamiltonian system with

$$H(x, y) = \frac{1}{2} \cdot \begin{cases} x^2 + y^2 - 2a \cdot \ln\left|(1 - x)/a\right| , \\ \qquad\qquad \text{if } |1 - x| > a \\ y^2 + 2x + a^2 - 1, \text{if } |1 - x| \le a \end{cases} \quad (18)$$

The evolution of the perturbed system (16) is graphed in Fig. 2 for the initial condition (0,0), that is the steady state of unperturbed pendulum, with a=0.1 and the trajectory is graphed from t=0 to t=10. Heavy oscillation is clearly shown in Fig. 2a (ε=0.1). One interesting aspect of that trajectory is that it does not oscillate around the fixed point (0,0), but rather has acquired an orbit that winds around another point. In Fig. 2b the same simulation is repeated for ε=0.01. One can see a significant reduction in oscillation but the perturbation still cause the orbit to wind around in less than perfect circles. Finally, Fig. 3 shows the averaged trajectory of Eq. (17). As expected, all the oscillations had been averaged out of the system, and the curve winds around in a closed circle defined by the Hamiltonian (18).

2.2 Averaging of (0,2) *ROS*

The (0,2) ROS is the following

$$\dot{y}_i = \Phi_i\left(t, y_1, y_2, \varepsilon^{-1}\varphi_i(t, y_i)\right), \quad i = 1,2. \tag{19}$$

where both $\Phi_i(t, y_1, y_2, \eta)$ are periodic on η with period one. Let us define the functions

$$u_i(t, y_i) = \frac{[\varphi_i]'_t(t, y_i)}{[\varphi_i]'_{y_i}(t, y_i)} \tag{20}$$

and

Non-Resonance Condition for (0,2) ROS*: numerator and denominator of (20) do not vanish simultaneously*.

With technical conditions formulated in [Belikov, S., and Belikov, R. (1996)], including the non-resonance condition,

T*he averaging of (0,2) ROS described by (19) is the following*

$$\dot{y}_i = \left[\int_0^1 \frac{d\eta}{u_i(t, y_i) + \Phi_i(t, y_1, y_2, \eta)}\right]^{-1} - u_i(t, y_i), i = 1,2 \tag{21}$$

We will consider example of (0,2) ROS in Section 4.

3. WEAK RESONANCE CONDITIONS

A lot of unusual things may happen when non-resonance condition for (1,1) ROS (11) or for (0,2) ROS is not satisfied. Recall that these are the conditions that the numerator and denominator of (12) for (1,1) ROS or (20) for (0,2) ROS, do not vanish simultaneously, i.e. the fractions belong to the projective space $RP^1=(R^2\backslash 0)/(R\backslash 0)$ [Arnold, V.I. (1988): p. 20]:

$$u(t, x, y) = \frac{\varphi'_t(t, x, y) + \varphi'_x(t, x, y) f(t, x, y)}{\varphi'_y(t, x, y)} \in RP^1$$

$$u_i(t, y_i) = \frac{[\varphi_i]'_t(t, y_i)}{[\varphi_i]'_{y_i}(t, y_i)} \in RP^1$$

Hence the averaging formula (13) for (1,1) ROS is only valid outside the set defined by equations

$$\begin{cases} \varphi'_t(t, x, y) + \varphi'_x(t, x, y) f(t, x, y) = 0 \\ \varphi'_y(t, x, y) = 0 \end{cases} \tag{22}$$

and the averaging formula (21) is only valid outside the set defined by equations

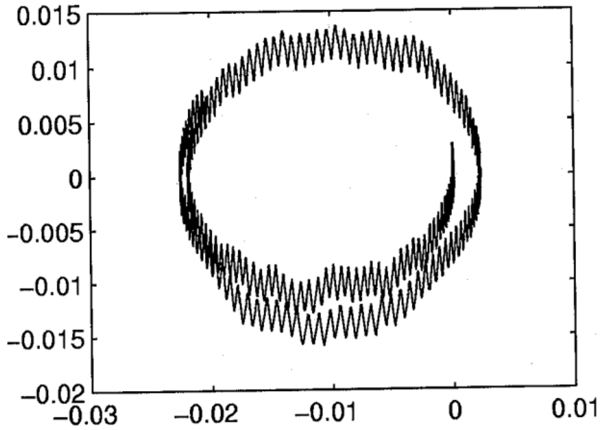
$$\begin{cases} [\varphi_i]'_t(t, y_i) = 0 \\ [\varphi_i]'_{y_i}(t, y_i) = 0 \end{cases} \tag{23}$$

Let us assume the following:

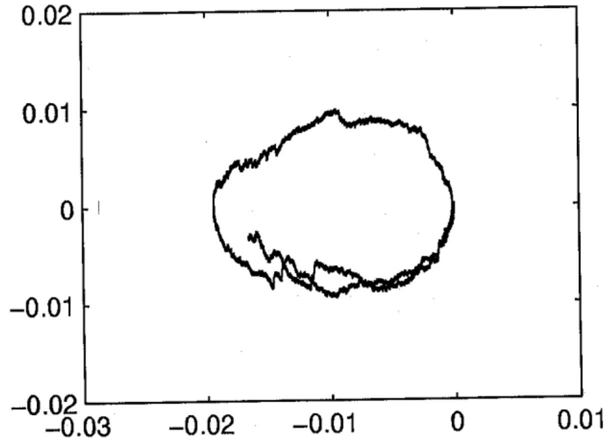
The solutions of Eq. (22) for (1,1) ROS, and Eq. (23) for (0,2) ROS form stratified manifolds

as illustrated in Fig. 3 for (1, 1) ROS with $\varphi(x,y)$ and $f(x,y)$.

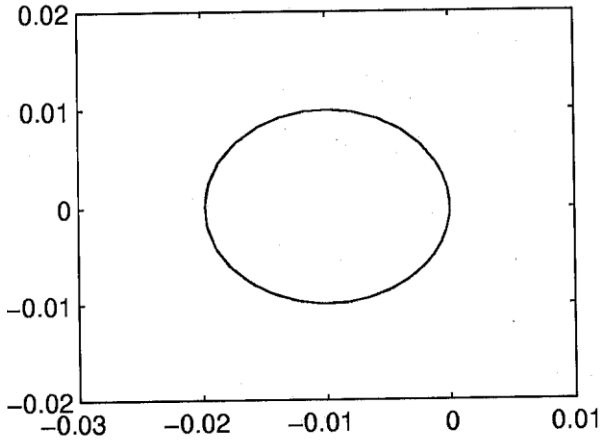
Recall that a *stratified manifold* M [Arnold, V.I. (1988): p. 230] (or *stratified subvariety of smooth manifold*) is a finite



a. Numerically calculated trajectory of Eq. (16) with a=0.1 and ε=0.01



b. Numerically calculated trajectory of Eq. (16) with a=0.1 and ε=0.001



c. Asymptotic ($\varepsilon \to 0$) solution described by averaged system (17).

Fig. 2. Trajectories of perturbed pendulum and its averaging system

union of mutually disjoint smooth manifolds (*strata*) satisfying the following condition:

The closure of every stratum consists of the stratum itself and a finite union of strata of smaller dimensions.

The stratified manifold for (1,1) ROS with $\varphi=\varphi(x,y)$ and $f=f(x,y)$ is shown in Fig. 3a. It consists of only zero-dimensional strata where the curve $\varphi'_y(x,y)=0$ intersects either $\varphi'_x(x,y)=0$ or $f(x,y)=0$.

We illustrate the concept on examples of the following (1,1) ROS:

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=\Phi\left(x,y,\varepsilon^{-1}\varphi(x)\right)\end{aligned} \qquad (24)$$

For this system, $u(x,y)=y\varphi'(x)/0$ and the resonance set is described by

$$y\varphi'(x)=0 \qquad (25)$$

Remark. When denominator of $u(t,x,y)$ is zero, as in (24), calculations in (13) should be made for $u\to\infty$:

$$\lim_{u\to\infty}\left(\left[\int_0^1[u+\Phi(\eta)]^{-1}d\eta\right]^{-1}-u\right)=\lim_{v\to0}\frac{1}{v}\left(\left[\int_0^1[1+v\Phi(\eta)]^{-1}d\eta\right]^{-1}-1\right)$$
$$=\frac{d}{dv}\left[\int_0^1[1+v\Phi(\eta)]^{-1}d\eta\right]^{-1}\Bigg|_{v=0}=\int_0^1\Phi(\eta)d\eta$$

This is the closest to KBM averaging (3) for periodic functions. In this case the averaging does not depend on φ.

The stratified manifold for (1,1) ROS (24) is shown in Fig. 3b. It consists of the zero-dimensional strata $\mathrm{M}^0=\{y=\varphi'(x)=0\}$, one dimensional (horizontal) strata $\mathrm{M}^{1h}=\{y=0\ \&\ \varphi'(x)\neq0\}$ and one-dimensional (vertical) strata $\mathrm{M}^{1v}=\{y\neq0\ \&\ \varphi'(x)=0\}$. Now, we formulate

Weak Resonance Condition for (1,1) ROS. *At every point* (t,x,y) *of the stratified manifold* M *associated with Eq.* (22) *and every* $\eta\in[0,1]$ *the vector* $(t,f(t,x,y),\Phi(t,x,y,\eta))$ *does not belong to the tangent hyperplane* $T_{(t,x,y)}\mathrm{M}$ *to the appropriate stratum.*

Let us review the (1,1) ROS (24). The weak resonance condition for M^0 is $\forall\eta\in[0,1]\forall x_i\in\{\varphi'(x_i)=0\}$: $(0,\Phi(x_i,0,\eta))\neq(0,0)$, i.e. $\Phi(x_i,0,\eta)\neq0$ for these η and x_i. The weak resonance condition for M^{1h} is $\forall\eta\in[0,1]\forall x\in\{\varphi'(x)\neq0\}$: $(0,\Phi(x,0,\eta))\notin T_{(x,0)}\mathrm{M}^{1h}=x-\text{axis}$, i.e. $\Phi(x,0,\eta)\neq0$ for $\eta\in[0,1]$. The weak resonance condition for M^{1v} is $\forall\eta\in[0,1]\forall x_i\in\{\varphi'(x_i)=0\}\forall y\neq0$: $(y,\Phi(x_i,y,\eta))\notin T_{(x_i,y)}\mathrm{M}^{1v}=(y-\text{axis})$, that is satisfied automatically because the x-component of $(y,\Phi(x_i,y,\eta))$ is $y\neq0$ on M^{1v}. Analyzing these results we conclude that the Weak Resonance Condition for (1,1) ROS (24) is

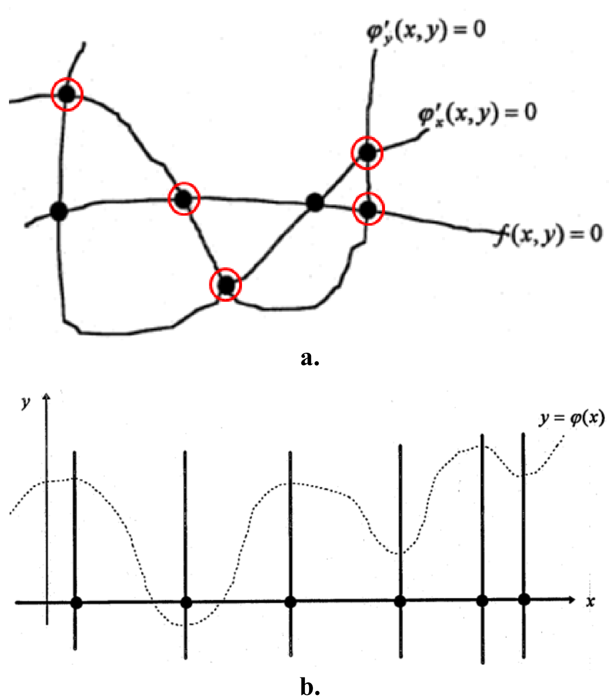


Fig. 3. Stratified manifolds: of (1,1) ROS described by Eq. (11) with $\varphi=\varphi(x,y)$, $f=f(x,y)$ –only red circled dots (dim 0) (**a.**) and with $\varphi=\varphi(y)$, $f(t,x,y)=y$ –dots (dim 0) and open vertical and horisontal segments (dim 1) (**b.**)

$$\Phi(x,0,\eta)\neq0,\quad -\infty<x<\infty, 0\leq\eta<1 \qquad (26)$$

Now, imagine a system that does not satisfy the non-resonance condition but does satisfy the weak resonance condition. Then all the trajectories of averaged system intersect the resonance set M transversely and spent time of measure 0 (as $\varepsilon\to0$) on it. Since M is also a boundary of non-resonance sets (on which the averaging does exists), the limiting space trajectories of (1,1) ROS (11) must be piecewise composed of the solutions to the averaged system (13) on the non-resonant sets. This justifies the following theorem:

Averaging of* (1,1) *ROS satisfying the Weak Resonance Condition. *If the Weak Resonance Condition is satisfied for* (1,1) *ROS governed by Eq.* (11)*, then any limiting as* $\varepsilon\to0$ *solution of* (11) *does not spend time on the resonance set* M *and any limiting trajectory is a closure of the union of some phase trajectories of the averaged system* (13) *on the non-resonance sets.*

Example 4. The following perturbation of an integrable Hamiltonian system in coordinates (I,ψ) –(action, angle) has been analized in [Arnold, V.I. (1988), p. 156] using classical technique for a=1:

$$\dot{I}_1=\varepsilon a,\ \dot{I}_2=\varepsilon\cdot\cos(\psi_1-\psi_2),\ \dot{\psi}_1=I_1,\ \dot{\psi}_2=I_2 \qquad (27)$$

The simplest classical resonance condition is equality of the frequencies, i.e. $I_1=I_2$. It has been shown in [Arnold, V.I. (1988)] for a=1, that the averaged dynamics { $\dot{I}_1=\varepsilon,\ \dot{I}_2=0$ } approximates the evolution of (I_1, I_2) on the time interval of the order of $1/\varepsilon$ outside of the resonance set $I_1 = I_2$. However, the system may be "caught by resonance", and it is definitely happened if $\psi_1 = \psi_2$ at the time when $I_1 = I_2$. In this case the averaged dynamics $\dot{I}_2 = 0$ does not approximate the real solution of $\dot{I}_2 = \varepsilon \cdot \cos 0 = \varepsilon$ on the interval of the order of $1/\varepsilon$.

By the change of variables $\tau = \varepsilon t$, $x_i = \varepsilon\psi_i$ Eq. (27) becomes

$$\begin{cases} dx_1/d\tau = I_1,\ dx_2/d\tau = I_2,\ dI_1/d\tau = a, \\ dI_2/d\tau = \cos\left(\varepsilon^{-1}(x_1 - x_2)\right) \end{cases}$$

It is (3,1) ROS with $\varphi = x_1 - x_2$, and $u = \left(\varphi'_{x_1} I_1 + \varphi'_{x_2} I_2\right)/0 = (I_1 - I_2)/0$. Thus, the ROS resonance condition $I_1 - I_2 = 0$ coincides with the classical one. Using the variables $x = x_1 - x_2$ and $y = I_1 - I_2$, we get the following (1,1) ROS of the type (24):

$$dx/d\tau = y,\ dy/d\tau = a - \cos\left(\varepsilon^{-1}x\right) \quad (28)$$

For (28), $\varphi = x$, $u = y/0$, and the ROS resonant set is $\{y=0\}$. The Weak Resonant Condition is not satisfied for this ROS with $|a| \le 1$, and as a result we have the problem of approximating the evolution of (I_1, I_2) on the time interval of the order of $1/\varepsilon$. However, the condition is satisfied for $|a|>1$.

Example 5. The following equation of the pendulum with small parameter ε was analyzed in [Arnold, V.I. (1988), p. 157]: $\ddot{\psi} = \varepsilon\left(a + \sin\psi - \dot{\psi}\right)$ with $a>0$. It can be written as

$$\dot{\psi} = y,\ \dot{y} = \varepsilon(a + \sin\psi - y)$$

Remark. As shown in [Arnold, V.I. (1988), p. 157], by using time $\nu = \sqrt{\varepsilon t}$ and denoting by the prime the derivation with respect to ν, this equation describes a pendulum with small friction $\psi'' = a + \sin\psi - \sqrt{\varepsilon}\psi'$.

Classical resonance is $y=0$. By the change of variables $\tau = \varepsilon t$, $x = \varepsilon\psi$, the equation becomes the following (1,1) ROS of the type (24):

$$dx/d\tau = y,\ dy/d\tau = a - y + \sin\left(\varepsilon^{-1}x\right) \quad (29)$$

For (29), as for (28), $\varphi = x$, $u = y/0$, and the ROS resonant set is $\{y=0\}$. The Weak Resonant Condition is not satisfied for this ROS with $|a| \le 1$, and as a result we have the problem of approximating the evolution of (ψ, y) on the time interval of the order of $1/\varepsilon$. However, the condition is satisfied for $|a|>1$.

These examples demonstrate that ROS approach can unify and simplify the analysis of some classical systems –in [Arnold, V.I. (1988)] these examples were analyzed by different model-specific approaches. The following example presents Weak Resonance in a non-classical ROS.

Example 6. Let us consider the following (1,1) ROS:

$$dx/d\tau = 1,\ dy/d\tau = \cos\left(2\pi\varepsilon^{-1}\left(|y|^\alpha - |x|^\alpha\right) + a\right) \quad (30)$$

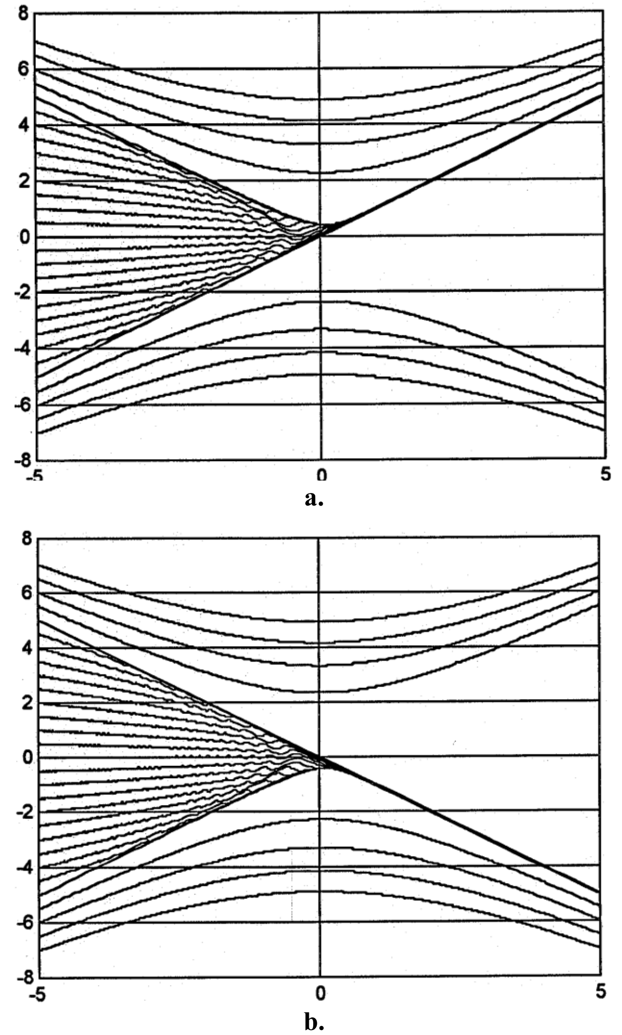


Fig. 4. Phase trajectories for the (1,1) ROS (31) with $\alpha=2$; $a=0$ (**a.**) $a=\pi$ (**b.**)

with $\alpha \ge 1$. The system depends on parameter a. We have

$$\varphi(x,y) = |y|^\alpha - |x|^\alpha \Rightarrow u(x,y) = \frac{|x|^{\alpha-1}\operatorname{sgn} x}{|y|^{\alpha-1}\operatorname{sgn} y} \quad (31)$$

The resonant set contains the origin $(x,y) = (0,0)$. The weak resonance condition is satisfied ($dx/d\tau = 1 \neq 0$), and limiting trajectory is a closure of the union of some phase trajectories of the following averaged system on the whole plane excluding the origin:

$$\dot{x} = 1, \quad \dot{y} = \begin{cases} \operatorname{sgn} u \cdot \sqrt{u^2 - 1} - u, \text{ if } |u| > 1 \\ -u, \quad \text{if } |u| \le 1 \end{cases} \quad (32)$$

Although the averaged system (32) does not depend on the parameter a, the phase portrait does, as shown in Fig. 4 for different values of the parameter. Thus, a non-unique possibility of connecting the limiting trajectories at the resonant point (0,0) does depend on the parameter. Fig. 4 also suggests that at a certain value of the parameter canards should be observed as discussed in Introduction.

4. ROS ASYMPTOTIC CONTROL

As shown in previous sections, non-classical ROS has a unique feature that allows to control macro-system (averaged) by manipulating the functions $\varphi(x,y)$ at micro-level. A method of designing ROS asymptotic control has been developed in [Belikov, S., and Belikov, R. (1996)] and illustrated in the bipedal walk problem. Other applications are described in [Belikov, S., and Belikov, R. (1999)] and [Belikov, R., and Belikov, S. (2001)]. The aim of the following example is to illustrate the method on the (0,2) ROS and connect it to a classical optimal control example [Boltyanskii, V.G. (1971): p. 204].

Example 7. Let us consider the following (0,2) ROS:

$$\begin{aligned} \dot{y}_1 &= y_2 \sin\left(2\pi\varepsilon^{-1}\varphi_1(t, y_1)\right) \\ \dot{y}_2 &= \sin\left(2\pi\varepsilon^{-1}\varphi_2(t, y_2)\right) \end{aligned} \quad (33)$$

Associated functions $u_i(t,y_i)$, i=1,2, are calculated by (20) and according to equation (21) the averaged system is

$$\begin{aligned} \dot{y}_1 &= y_2 \cdot v\left(u_1(t, y_1)/ y_2\right) \\ \dot{y}_2 &= v\left(u_2(t, y_2)\right) \end{aligned} \quad (34)$$

where (see remark after formula (15))

$$v(u) = \begin{cases} \operatorname{sgn} u \cdot \sqrt{u^2 - 1} - u, & \text{if } |u| > 1 \\ -u, & \text{if } |u| \le 1 \end{cases} \quad (35)$$

and $|v(u)|\le 1$ by definition.

Let us consider an asymptotic time-optimal control problem for the system (33). According to [Belikov, S., and Belikov, R. (1996)], the asymptotically time-optimal control functions $\varphi_i(t,y_i)$, i=1,2, can be calculated by solving equations (20) with $u_i(t,y_i)$, i=1,2, equal to the time-optimal controls for the averaged system (34).

Time-optimal controls for system (34) can be found by using Pontryagin maximum principle [Boltyanskii, V.G. (1971)]. Applying it to the time-optimal problem for the system

$$\dot{y}_1 = y_2 \cdot v_1, \quad \dot{y}_2 = v_2; \quad |v_i| \le 1, \quad i = 1,2 \quad (36)$$

gives the following solution [Boltyanskii, V.G. (1971): p. 204]:

$$v_1(y_1, y_2) = \begin{cases} +1, & (y_1, y_2) \in V_1 \cup V_2 \cup V_4 \cup V_6 \cup \sigma_2 \cup \sigma_4 \\ -1, & (y_1, y_2) \in V_3 \cup V_5 \cup \sigma_1 \cup \sigma_3 \end{cases} \quad (37)$$

$$v_2(y_1, y_2) = \begin{cases} +1, & (y_1, y_2) \in V_2 \cup V_3 \cup V_4 \cup \sigma_3 \cup \sigma_4 \\ -1, & (y_1, y_2) \in V_1 \cup V_5 \cup V_6 \cup \sigma_1 \cup \sigma_2 \end{cases} \quad (38)$$

where σ_i, i=1,2,3,4, are the branches of parabolas shown in Fig. 5 that separate the areas V_i, i=1,2,3,4,5,6, on the plane (y_1, y_2). Union of σ_i, i=1,2,3,4, is a stratified manifold where at least one of v_i, i=1,2, changes sign, and inside each V_i, i=1,2,3,4,5,6, v_1 and v_2 are constant (+1 or −1). Arrows in Fig. 5 indicate the directions of the time-optimal trajectories inside V_is.

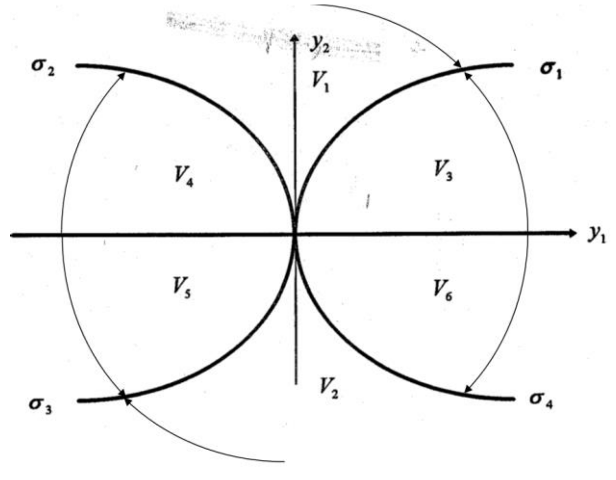


Fig. 5. Definition of sets V_1,..., V_6, and σ_1,..., σ_4, used in equations (37)-(38).

From equations (34)-(38), we have

$$u_1(t, y_1) = -v_1(y_1, y_2) \cdot y_2, \quad u_2(t, y_2) = -v_2(y_1, y_2) \quad (39)$$

Eq. (39) gives the solution in the form of full feedback control law. The requirements that u_1 depends explicitly only on t and y_1, and u_2 depends explicitly only on t and y_2 can be satisfied by substituting wherever necessary y_1, and y_2 for the solutions $\bar{y}_1(t), \bar{y}_2(t)$ of the system governed by equations (36)-(38), i,e.

$$\begin{cases} u_1(t, y_1) = -v_1(y_1, \bar{y}_2(t)) \cdot \bar{y}_2(t) \\ u_2(t, y_2) = -v_2(\bar{y}_1(t), y_2) \end{cases} \quad (40)$$

This means that $u_i(t,y_i)$, i=1,2, depend also on initial states $\bar{y}_1(0), \bar{y}_2(0)$.

Having u_1 and u_2 calculated by (40), time-optimal asymptotic control functions $\varphi_i(t, y_i), i = 1,2$, can be calculated by solving partial differential equations of the first order (20) by the method of characteristics [Arnold, V.I. (1988)].

From (20) and the first equation of (40) we have

$$\frac{\partial \varphi_1(t, y_1)}{\partial t} + v_1 \cdot \bar{y}_2(t) \frac{\partial \varphi_1(t, y_1)}{\partial y_1} = 0 \quad (41)$$

where v_1 calculated by (37) is constant in every V_i, i=1,…,6, and σ_i, i=1,…,4, and from (36)

$$\bar{y}_2(t) = \bar{y}_2(0) + v_2 t \quad (42)$$

where v_1 is calculated by (37) and v_2 is calculated by (38).

Then a solution of (41) inside the V_i or σ_i that contains $(\bar{y}_1(0), \bar{y}_2(0))$ is

$$\varphi_1(t, y_1) = (\bar{y}_2(0)\cdot t + v_2 \cdot t^2/2)\cdot v_1 - y_1 \quad (43)$$

From (20) and the second equation of (40) we have

$$\frac{\partial \varphi_2(t, y_2)}{\partial t} + v_2 \cdot \frac{\partial \varphi_2(t, y_2)}{\partial y_2} = 0 \quad (44)$$

where v_2 calculated by (38) is constant in every V_i, i=1,…,6, and σ_i, i=1,…,4. A solution of (44) inside the V_i or σ_i that contains $(\bar{y}_1(0), \bar{y}_2(0))$ is

$$\varphi_2(t, y_2) = t \cdot v_2 - y_2 \quad (45)$$

5. CONCLUSIONS

Understanding specifics and techniques of Rapidly Oscillating Systems (ROS), both classical and non-classical is important in applications. We demonstrated typical methods on the ROS on the plane, classified as (1,1) and (0,2) ROS.

There are many interesting properties that are virtually unique to non-classical ROS, such as weak resonance conditions, rapid bifurcations (canards in ROS), and asymptotic control. However, non-classical ROS technique is often useful in analysis of classical problems as illustrated in the paper.

The technique of asymptotic control is to find the control of the perturbed system by solving the control problem on its averaging (“macro” system), and then calculate the control functions of the perturbed system (“micro”) that give the same optimal averaged dynamics when the small parameter converges to zero. This has already been applied to several engineering tasks.

Among challenging unsolved problems is an extension of the averaging of (0,2) ROS governed by (19) to a system

$$\dot{y}_i = \Phi_i(t, y_1, y_2, \varepsilon^{-1}\varphi_i(t, y_1, y_2)), \quad i = 1,2. \quad (46)$$

where the functions φ_i, i=1,2, depend on the state variables y_1 and y_2 of both oscillating blocks. There are important applications, including Atomic Force Microscopy, that require this extension.

Acknowledgment: This work was supported by Defense Microelectronics Activity under Contract HQ072720P0003.

REFERENCES

Albeverio, S., Fenstad, J.E, Hoegh-Krohn, and Lindstrom, T. (1986) *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, 1989 Dover Publications.

Arnold, V.I. (1988). *Geometrical Methods in the Theory of Ordinary Differential Equations* –Second Edition, Springer.

Belikov, S., and Samborskii, S. (1989). Duck-cycles of fast-slow vector fields with one dimensional slow variable, *C.R. Acad. Sci. Paris, t. 309, Serie I, pp. 673-677.*

Belikov, S. (1989). Averaging of Systems Containing Strongly Oscillating Functions of Phase Variables, *Ukrainian Math. Journal*, 1990 Plenum Publishing Corp., pp. 1011-1016.

Belikov, S., and Samborskii, S. (1991). Canard Cycles of Fast-Slow Fields with One-Dimensional Slow Component, translated from *Mathematical Notes* (*Matematicheskie Zametki*), Vol. 49, No. 4, 1991 Plenum Publishing Corporation, pp. 339-346

Belikov, S., and Belikov, R. (1996). Rapidly Oscillating Systems: Averaging and Control, *1996 Conference on Decision and Control*, pp. 2169-2174.

Belikov, S., and Belikov, R. (1997). Rapidly Oscillating Systems on the Plane: Resonances and Rapid Bifurcations, 1997 *Conference on Decision and Control*, pp. 1153-1154.

Belikov, S., and Belikov, R. (1999). Rapidly Oscillating Systems for Frequency Demodulation, *1999 American Control Conference*, pp. 4451-4455.

Belikov, R., and Belikov, S. (2001). Averaging and Control of Wave Propagation in Rapidly Varying Media, *2001American Control Conference*, pp. 1954-1959.

Belikov, S., Alexander, J., Surtchev, M., and Magonov, S. (2016). Implementation of Atomic Force Microscopy Resonance Modes based on Asymptotic Dynamics using Costas Loop, *2016 American Control Conference*, pp. 6201-6208.

Belikov, S., and Magonov, S. (2017). Modeling of Cantilever Beam under Dissipative and Nonlinear Forces with Application to Multi-Resonant Atomic Force Microscopy, 2017 *IFAC Congress*, pp. 8984-8991.

Belikov, S., and Magonov, S. (2019). Simulation of asymptotic amplitude-phase dynamics for AFM resonant modes, *2019 American Control Conference*, pp. 2477-2482.

Bellman, R. (1966). *Perturbation Techniques in Mathematics, Engineering & Physics*, Dover Publications.

Benoit, E., Callot, J.L, Diener, F., and Diener, M. (1981), Chase au canard, *Collect. Math*., 32.

Bogoliubov, N., and Mitropolsky, Yu. (1961). *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Gordon&Breach, NY.

Boltyanskii, V.G. (1971). *Mathematical Methods of Optimal Control*, Holt, Rinehart and Windson, Inc.

Dwight, H.B. (1961). *Tables of integrals and other mathematical data*, The MacMillan Company.

O’Malley R.E. (2016). *Historical Development of Singular Perturbation*, Springer.

Sanders, J.A., Verhulst, F., and Murdock, J. (2007). *Averaging Methods in Nonlinear Dynamical Systems – Second Edition*, Springer.

Sari, T. (1983). Sur la theory asymptotique des oscillations non stationnaires, *Asterisque*, Soc. Math. France, 109-110, pp. 141-158.

Zvonkin, A.K., and Shubin M.A. (1984). Nonstandard analysis and singular perturbation of ordinary differential equations, *Russian Math. Surveys*, 39.