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Ergodic Linear-Quadratic Control for a Two Dimensional Stochastic System Driven by a Continuous Non-Gaussian Noise

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Abstract:
In this paper an infinite time horizon (ergodic) quadratic cost control problem for a linear two dimensional stochastic system with a two dimensional Rosenblatt noise process is solved by providing an explicit expression to determine the optimal feedback. The system has some symmetry properties that allow for an explicit determination of an optimal control. The controls are the family of constant linear feedbacks which is known to be the natural family of controls for a Brownian motion noise to determine optimality. This family of constant linear feedback controls allows for practical implementation of the optimal control. Rosenblatt processes are continuous, non-Gaussian processes that have a long range dependence and a useful stochastic calculus and they are generated by double Wiener-Itô integrals with singular kernels. The long range dependence property of the Rosenblatt processes is a natural generalization from an important subfamily of (Gaussian) fractional Brownian motions. Long range dependent processes have been identified empirically in a significant variety of physical phenomena. An expression is obtained to determine explicitly the optimal ergodic control. The ergodic control result in this paper seems to be the first explicit ergodic control result for a multidimensional control system with a continuous, non-Gaussian noise. Furthermore it seems to be the first solution for a multi-dimensional game problem with a Rosenblatt noise.

Keywords: stochastic systems, ergodic control, non-Gaussian noise, Rosenblatt processes, explicit stochastic optimal controls

1. INTRODUCTION

Since noise in stochastic systems is typically used to model perturbations or unmodeled dynamics of the physical systems, it is important to have physically justifiable noise models. Historically noise in continuous time physical systems was modeled as white Gaussian noise (the formal derivative of Brownian motion). This noise model for stationary systems was often justified by the fact that the nonzero region of the spectral density of the noise was significantly broader than the frequency description of the dynamics of the system so a constant spectral density was chosen on $\mathbb{R}$ that relates to a white noise and that a suitable Central Limit Theorem should be applicable to justify a Gaussian process. In the last few decades, a family of Gaussian processes have been introduced as an attempt to describe more effectively the empirical properties of a noise in a variety of physical systems. This family of processes indexed by the Hurst parameter $H \in (0, 1)$ is called fractional Brownian motions and includes Brownian motion for $H = \frac{1}{2}$ and some long range dependent Gaussian processes for $H \in (\frac{1}{2}, 1)$ that describe more effectively some observed behavior from many physical phenomena such as rainfall, turbulence, earthquakes, cognition, and epileptic seizures. The authors have worked on these fractional Brownian processes in both finite and infinite dimensions by developing some stochastic calculus for these processes and solving a variety of control problems e.g. Duncan, Jakubowski and Pasik-Duncan [2006], Duncan, Maslowski and Pasik-Duncan [2012], Duncan and Pasik-Duncan [2013] as well as solving adaptive control problems with these noise processes. However all of these processes are Gaussian though significant empirical evidence from physical systems demonstrates that typically the noise in real world control systems is not Gaussian (Domański [2015]) and furthermore there are mathematical justifications from non-Gaussian limit theorems e.g. Dobrushin, Major [1979]. Specifically this claim of non-Gaussian noise is partially based on data from several hundred control loops operating in different process industries located in many sites throughout the world (Domański [2015]). Thus the applicability of results from control system models with Gaussian noise can be questioned when applied to a variety of nonlocal physical systems. Furthermore it seems both mathematically and empirically important to have results for processes that have self-similarity properties for scaling and long range time dependence. Self-similar processes have probability

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laws that are invariant under suitable scaling and have long range dependence that implies a relatively slow decay to zero asymptotically which is often described in terms of the asymptotic behavior of the corresponding covariance function. The range of applications of self-similar processes is significant. They have been identified in telecommunications, hydrology, biophysics, geophysics, atmospheric sciences, cognition, and finance (see the bibliographical guides on the applications of self-similar processes by Taqqu [1986]; and Willinger, Taqqu, and Erramili [1996] that contain many references). An extensive review of some turbulence models in given in Wilson [1998]. A fractional Brownian motion (fBm), with the process denoted by B_H, is probably the most well known example of a self-similar process. The fractional Brownian motions (fBm) are indexed by the Hurst parameter $H \in (0, 1)$. The reasons for their importance are at least two-fold. First, the fact that a fractional Brownian motion is self-similar, has stationary increments, and exhibits long-range dependence when the Hurst parameter H is in the interval $(1/2, 1)$ makes it an attractive model for many physical phenomena. The second major reason is the fact that each fractional Brownian motion is a Gaussian process. The stochastic calculus for a fractional Brownian motion has been developed especially for $H \in (\frac{1}{2}, 1)$. However, to generalize a noise to non-Gaussian models, some major impediments arise in the analysis and application of these processes for the solutions of control problems such as having an effective stochastic calculus to make useful models and determining explicit expressions for control problem solutions for systems driven by these non-Gaussian processes. A Rosenblatt process R_H with the Hurst parameter $H \in (1/2, 1)$ is a stochastic process which appears as the non-Gaussian limit of a suitable limit theorem e.g. Dobrushin, Major [1979]. This process shares some properties with a fractional Brownian motion with the Hurst parameter $H \in (1/2, 1)$. In particular, a Rosenblatt process has a version with non-differentiable Hölder continuous sample paths up to the exponent H, stationary increments, and it also exhibits long-range dependence and self-similarity of order H. In fact, a fractional Brownian motion and a Rosenblatt process are particular cases of the so-called family of Hermite processes and as such, they have the same autocovariance function. All of the Hermite processes can be described by multiple singular Wiener-Itô integrals

Historically noise in stochastic systems has been modeled by a Brownian motion or a formally equivalent Gaussian white noise. More recently there has been some work on linear systems with fractional Brownian motion noise. However all of these processes are Gaussian. The authors are not aware of any work on ergodic quadratic control problems with continuous non-Gaussian noise other than Čoupek, Duncan, Maslowski, and Pasik-Duncan [2019] where an ergodic control is explicitly determined for a scalar stochastic system with a Rosenblatt noise. The problem considered here is a two dimensional linear stochastic system where the two dimensional noise is two real-valued Rosenblatt processes that are independent. Some special symmetry assumptions are made to obtain explicit solutions though the results can be extended to other linear equations that lack the assumed symmetries. However the results will not be as explicit as for the case that is considered here. Furthermore some generalizations to higher dimensional systems can be made. It seems that no results for the optimal control of multidimensional stochastic equations driven by Rosenblatt processes or other continuous non-Gaussian and non-Markovian processes are available. Since the Rosenblatt processes are not Markov, Hamilton-Jacobi-Bellman equations are not applicable for the control problem considered here. Furthermore a stochastic maximum principle with forward-backward stochastic differential equations is not available. Thus it seems necessary to apply a direct method that has been successfully used for linear-quadratic control problems with Brownian motions and fractional Brownian motions e.g. Duncan and Pasik-Duncan [2013], Duncan, Maslowski and Pasik-Duncan [2012], linear-exponential quadratic control and games Duncan [2013] Duncan [2016] and control with Gauss-Volterra noise Duncan, Maslowski and Pasik-Duncan [2017].

The probabilistic approach for the systems with a Rosenblatt noise process that occurs here uses a stochastic calculus for these processes that is an evolution from the stochastic calculus for Brownian motion and a family of fractional Brownian motions Čoupek, Duncan and Pasik-Duncan [2019]. An alternative approach to Rosenblatt processes has used a white noise approach Arras [2015], Arras [2016] that is an evolution from the white noise approach for Brownian motion Hida,Kuo,Potthoff,and Streit [1994].

2. AN ERGODIC CONTROL PROBLEM

An ergodic control problem is chosen because it is important not only for infinite time problems but even for long range time problems and the limiting operation for $T \to \infty$ simplifies some of the calculations that appear from the Rosenblatt processes as is even the case for a Brownian motion noise. For example an integral on the half line to determine the optimal cost can be expressed as a Gamma function while the integral on a bounded interval cannot be related to a classical function. The ergodic control problem considered here is formulated with a two dimensional stochastic system and an ergodic quadratic cost. The two dimensional aspect provides an indication for possible generalizations to higher dimensions and how the multidimensional system responds to the noise vector. Some symmetries are assumed on the system that allow for explicit computations and also imply the existence of optimal ergodic costs for systems without these symmetries. However even with the assumed symmetries the control problem does not separate into two scalar problems. The controlled stochastic system satisfies the following stochastic equation

$$dX(t) = AX(t)dt + CU(t)dt + dR_H(t) \qquad (1)$$

$$X(0) = x_0 \qquad (2)$$

where $X(t) \in \mathbb{R}^2, A \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2), A = A^T, C \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ and is $C = I$, and $(R_H(t), t \geq 0)$ is a standard two dimensional Rosenblatt process with parameter H for both independent components of the two dimensional Rosenblatt process. The term U is the control. The ergodic quadratic cost, $J_\infty(U)$, is

$$J_\infty(U) = \lim \sup_{T\to\infty} \frac{1}{T}\mathbb{E}[\int_0^T (\langle QX(t), X(t)\rangle + \langle RU(t), U(t)\rangle)dt] \tag{3}$$

where Q, R are symmetric and $Q > 0$ and $R > 0$. The family of admissible controls, $\mathcal{U}$, is the collection of constant linear feedbacks of the state X, that is,

$$\mathcal{U} = \{U(t) = KX(t) | K \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)\} \tag{4}$$

This family of feedback controls is quite natural from the result for a Brownian motion noise. However allowing the controls to be adapted to the past of the process here would imply functional dependence on the past of the control because the control would be predicting the future of the state process as is the case with fractional Brownian motions e.g. Duncan and Pasik-Duncan [2013]. Such controls are not easily implementable. While the assumption that $C = I$ is restrictive it does not imply that the problem reduces to two distinct scalar systems because the cost functional cannot be split as the sum of two separate costs for each scalar system without some additional assumptions on the terms in the cost.

Initially some definitions of the Rosenblatt process, a related fractional Brownian motion, and some differential operators are given that are used in the change of variables formula. Let $(u)_+ = \max(u, 0)$ be the positive part of u and define h_k^H as

$$h_k^H(u, y) = \prod_{j=1}^{k} (u - y_j)_+^{\frac{H}{k} - \left(\frac{1}{k} + \frac{1}{2}\right)} \tag{5}$$

for $H \in (\frac{1}{2}, 1), u \in \mathbb{R}$, $k \in (1, 2)$ and $y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$.

Some fractional Brownian motions and Rosenblatt processes are defined now. The fractional Brownian motions are naturally associated with the Rosenblatt process.

Definition. Let $H \in (1/2, 1)$ be fixed. A real-valued *fractional Brownian motion* $B_H = (B_H(t), t \in \mathbb{R})$ is defined as follows

$$B_H(t) = C_H^B \int_{\mathbb{R}} \left(\int_0^t h_1^H(u, y) du \right) dW(y) \tag{6}$$

for $t \geq 0$ (and similarly for $t < 0$) where C_H^B is a constant given below such that $\mathbb{E}(B_H(1))^2 = 1$ and W is a standard Wiener process on the probability space.

Definition. Let $H \in (1/2, 1)$. A real-valued *Rosenblatt process* $R_H = (R_H(t), t \in \mathbb{R})$ is defined as follows

$$\begin{aligned} R_H(t) &= C_H^R \int_{\mathbb{R}^2} \left(\int_0^t h_2^H(u, y_1, y_2) du \right) dW(y_1) dW(y_2) \quad (7) \\ &= C_H^R \int_0^s u_{(\frac{H}{2}-\frac{1}{4})}(y_1) I_{s-}^{-(\frac{H}{2}-\frac{1}{4})} I_{t-}^{(\frac{H}{2}-\frac{1}{4})} u_{(\frac{H}{2}-\frac{1}{4})}(y_1) \\ &\times 1_{[s,t)}(y_1) u_{(\frac{H}{2}-\frac{1}{4})}(y_2) I_{s-}^{-(\frac{H}{2}-\frac{1}{4})} I_{t-}^{(\frac{H}{2}-\frac{1}{4})} u_{(\frac{H}{2}-\frac{1}{4})}(y_2) \\ &\times 1_{[s,t)}(y_2) du \; dW(y_1) dW(y_2) \end{aligned}$$

for $t \geq 0$ (and similarly for $t < 0$) where C_H^R is a constant such that $\mathbb{E}(R^H(1)^2) = 1$, I_{s-}, I_{t-} are fractional integrals (Samko, Kilbas, and Marichev [1992]) and the integral is a Wiener-Itô multiple integral (Itô [1951]) of order two with respect to the Wiener process (standard Brownian motion) W.

The normalizing constants C_H^B and C_H^R in the above two definitions are given explicitly as

$$C_H^B = \sqrt{\frac{H(2H-1)}{\mathrm{B}\left(2-2H, H-\frac{1}{2}\right)}}, \qquad C_H^R = \frac{\sqrt{2H(2H-1)}}{2\mathrm{B}\left(1-H, \frac{H}{2}\right)}$$

where B is the Beta function. For the subsequent Itô-type formula (change of variables), it is also convenient to define the following constants

$$c_H^B = C_H^B \, \Gamma\left(H - \frac{1}{2}\right), \qquad c_H^R = C_H^R \, \Gamma\left(\frac{H}{2}\right)^2,$$

and

$$c_H^{B,R} = \frac{c_R^H}{c_{\frac{H}{2}+\frac{1}{2}}^B} = \sqrt{\frac{(2H-1)}{(H+1)} \frac{\Gamma\left(1-\frac{H}{2}\right)\Gamma\left(\frac{H}{2}\right)}{\Gamma(1-H)}} \tag{8}$$

where Γ is the Gamma function.

A change of variables (Itô formula) is described that is important for the control solution and is verified in Čoupek, Duncan and Pasik-Duncan [2019]. The subsequent change of variables formula contains the following two differential operators,

$$\nabla^{\frac{H}{2}} = I_+^{\frac{H}{2}} D \tag{9}$$

$$\nabla^{\frac{H}{2},\frac{H}{2}} = I_{+,+}^{\frac{H}{2},\frac{H}{2}} D^2. \tag{10}$$

where D is the Malliavin derivative and

$$I_+^\alpha(f(x)) = \int_{-\infty}^{x} f(u)(x-u)^{\alpha-1} du \tag{11}$$

$$(I_{+,+}^{\alpha_1,\alpha_2} f)(x_1, x_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u,v) (x_1-u)^{\alpha_1-1}(x_2-v)^{\alpha_2-1} dudv \tag{12}$$

These two differential operators occur in the description of the Skorokhod integral for the Rosenblatt process from the notion of the forward integrals (Russo and Vallois [1993]). Specifically there is the following equality where the left side of the equality has the forward stochastic integral and right side has the Skorokhod (stochastic) integral and the integrals depending on the above two differential operators.

$$\begin{aligned} \int_0^t g_s \, \mathrm{d}^- R_s^H &= \int_0^t g_s \delta R_s^H \qquad (13) \\ &+ 2c_H^{B,R} \int_0^t (\nabla^{\frac{H}{2}} g_s)(s) \delta B_s^{\frac{H}{2}+\frac{1}{2}} \\ &+ \int_0^t (\nabla^{\frac{H}{2},\frac{H}{2}} g_s)(s,s) \, \mathrm{d}s \end{aligned}$$

A change of variables formula is given for a time-varying quadratic polynomial of the solution of a stochastic equation driven by a two dimensional Rosenblatt process.

Theorem. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a quadratic function and $g : \mathbb{R} \to \mathbb{R}$ is a smooth function and let $(X(t), t \geq 0)$ satisfy (1). Then stochastic process $(Y(t) = g(t)f(X(t)), t \geq 0)$ satisfies the following stochastic equation.

$$Y(t) = Y_0 + \int_0^t \tilde{\vartheta}_s \, ds + 2c_H^{B,R} \int_0^t \tilde{\varphi}_s \delta B_s^{\frac{H}{2}+\frac{1}{2}} \quad (14)$$
$$+ \int_0^t \tilde{\psi}_s \delta R_s^H$$

where

$$\tilde{\vartheta}_s = f\frac{\partial g}{\partial s}(s, y_s) + \frac{g\partial f}{\partial x}(s, y_s)\vartheta_s + c_H^R \frac{g\partial^2 f}{\partial x^2}(s, y_s)(\nabla^{\frac{H}{2},\frac{H}{2}} y_s)(s, s)$$
$$\tilde{\varphi}_s = \frac{g\partial^2 f}{\partial x^2}(s, y_s)(\nabla^{\frac{H}{2}} y_s)(s)$$
$$\tilde{\psi}_s = g\frac{\partial f}{\partial x}(s, y_s).$$

and $\vartheta_s = Ax + Cu$. The linear operators $\nabla^{\frac{H}{2},\frac{H}{2}}, \nabla^{\frac{H}{2}}$ are defined in (9), (10). The two stochastic integrals with respect to a fractional Brownian motion and a Rosenblatt process in the expression for Y are Skorokhod integrals so they have expectation zero. This change of variables (Itô formula) is verified in Čoupek, Duncan and Pasik-Duncan [2019]. For a general smooth function of X a third derivative term of X also appears.

3. OPTIMAL FEEDBACK CONTROL

The solution of the optimal feedback control for the ergodic control problem described by (1), (3) is given in the following theorem which is the main result in this paper.

Theorem. The stochastic control problem with the stochastic equation (1), the ergodic quadratic cost (3), and the family of controls (4) has an optimal feedback control, K^*, given by the minimum of the following expression which can be obtained by differentiation. The expression, $g(K)$, is strictly convex in K so the optimal K is determined by the unique zero of the derivative.

$$g(K) = \int_0^T |R^{-\frac{1}{2}}(RKX + C^T PX)|^2 dt \quad (15)$$
$$+ \tilde{C}_H \int_0^T \mathrm{e}^{(A+CK+A^T+K^T C^T)r} r^{2H-2} dr$$

Proof. The change of variables formula is applied to the real-valued process $(\langle P(t)X(t), X(t)\rangle, t \geq 0)$ where P satisfies a Riccati equation given subsequently. Initially a change of variablez formula for Rosenblatt processes is applied to $\langle PX, X\rangle$ using the result in Čoupek, Duncan and Pasik-Duncan [2019] described above.

$$\langle P(T)X(T), X(T)\rangle - \langle P(0)x_0, x_0\rangle \quad (16)$$
$$= \int_0^T [\langle P(A + CK + A^T + K^T C^T)X, X\rangle$$
$$+ 2c_H tr(\nabla^{\frac{H}{2},\frac{H}{2}} X_s(s, s))]ds$$
$$+ 2\int_0^T \langle \nabla^{\frac{H}{2}} X_s(s), dB_H\rangle$$
$$+ 2\int_0^T \langle X, dR_H\rangle$$
$$+ \int_0^T \langle \frac{dP}{dt} X(s), X(s)\rangle ds$$

Now take expectation of the equality (16) to obtain the following equality.

$$\mathbb{E}\langle P(T)X(T), X(T)\rangle - \langle P(0)x_0, x_0\rangle \quad (17)$$
$$= \mathbb{E}[\int_0^T [\langle P(A + CK + A^T + K^T C^T)X, X\rangle$$
$$+ 2c_H tr(\nabla^{\frac{H}{2},\frac{H}{2}} X_s(s, s))]ds$$
$$+ \int_0^T \langle \frac{dP}{dt} X(s), X(s)\rangle ds]$$

The two stochastic integrals with respect to B_H and R_H are Skorokhod integrals (Skorokhod [1975]) so they have expectation zero. It is necessary to compute $\nabla^{\frac{H}{2},\frac{H}{2}} X_t(u, u)$ Initially the process X for this differential operator is replaced by the Rosenblatt process R_H. Recall that the noise process in the stochastic equation (1) is dR_H so it is necessary to compute in the change of variables formula, $\nabla^{\frac{H}{2},\frac{H}{2}} R_H$ where R_H is a two-vector of independent real-valued Rosenblatt processes. This computation is the following

$$\nabla^{\frac{H}{2},\frac{H}{2}} R_{H,t}(u, u) = \tilde{C}_H \int_0^t |u - r|^{2H-2} dr \quad (18)$$

where the constant, $\tilde{C}_H$, is given by

$$\tilde{C}_H = 2c_H^R \frac{\mathrm{B}\left(\frac{H}{2}, 1 - H\right)^2}{\Gamma\left(\frac{H}{2}\right)^2}. \quad (19)$$

and B is the beta function. Note that the integral on the right hand side is a two-vector each element having the same integral because the two components of the Rosenblatt process are independent.

Let $\Xi_t(u) = \nabla^{\frac{H}{2},\frac{H}{2}} X_t(u, u)$. Then it follows from the solution of (1) that

$$\Xi_t(u) = \int_0^t [(A + CK) + (A^T + K^T C^T)]\Xi_s(u)ds \quad (20)$$
$$+ \nabla^{\frac{H}{2},\frac{H}{2}} R_{H.t}(u, u)$$

because the operator $\nabla^{\frac{H}{2},\frac{H}{2}} X_t(u, u)$ is symmetric. Solving this affine integral equation, it follows directly that

$$\Xi_t(u) = \tilde{C}_H \int_0^t \mathrm{e}^{(A+CK+A^T+K^T C^T)(t-r)} \quad (21)$$
$$\times |u - r|^{2H-2} dr$$

which by an elementary change of variables is

$$\Xi_s(s) = \nabla^{\frac{H}{2},\frac{H}{2}} X_s(s, s) \quad (22)$$
$$= \tilde{C}_H \int_0^s \mathrm{e}^{(A+CK+A^T+K^T C^T)r} r^{2H-2} dr$$

Note that the term $|u - r|^{2H-2}$ in (21) is a two vector which has this same scalar term in both elements. Since it is assumed that $C = I$ and $A = A^T$, it can be assumed that $K = K^T$. From the formula for the roots of a quadratic polynomial, the three terms that determine K occur explicitly in the roots. Let the roots be denoted c_1, c_2.

The Riccati equation that is used for some computations here is the one used for a Brownian motion noise so it is not intrinsic for a Rosenblatt noise but it suffices for some computations. Furthermore while some terms in the equation are diagonal the Riccati equation is not. It is the following equation.

$$\frac{dP}{dt} - -A^T P - PA + PC^T R^{-1} CP - Q \quad (23)$$

$$P(T) = 0 \quad (24)$$

Substituting the Riccati equation (23) in (17) and taking expectation, the following equation results.

$$\begin{aligned} &\mathbb{E}\langle P(T)X(T), X(T)\rangle + \mathbb{E}\int_0^T \langle QX, X\rangle dt \quad (25) \\ &+\mathbb{E}\int_0^T \langle RKX, KX\rangle dt \\ &= \mathbb{E}\langle P(0)x_0, x_0\rangle + \mathbb{E}\int_0^T (\langle RKX, KX\rangle dt \\ &+\langle P(CK + K^T C^T)X, X\rangle dt \\ &+ \int_0^T tr(\tilde{C}_H \int_0^t \mathrm{e}^{(A+CK+A^T+K^T C^T)r} r^{2H-2} dr))dt \\ &= \mathbb{E}[\langle P(0)x_0, x_0\rangle + \int_0^T |R^{-\frac{1}{2}}(RKX + C^T PX)|^2] dt \\ &+\tilde{C}_H \int_0^T tr(\int_0^t \mathrm{e}^{(A+CK+A^T+K^T C^T)r} r^{2H-2} dr) dt \end{aligned}$$

Initially consider a limit of the inner integral for the last term on the right hand side, that is,

$$\begin{aligned} &lim_{t\to\infty} \tilde{C}_H tr(\int_0^t \mathrm{e}^{(A+CK+A^T+K^T C^T)r} r^{2H-2} dr) \quad (26) \\ &= \frac{\Gamma(2H-1)}{\Sigma_{i=1}^2 c_i^{2H-1}} \end{aligned}$$

where (c_1, c_2) are the eigenvalues of the symmetric transformation $(A + CK + A^T + K^T C^T)$. Clearly averaging of this result as $\frac{1}{T}\int_o^T$ converges to the same value. Now divide the previous equality by T and let $T \to \infty$.

$$\begin{aligned} &lim_{T\to\infty} \frac{1}{T}\mathbb{E}J_\infty(K) = \frac{1}{T}\mathbb{E}\int_0^T \langle QX, X\rangle dt \quad (27) \\ &+\mathbb{E}\int_0^T \langle RKX, KX\rangle dt \\ &= lim_{T\to\infty} \int_0^T |R^{-1}(RKX + C^T PX)|^2 dt \\ &+ \frac{\Gamma(2H+1)}{\Sigma_{i=1}^2 c_i^{2H-1}} \end{aligned}$$

The minimization of the sum of the two integrals as $T \to \infty$ can be done with respect to the three distinct elements in K by computing the derivative with respect to these variables and likewise for the quadratic expression in the other integral term. The characteristic polynomial of $2A + 2K$ is a polynomial of degree two so it can be solved with explicit dependence on the elements of K that is k_1, k_2, k_3 where k_3 is the off diagonal term. It follows directly from the form of the two terms in (27) that both of them are strictly convex functions of the elements of K and is the sum so the minimum is determined by setting the derivative equal to zero.
End of proof.
Thus the optimal feedback, K, can be explicitly determined.

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4. CONCLUDING REMARKS

The result in this paper allows for the use of a Rosenblatt noise for an ergodic control problem so that the noise can better model the noise that is observed in control systems for physical systems. For some generalizations it is important to obtain optimal feedback control results for higher dimensional stochastic systems and to eliminate the symmetry conditions that are assumed here for A and thereby K and to allow a general linear transformation for C instead of I. Furthermore it is important to consider the case where the Rosenblatt noise components are correlated and likewise to study the corresponding multidimensional finite time horizon control problems with Rosenblatt noise. The finite time horizon linear-quadratic control problems for a Rosenblatt noise are more complicated because the term here that gave a Gamma function to determine an optimal feedback control is an integral over a finite interval for the finite time problem so this integral does not yield a Gamma function or another classical function for its integral. Given these optimal ergodic control results, it is natural to consider some adaptive control problems where some unknown parameters appear in the drift term of the system equation. Another important generalization is to consider some higher order Hermite processes. Rosenblatt processes are order two Hermite processes because they are defined by a double Wiener-Itô integral. However a stochastic calculus needs to be developed for third or higher order Hermite processes. Such a result would provide extra flexibility in the choice of a noise model for a controlled stochastic system. For the scalar ergodic control result for a Rosenblatt noise process in Čoupek, Duncan, Maslowski, and Pasik-Duncan [2019] it is important to understand the scalar Riccati equation given there whose solution determines the optimal feedback control and which has an explicit dependence on the parameter H that determines the optimal feedback control.

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