Distributed Fault Diagnosis for a Class of Time-Varying Systems over Sensor Networks with Stochastic Protocol

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Abstract: This paper is concerned with the distributed fault diagnosis problem for a class of time-varying systems over sensor networks with nonlinearity and uncertainty. For the purpose of solving the problem of data conflict, the stochastic protocol is used to determine which node has the right to send data to the estimator at a certain transmission time. The aim of this paper is to design a set of distributed estimators to detect, isolate and estimate fault signals. The upper bound of estimation error covariance is obtained by solving two recursive matrix equations and the upper bound can be minimized by designing appropriate estimator gain at each step. Finally, a numerical example is provided to show the effectiveness of the proposed design scheme.

Keywords: distributed fault diagnosis, Riccati-like difference equations, sensor networks, stochastic protocol

1. INTRODUCTION

Over the past decades, sensor networks have received much attention from researchers due to its wide applications in various fields such as military, industry, urban management and environment and so on, see, e.g. Ge et al. (2019), Zuo et al. (2018). As a special type of complex network, the sensor network is a distributed intelligent network system consisting of a set of sensor nodes with wireless communication and computing capabilities. The fault detection and fault estimation problems have received increasing research attention, see Chao et al. (2018), Li et al. (2015). In Dong et al. (2014), the problem of finite-horizon estimation was investigated for a class of nonlinear time-varying systems with randomly occurring faults. In Witczak et al. (2017), the problems of state and actuator fault estimation were studied for nonlinear discrete-time systems with unknown input decoupling based on neural network approach. By introducing spectral decomposition and coordinate transformation, an integrated design method was proposed in Liu and Yang (2019) for fault estimation and fault-tolerant control of linear multi-agent systems. It is worth mentioning that the problem of fault isolation is seldom addressed in the existing literature on fault diagnosis.

In the actual physical system, nonlinearity and model uncertainty are inevitable and have attracted wide research attention in the past decades Hu et al. (2013), Ding et al.

(2015). For example, the joint state and fault estimation problem was studied in Hu et al. (2018) for a class of discrete time-varying systems with uncertainty and nonlinearity. In Huo et al. (2017), the problems of non-fragile mixed H_∞ and passive asynchronous state estimation were dealt with for discrete-time Markov jump neural networks with randomly occurring uncertainties and sensor nonlinearity. In Dong et al. (2015), the finite-horizon reliable control problem was studied for time-varying systems with randomly occurring uncertainties, nonlinearities and measurement quantization by employing the approach of recursive linear matrix inequality. However, up to now, little research attention has been paid on the fault diagnosis problem for systems over sensor networks with uncertainty and nonlinearity, and still remains open.

It is worth noting that, distributed information transmission is very important in sensor networks because nodes are distributed in different locations in space. In the existing literature on sensor networks, it is assumed that each sensor node is allowed to simultaneously receive the information from all neighboring nodes. Unfortunately, this assumption reduces network performance in most practical projects due to the fact that multiple transmission at the same time would lead to inevitable data collisions in the case of limited bandwidth. In order to prevent data conflicts, it is necessary to use some communication protocol to arrange the signal transmission. Up to now, the communication protocols that have been widely used in practical systems include, but are not limited to, Round-Robin protocol, stochastic communication protocol, Try-once-discard protocol. Among them, the stochastic protocol is one of the most widely used protocols in industry and its main idea is to randomly select the transmitted sensor under a certain probability. In recent years, the

problems of analysis and synthesis for networked systems subject to communication protocols have attracted increasing research attention, see Gao et al. (2019), Liu et al. (2015). For example, the problem of neural network based output feedback control was investigated in Ding et al. (2019) for nonlinear systems under stochastic communication protocols. Unfortunately, the fault diagnosis for time-varying systems over sensor networks with stochastic protocol scheduling has not been fully addressed.

Based on the above discussion, in this paper, we aim to investigate the problem of fault diagnosis for a class of time-varying systems over sensor networks with nonlinearity and uncertainty under stochastic protocol. The main contributions of this paper as follows: (1) A comprehensive system that accounts for nonlinearity, uncertainty and stochastic protocol is considered; (2) A set of distributed estimators that can detect, isolate and estimate faults signals are designed, and estimators parameters can be obtained by minimizing upper bound.

Notation The notations used throughout the paper are standard expect otherwise noted. $\mathbb{R}^{n \times m}$ and $\mathbb{R}^n$ denote, respectively, the set of all $n \times m$ real matrix space and the n-dimensional Euclidean space. $X \geq 0$ represents that X is a positive semi-definite matrix. X^T represents the transpose of a matrix X and X^{-1} denotes the inverse of X. I denotes the unit matrix with appropriate dimensions. $X \geq Y$ means that $X - Y$ is positive semi-definite matrix. $\mathbb{E}\{x\}$ means the mathematical expectation of the stochastic variable x. $\text{tr}\{A\}$ represents the trace of the matrix A. $\text{diag}\{\cdots\}$ is a block diagonal matrix. $\|\cdot\|$ denotes the Euclidean norm of a vector. $\circ$ denotes the Hadamard product.

2. STATEMENT OF THE PROBLEM

In this paper, the sensor network with M sensor nodes is considered. The topology is represented by a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ where $\mathcal{N} = (1, 2, \cdots, M)$ denotes the set of nodes, $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of edges, and $\mathcal{A} = [a_{ij}]$ stands for the weighted adjacency matrix with non-negative element a_{ij}. An edge of $\mathcal{G}$ is denoted by the ordered pair (i, j). The adjacency elements associated with the edges of the graph are positive, i.e., $a_{ij} > 0 \Longleftrightarrow (i, j) \in \mathcal{E}$, which represents that sensor i can receive information from sensor j. The self edge (i, i) is allowed, i.e., $(i, i) \in \mathcal{E}$ and $a_{ii} = 1$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j : (i, j) \in \mathcal{E}, j \neq i\}$.

Consider the following time-varying uncertain nonlinear system

$$x_{k+1} = (A_k + \Delta A_k)x_k + h(x_k) + B_k w_k, \qquad (1)$$

over a sensor network with M sensor nodes, and the measurement output of i-th node is given by

$$y_{i,k} = C_{i,k} x_k + E_{i,k} v_{i,k} + F_{i,k} f_{i,k}, \ i = 1, 2, \cdots, M \quad (2)$$

where $x_k \in \mathbb{R}^{n_x}$, $y_{i,k} \in \mathbb{R}^{n_y}$ respectively represent the state vector, the measurement output of the i-th node. $f_{i,k} \in \mathbb{R}^{n_f}$ is the additive fault of the i-th node. $\omega_k \in \mathbb{R}^{n_v}$ is a Gaussian white process noise with $\mathbb{E}\{\omega_k\} = 0$ and $\mathbb{E}\{\omega_k \omega_k^T\} = R_k$. $v_{i,k} \in \mathbb{R}^{n_v}$ is a Gaussian white measurement noise with $\mathbb{E}\{v_{i,k}\} = 0$ and $\mathbb{E}\{v_{i,k} v_{i,k}^T\} = S_{i,k}$. Moreover, ω_k and $v_{i,k}$ are assumed to be mutually independent. $A_k, B_k, C_{i,k}, E_{i,k}$ and $F_{i,k}$ are known, real,

time-varying matrices with appropriate dimensions. The uncertain time-varying matrix ΔA_k satisfies

$$\Delta A_k = H_k Q_k J_k, \qquad Q_k^T Q_k \leq I, \qquad (3)$$

where H_k, J_k are known real time-varying matrices, and Q_k is an unknown real time-varying matrix. The nonlinear function $h(x_k)$ satisfies

$$\| h(x) - h(y) \| \leq \| G(x - y) \|, \qquad (4)$$

where G is a known matrix.

Assumption 1. In system (1)-(2), only one node occurs fault at a certain time. Moreover, it is assumed that $f_{i,k+1} = f_{i,k} + \Delta f_{i,k}$, and $\Delta f_{i,k}$ is a constant amplitude.

Define an augmented state

$$\bar{x}_{i,k}^{(m)} = \begin{bmatrix} x_k \\ L_i^{(m)} f_{i,k} \\ L_i^{(m)} \Delta f_{i,k} \end{bmatrix} \qquad (5)$$

where

$$L_i^{(m)} = \begin{cases} I, & i = m \\ 0, & i \neq m \end{cases}$$

Then, system (1)-(2) can be rewritten as:

$$\begin{cases} \bar{x}_{i,k+1}^{(m)} = (\bar{A}_k + \bar{\Delta} A_k) \bar{x}_{i,k}^{(m)} + \bar{h}(\bar{x}_{i,k}^{(m)}) + \bar{B}_k w_k, \\ y_{i,k}^{(m)} = \bar{C}_{i,k} \bar{x}_{i,k}^{(m)} + E_{i,k} v_{i,k}, \\ \qquad i = 1, 2, \cdots, M, \ m = 0, 1, \cdots, M \end{cases} \qquad (6)$$

where

$$\bar{A}_k = \begin{bmatrix} A_k & 0 & 0 \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix}, \ \bar{B}_k = \begin{bmatrix} B_k \\ 0 \\ 0 \end{bmatrix}, \ \bar{H}_k = \begin{bmatrix} H_k \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{h}(\bar{x}_{i,k}^{(m)}) = \begin{bmatrix} h(x_k) \\ 0 \\ 0 \end{bmatrix}, \ \bar{\Delta} A_k = \begin{bmatrix} \Delta A_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\bar{C}_{i,k} = [C_{i,k} \ F_{i,k} \ 0], \ \bar{J}_k = [J_k \ 0 \ 0].$$

To avoid data collisions, the stochastic protocol is utilized to select the neighboring node which obtains permission to send information to the node i. A set of adjacent nodes except the node itself of the i-th node is denoted by $\mathcal{N}_i = \{j : (i, j) \in \mathcal{E}, i \neq j\}$. Let node $\rho_{i,k} \in \mathcal{N}_i$ denote the node which obtains the right to communicate to the i-th node at the k-th transmission step. In the stochastic protocol, the probability distribution of $\rho_{i,k}$ is

$$\text{Prob}\{\rho_{i,k} = j\} = p_{ij,k}, \qquad (7)$$

where $p_{ij,k} > 0 (j \in \mathcal{N}_i, \ i = 1, 2, \cdots, M)$ is the occurrence probability that node j is selected to send measurement data to node i. Suppose that $p_{ij,k} = 0 \ (\forall j \notin \mathcal{N}_i)$ and $\sum_{j \in \mathcal{N}_i} p_{ij,k} = 1 (\ i = 1, 2, \cdots, M)$.

Then, the distributed estimator subject to the scheduling of stochastic protocol is designed as follows:

$$\hat{x}_{i,k+1|k}^{(m)} = \bar{A}_k \hat{x}_{i,k|k}^{(m)} + \bar{h}(\hat{x}_{i,k|k}^{(m)}), \qquad (8)$$

$$\hat{x}_{i,k+1|k+1}^{(m)} = \hat{x}_{i,k+1|k}^{(m)} + \sum_{j \in \mathcal{N}_i} a_{ij} K_{ij,k+1}^{(m)} \Big(\delta(\rho_{i,k+1} - j)$$

$$\times y_{i,k+1}^{(m)} - p_{ij,k+1} \bar{C}_{j,k+1} \hat{x}_{j,k+1|k}^{(m)} \Big)$$

$$+ a_{ii} K_{ii,k+1}^{(m)} \Big(y_{i,k+1}^{(m)} - \bar{C}_{i,k+1} \hat{x}_{i,k+1|k}^{(m)} \Big),$$

$$m = 0, 1, \cdots, M, \ i = 1, 2, \cdots, M, \qquad (9)$$

where $\hat{x}_{i,k|k}^{(m)}$ is the estimate of the state $\bar{x}_{i,k}^{(m)}$ with $\hat{x}_{i,0}^{(m)} = \mathbb{E}\{[x_0\ 0\ 0]^T\}$, $\hat{x}_{i,k+1|k}^{(m)}$ is the one-step estimate of the state $\bar{x}_{i,k}^{(m)}$, $K_{ij,k}^{(m)}$ is the estimator parameter to be determined and $\delta(\cdot) \in \{0,1\}$ is the Kronecker delta function.

Define one-step prediction error of the i-th node as $e_{i,k+1|k}^{(m)} = \bar{x}_{i,k+1}^{(m)} - \hat{x}_{i,k+1|k}^{(m)}$, and the estimation error of the i-th node as $e_{i,k+1|k+1}^{(m)} = \bar{x}_{i,k+1}^{(m)} - \hat{x}_{i,k+1|k+1}^{(m)}$. Then, subtracting the first equation in (9) from (6), we have

$$e_{i,k+1|k}^{(m)} = \bar{A}_k e_{i,k|k}^{(m)} + \left[\bar{h}(\bar{x}_{i,k}^{(m)}) - \bar{h}(\hat{x}_{i,k|k}^{(m)})\right]$$
$$+ \Delta \bar{A}_k \bar{x}_{i,k}^{(m)} + \bar{B}_k w_k. \qquad (10)$$

Similarly, the estimation error can be derived as follows

$$e_{i,k+1|k+1}^{(m)} = e_{i,k+1|k}^{(m)} - \sum_{j \in \mathcal{N}_i} a_{ij} K_{ij,k+1}^{(m)} \Big(\delta(\rho_{i,k+1} - j)$$
$$\times y_{j,k+1}^{(m)} - p_{ij,k+1}\bar{C}_{j,k+1}\hat{x}_{j,k+1|k}^{(m)}\Big)$$
$$- a_{ii} K_{ii,k+1}^{(m)} \Big(y_{i,k+1}^{(m)} - \bar{C}_{i,k+1}\hat{x}_{i,k+1|k}^{(m)}\Big). \quad (11)$$

For simplicity, setting

$$\bar{x}_k^{(m)} = \text{col}_M\left\{\bar{x}_{i,k}^{(m)}\right\}, \ \hat{x}_{k|k}^{(m)} = \text{col}_M\left\{\hat{x}_{i,k|k}^{(m)}\right\},$$
$$\hat{x}_{k+1|k}^{(m)} = \text{col}_M\left\{\hat{x}_{i,k+1|k}^{(m)}\right\}, \ e_{k+1|k}^{(m)} = \text{col}_M\left\{e_{i,k+1|k}^{(m)}\right\},$$
$$e_{k|k}^{(m)} = \text{col}_M\left\{e_{i,k|k}^{(m)}\right\}, \ \hat{h}_{k|k}^{(m)} = \text{col}_M\left\{\bar{h}(\hat{x}_{i,k|k}^{(m)})\right\},$$
$$y_k^{(m)} = \text{col}_M\left\{y_{i,k}^{(m)}\right\}, \ \bar{h}_k^{(m)} = \text{col}_M\left\{\bar{h}(\bar{x}_{i,k}^{(m)})\right\},$$
$$\tilde{w}_k = \text{col}_M\{w_k\}, \ v_k = \text{col}_M\{v_{i,k}\}, \ \tilde{A}_k = \text{diag}_M\{\bar{A}_k\},$$
$$\Delta \tilde{A}_k = \text{diag}_M\{\Delta \bar{A}_k\}, \ \tilde{B}_k = \text{diag}_M\{\bar{B}_k\},$$
$$\bar{C}_k = \text{diag}_M\{\bar{C}_{i,k}\}, \quad \bar{E}_k = \text{diag}_M\{E_{i,k}\}.$$

Then, system (6) can be written as

$$\begin{cases} \bar{x}_{k+1}^{(m)} = \left(\tilde{A}_k + \tilde{\Delta}A_k\right)\bar{x}_k^{(m)} + \bar{h}_k^{(m)} + \tilde{B}_k w_k, \\ y_k^{(m)} = \bar{C}_k \bar{x}_k^{(m)} + \bar{E}_k v_k, \ m = 0, 1, \cdots, M. \end{cases} \quad (12)$$

The m-th one-step prediction error $e_{k+1|k}^{(m)}$ is

$$e_{k+1|k}^{(m)} = \tilde{A}_k e_{k|k}^{(m)} + \tilde{h}_k^{(m)} + \Delta \tilde{A}_k \tilde{x}_k^{(m)} + \tilde{B}_k \tilde{w}_k \qquad (13)$$

where $\tilde{h}_k^{(m)} = \bar{h}_k^{(m)} - \hat{h}_{k|k}^{(m)}$ and the estimation error as follows

$$e_{k+1|k+1}^{(m)} = \Psi_k^{(m)} e_{k+1|k}^{(m)} - \sum_{i=1}^M \kappa_i K_{k+1}^{(m)} \Phi_{i,k+1}\bar{E}_{k+1}\bar{v}_{k+1}$$
$$- \sum_{i=1}^M \kappa_i K_{k+1}^{(m)} \tilde{\Phi}_{i,k+1}\bar{C}_{k+1}\tilde{x}_{k+1}^{(m)}, \qquad (14)$$

where

$$\Psi_k^{(m)} = I - \sum_{i=1}^M \kappa_i K_{k+1}^{(m)} \bar{\Phi}_{i,k+1}\bar{C}_{k+1},$$

$$\kappa_i = \text{diag}\left\{\underbrace{0, \cdots, 0}_{i-1}, I, \underbrace{0, \cdots, 0}_{M-i}\right\}, \ K_k^{(m)} = \left\{K_{ij,k}^{(m)}\right\}_{M \times M}$$

$$\bar{\Phi}_{i,k} = \text{diag}\{\bar{p}_{i1,k}, \cdots \bar{p}_{iM,k}\}, \ \tilde{\Phi}_{i,k+1} = \Phi_{i,k+1} - \bar{\Phi}_{i,k+1},$$
$$\Phi_{i,k} = \text{diag}\{a_{i1}\sigma_{i1,k}, \cdots, a_{iM}\sigma_{iM,k}\},$$

$$\bar{p}_{ij}(k) = \begin{cases} 1, & j = i \\ p_{ij,k}, & j \in \mathcal{N}_i \\ 0, & j \notin \mathcal{N}_i \end{cases} \quad \sigma_{ij,k} = \begin{cases} 1, & j = i \\ \delta(\rho_{i,k} - j), & j \in \mathcal{N}_i \\ 0, & j \notin \mathcal{N}_i. \end{cases}$$

Let $P_{k+1|k}^{(m)} = \mathbb{E}\{e_{k+1|k}^{(m)}(e_{k+1|k}^{(m)})^T\}$ and $P_{k+1|k+1}^{(m)} = \mathbb{E}\{e_{k+1|k+1}^{(m)}(e_{k+1|k+1}^{(m)})^T\}$ be the one-step prediction error covariance and estimation error covariance, respectively. The aim of this paper is to design a set of estimators for the system (1)-(2) such that the upper bound of the estimation error covariance is guaranteed, that is, there exists a sequence of positive-definite matrices $Z_{k+1|k+1}^{(m)}$ satisfying

$$\mathbb{E}\{e_{k+1|k+1}^{(m)}(e_{k+1|k+1}^{(m)})^T\} \le Z_{k+1|k+1}^{(m)}. \qquad (15)$$

Moreover, the upper bound can be minimized by designing appropriate estimator gains a. Finally, the residual evaluation function and fault isolation logic are proposed to detect, isolate and estimate the occurred sensor fault.

The following lemmas are needed in the subsequent sections.

Lemma 1. For given two vectors $x, y \in \mathbb{R}^n$, the following inequality holds:
$$xy^T + yx^T \le \varepsilon xx^T + \varepsilon^{-1} yy^T$$
where $\varepsilon > 0$ is any scalar.

Lemma 2. For given matrices M, N, L and Y with the appropriate dimensions, the following equality holds:
$$\frac{\partial \text{tr}\{MYN\}}{\partial Y} = M^T N^T, \qquad \frac{\partial \text{tr}\{MY^T N\}}{\partial Y} = NM,$$
$$\frac{\partial \text{tr}\{MYNY^T L\}}{\partial Y} = M^T N^T Y L^T + LMYN.$$

Lemma 3. Letting $A = [a_{ij}]_{n \times n}$ be a real-valued matrix and $B = \text{diag}\{b_1, b_2, \cdots, b_n\}$ be a diagonal stochastic matrix, then

$$\mathbb{E}\{BAB^T\} = \begin{bmatrix} \mathbb{E}\{b_1^2\} & \mathbb{E}\{b_1 b_2\} & \cdots & \mathbb{E}\{b_1 b_n\} \\ \mathbb{E}\{b_2 b_1\} & \mathbb{E}\{b_2^2\} & \cdots & \mathbb{E}\{b_2 b_n\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}\{b_n b_1\} & \mathbb{E}\{b_n b_2\} & \cdots & \mathbb{E}\{b_n^2\} \end{bmatrix} \circ A.$$

where $\circ$ is the Hadamard product.

3. MAIN RESULTS

3.1 Design of the estimator gain

In this subsection, the one-step prediction error covariance and estimation error covariance are obtained based on (13) and (14). Subsequently, their upper bound are computed and minimized by selecting suitable gain matrices.

Lemma 4. For given positive scalars λ_1, λ_2 and λ_3, the state covariance matrix $\Omega_{k+1}^{(m)} = \mathbb{E}\{\bar{x}_{k+1}^{(m)}(\bar{x}_{k+1}^{(m)})^T\}$ satisfies

$$\Omega_{k+1}^{(m)} \le (1 + \lambda_1 + \lambda_2)\tilde{A}_k \Omega_k^{(m)} \tilde{A}_k^T + \tilde{B}_k \tilde{R}_k \tilde{B}_k^T$$
$$+ (1 + \lambda_1^{-1} + \lambda_3)\text{tr}\{\tilde{J}_k \Omega_k^{(m)} \tilde{J}_k\} \tilde{H}_k \tilde{H}_k^T$$
$$+ (1 + \lambda_2^{-1} + \lambda_3^{-1})\bar{G}\Omega_k^{(m)}\bar{G}^T \qquad (16)$$

where
$$\bar{G} = \text{diag}_M\{\text{diag}\{G, 0\}\}, \ \tilde{R}_k = \text{diag}_M\{R_k\},$$
$$\tilde{J}_k = \text{diag}_M\{\bar{J}_k\}, \ \tilde{H}_k = \text{diag}_M\{\bar{H}_k\}.$$

Based on Lemma 4, the upper bound of the one-step prediction error covariance and estimation error covariance are given in the following theorem.

Theorem 1. Let $\lambda_4, \lambda_5, \lambda_6$ and λ_7 be positive scalars. For the system (1)-(2), if the following difference equations have solution under the initial condition $P_{0|0}^{(m)} \leq Z_{0|0}^{(m)}$:

$$Z_{k+1|k}^{(m)} = (1 + \lambda_4 + \lambda_5)\tilde{A}_k Z_{k|k}^{(m)} \tilde{A}_k^T + \tilde{B}_k \tilde{R}_k \tilde{B}_k^T$$
$$+ (1 + \lambda_4^{-1} + \lambda_6)\text{tr}\left\{\tilde{J}_k \Omega_k^{(m)} \tilde{J}_k\right\} \tilde{H}_k \tilde{H}_k^T$$
$$+ (1 + \lambda_5^{-1} + \lambda_6^{-1})\bar{G} Z_{k|k}^{(m)} \bar{G}^T, \qquad (17)$$

$$Z_{k+1|k+1}^{(m)} = (1 + \lambda_7)\Psi_k^{(m)} Z_{k+1|k}^{(m)}(\Psi_k^{(m)})^T$$
$$+ \sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)} \Pi_{1,i,k+1}\left(K_{k+1}^{(m)}\right)^T \kappa_i^T, \qquad (18)$$

where

$$\Pi_{1,i,k+1} = (1 + \lambda_7^{-1})\check{\Phi}_{i,k+1} \circ \left(\bar{C}_{k+1}\Omega_{k+1}^{(m)} \bar{C}_{k+1}^T\right)$$
$$+ \bar{\Phi}_{i,k+1} \circ \left(\bar{E}_{k+1} S_{k+1} \bar{E}_{k+1}^T\right),$$
$$\check{\Phi}_{i,k} = \mathbb{E}\left\{\tilde{\Phi}_{i,k}\left(\tilde{\Phi}_{i,k}\right)^T\right\}$$

then the estimation error covariance satisfies

$$P_{k+1|k}^{(m)} \leq Z_{k+1|k}^{(m)}, \ P_{k+1|k+1}^{(m)} \leq Z_{k+1|k+1}^{(m)}. \qquad (19)$$

Moreover, the upper bound of the error covariance matrix is minimized by selecting the appropriate gain

$$K_{ij,k+1}^{(m)} = \begin{cases} 0, & a_{ij}\sigma_{ij,k+1} = 0 \\ \left[N_{i,k+1}^{(m)}\left(M_{i,k+1}^{(m)}\right)^{\dagger}\right]^{\natural}, & a_{ij}\sigma_{ij,k+1} \neq 0 \end{cases} \qquad (20)$$

where

$$N_{i,k+1}^{(m)} = (1 + \lambda_7)\kappa_i Z_{k+1|k}^{(m)} \bar{C}_{k+1}^T \bar{\Phi}_{i,k+1}^T,$$
$$M_{i,k+1}^{(m)} = \Pi_{1,i,k+1} + \Pi_{2,i,k+1},$$
$$\Pi_{2,i,k+1} = (1 + \lambda_7)\bar{\Phi}_{i,k+1} \bar{C}_{k+1} Z_{k+1|k}^{(m)} \bar{C}_{k+1}^T \bar{\Phi}_{i,k+1}^T$$

and $[\mathcal{A}]^{\natural}$ extracts the submatrix from matrix $\mathcal{A}$ associated with $K_{ij,k+1}^{(m)}$.

Proof. Mathematical induction is used to prove this theorem. According to initial condition $P_{0|0}^{(m)} \leq Z_{0|0}^{(m)}$, assuming $P_{k|k}^{(m)} \leq Z_{k|k}^{(m)}$, then we should prove that $P_{k+1|k+1}^{(m)} \leq Z_{k+1|k+1}^{(m)}$.

Firstly, according to the definition of the one-step prediction error covariance, we have

$$P_{k+1|k}^{(m)} = \tilde{A}_k P_{k|k}^{(m)} \tilde{A}_k^T + \tilde{A}_k \mathbb{E}\left\{e_{k|k}^{(m)}\left(\bar{x}_k^{(m)}\right)^T\right\}\Delta\tilde{A}_k^T$$
$$+ \mathbb{E}\left\{\tilde{h}_k^{(m)}\left(\tilde{h}_k^{(m)}\right)^T\right\} + \tilde{A}_k \mathbb{E}\left\{e_{k|k}^{(m)}\left(\tilde{h}_k^{(m)}\right)^T\right\}$$
$$+ \mathbb{E}\left\{\tilde{h}_k^{(m)}\left(e_{k|k}^{(m)}\right)^T\right\}\tilde{A}_k^T + \Delta\tilde{A}_k \mathbb{E}\left\{\bar{x}_k^{(m)}\left(\tilde{h}_k^{(m)}\right)^T\right\}$$
$$+ \mathbb{E}\left\{\left(\tilde{h}_k^{(m)}\right)^T\left(\bar{x}_k^{(m)}\right)^T\right\}\Delta\tilde{A}_k^T + \Delta\tilde{A}_k \Omega_k^{(m)} \Delta\tilde{A}_k^T$$
$$+ \Delta\tilde{A}_k \mathbb{E}\left\{\bar{x}_k^{(m)}\left(e_{k|k}^{(m)}\right)^T\right\}\tilde{A}_k^T + \tilde{B}_k \tilde{R}_k \tilde{B}_k^T. \qquad (21)$$

Considering Assumption 1 and Lemma 1, we obtain

$$P_{k+1|k}^{(m)} \leq (1 + \lambda_4 + \lambda_5)\tilde{A}_k P_{k|k}^{(m)} \tilde{A}_k^T + \tilde{B}_k \tilde{R}_k \tilde{B}_k^T$$
$$+ (1 + \lambda_4^{-1} + \lambda_6)\text{tr}\left\{\tilde{J}_k \Omega_k^{(m)} \tilde{J}_k\right\} \tilde{H}_k \tilde{H}_k^T$$
$$+ (1 + \lambda_5^{-1} + \lambda_6^{-1})\bar{G} P_{k|k}^{(m)} \bar{G}^T \qquad (22)$$

where λ_4, λ_5 and λ_6 are positive scalars. Since $P_{k|k}^{(m)} \leq Z_{k|k}^{(m)}$, it is obvious that $P_{k+1|k}^{(m)} \leq Z_{k+1|k}^{(m)}$ from (22).

On the other hand, $P_{k+1|k+1}^{(m)}$ can be calculated as follows

$$P_{k+1|k+1}^{(m)} = \Psi_k^{(m)} P_{k+1|k}^{(m)}(\Psi_k^{(m)})^T + \mathbb{E}\left\{\sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)} \Phi_{i,k+1}\right.$$
$$\times \bar{E}_{k+1} S_{k+1} \bar{E}_{k+1}^T \Phi_{i,k+1}^T\left(K_{k+1}^{(m)}\right)^T \kappa_i^T\right\}$$
$$+ \mathbb{E}\left\{\sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)} \tilde{\Phi}_{i,k+1} \bar{C}_{k+1}\Omega_{k+1}^{(m)} \bar{C}_{k+1}^T \tilde{\Phi}_{i,k+1}^T\right.$$
$$\times \left(K_{k+1}^{(m)}\right)^T \kappa_i^T\right\} - \mathbb{E}\left\{\Psi_k^{(m)})e_{k+1|k}^{(m)}(\bar{x}_{k+1}^{(m)})^T\right.$$
$$\times \left(\sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)} \tilde{\Phi}_{i,k+1} \bar{C}_{k+1}\bar{x}_{k+1}^{(m)}\right)^T\right\}$$
$$- \mathbb{E}\left\{\sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)} \tilde{\Phi}_{i,k+1} \bar{C}_{k+1}\bar{x}_{k+1}^{(m)}\right.$$
$$\times \left(e_{k+1|k}^{(m)}\right)^T(\Psi_k^{(m)})^T\right\}. \qquad (23)$$

From Lemma 1, there exists a positive scalar λ_7 such that

$$P_{k+1|k+1}^{(m)} \leq (1 + \lambda_7^{-1})\mathbb{E}\left\{\sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)} \tilde{\Phi}_{i,k+1} \bar{C}_{k+1}\Omega_{k+1}^{(m)} \bar{C}_{k+1}^T\right.$$
$$\times \tilde{\Phi}_{i,k+1}^T\left(K_{k+1}^{(m)}\right)^T \kappa_i^T\right\} + \mathbb{E}\left\{\sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)} \Phi_{i,k+1}\right.$$
$$\times \bar{E}_{k+1} S_{k+1} \bar{E}_{k+1}^T \Phi_{i,k+1}^T\left(K_{k+1}^{(m)}\right)^T \kappa_i^T\right\}$$
$$+ (1 + \lambda_7)\Psi_k^{(m)} P_{k+1|k}^{(m)}(\Psi_k^{(m)})^T. \qquad (24)$$

Furthermore, according to Lemma 3

$$P_{k+1|k+1}^{(m)} \leq (1 + \lambda_7)\Psi_k^{(m)} P_{k+1|k}^{(m)}(\Psi_k^{(m)})^T + (1 + \lambda_7^{-1})$$
$$\times \sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)}\left(\check{\Phi}_{i,k+1} \circ \left(\bar{C}_{k+1}\Omega_{k+1}^{(m)} \bar{C}_{k+1}^T\right)\right)$$
$$\times \left(K_{k+1}^{(m)}\right)^T \kappa_i^T + \sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)}\left(\bar{\Phi}_{i,k+1} \circ \left(\bar{E}_{k+1}\right.\right.$$
$$\times S_{k+1}\bar{E}_{k+1}^T\right)\right)\left(K_{k+1}^{(m)}\right)^T \kappa_i^T. \qquad (25)$$

Therefore, from (22) and (25), it is not difficult to verify that $P_{k+1|k+1}^{(m)} \leq Z_{k+1|k+1}^{(m)}$.

Next, let us prove that the estimated gain in (20) can minimize the trace of the upper bound. Noticing (18), it can be rewritten in the following form:

$$Z_{k+1|k+1}^{(m)} = (1 + \lambda_7)Z_{k+1|k}^{(m)} - (1 + \lambda_7)\left(I - \Psi_k^{(m)}\right)Z_{k+1|k}^{(m)}$$
$$- (1 + \lambda_7)Z_{k+1|k}^{(m)}\left(I - \Psi_k^{(m)}\right)^T + \sum_{i=1}^{M} \kappa_i K_{k+1}^{(m)}$$
$$\times \left(\Pi_{1,i,k+1} + \Pi_{2,i,k+1}\right)\left(K_{k+1}^{(m)}\right)^T \kappa_i^T. \qquad (26)$$

According to Lemma 2, taking the partial derivative of the trace of (26) with respect to $K_{k+1}^{(m)}$ and letting the derivative be zero yields

$$\frac{\partial \mathrm{tr}\left\{Z_{k+1|k+1}^{(m)}\right\}}{\partial K_{k+1}^{(m)}} = 0 - 2(1+\lambda_7)\sum_{i=1}^{M}\kappa_i Z_{k+1|k}^{(m)}\bar{C}_{k+1}^T\bar{\Phi}_{i,k+1}^T$$

$$+ 2\sum_{i=1}^{M}\kappa_i K_{k+1}^{(m)}\left(\Pi_{1,i,k+1}+\Pi_{2,i,k+1}\right)$$

$$= 0 \tag{27}$$

which implies

$$K_{i,k+1}^{(m)}M_{i,k+1}^{(m)} = N_{i,k+1}^{(m)} \tag{28}$$

where $K_{i,k+1}^{(m)} = \left[K_{i1,k+1}^{(m)},\cdots,K_{iM,k+1}^{(m)}\right]$ is the i-th row of $K_{k+1}^{(m)}$. Since $M_{i,k+1}^{(m)}$ is not invertible, the gain $K_{ij,k+1}^{(m)}$ can be calculated by (20), and the proof of this theorem is complete.

3.2 Fault diagnosis strategy

In this subsection, we will propose a scheme to detect, isolate and estimate sensor faults for the system (1)-(2). Based on the designed estimator, the m-th residual signal is defined as $r_k^{(m)} = y_k - \bar{C}_k\hat{x}_{k|k}^{(m)}(m = 0,1,\cdots,M)$. The residual evaluation function is as follows

$$D_k = \left\{\frac{1}{L}\sum_{s=k-L+1}^{k}\left(r_s^{(0)}\right)^T r_s^{(0)}\right\}^{1/2}, \tag{29}$$

where k is current sample time and D_k represents the average residual in L steps.

Let

$$D_{\mathrm{th}} = \sup_{w_k\neq 0,v_{i,k}\neq 0,f_{i,k}=0} D_k \tag{30}$$

be the threshold, then the detection logic is designed as

$$\begin{cases} D_k \leq D_{\mathrm{th}}, & \text{no alarm,} \quad \text{fault free} \\ D_k > D_{\mathrm{th}}, & \text{alarm,} \quad \text{fault occuring.} \end{cases}$$

In order to isolate the occurred fault, the following cumulative value of the residual signal is defined

$$D_{\mathrm{iso},k}^{(m)} = \left\{\sum_{s=1}^{k}\left(r_s^{(m)}\right)^T r_s^{(m)}\right\}^{1/2}, \quad m = 1,\cdots,M. \tag{31}$$

By using the similar approach in Gao et al. (2019), the node where the fault occurs can be isolated by the following logic:

$$m = \arg\min_{1\leq m\leq M} D_{\mathrm{iso},k}^{(m)}.$$

Furthermore, the estimation of the fault can be derived from the output of the isolated augmented estimator.

4. SIMULATION RESULTS

In this section, we provide a numerical example to illustrate the validity of the proposed method. The topological structure of the sensor network shown in Fig. 1 can be represented by a directed graph $\mathcal{G} = (\mathcal{N},\mathcal{E},\mathcal{A})$. Consider
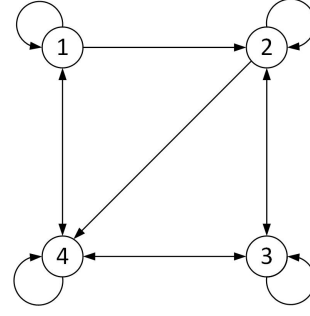

Fig. 1. Topological structure of the sensor network.

Table 1. The probability distribution of $\rho_{i,k}$

$p_{ij,k}$	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	0	0	0	1
$i=2$	0.4	0	0.6	0
$i=3$	0	0.7	0	0.3
$i=4$	0.2	0.4	0.4	0

the time-varying system (1)-(2) with four sensor nodes and the following parameters:

$$A_k = \begin{bmatrix} -0.44+\sin(0.12\pi k) & -0.05 \\ 0.3 & -0.35 \end{bmatrix}, \quad B_k = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$

$C_{1,k} = [0.82\ 0.82]$, $C_{2,k} = [0.65\ 0.6]$, $C_{3,k} = [0.34\ 0.45]$,

$C_{4,k} = [0.1\ 0.1]$, $H_k = [0.1\ 0.15]^T$, $J_k = [0.5\ 0.1]$,

$Q_k = \sin(0.5k)$, $E_{i,k} = 0.1$, $F_{i,k} = 1$, $i = 1,2,3,4$

and the time-varying nonlinear functions is chosen as

$$h(x_k) = \begin{bmatrix} 0.2\sin(x_k^1) \\ 0.1\sin(x_k^2) \end{bmatrix}$$

where $x_k = [x_k^1\ x_k^2]^T$ is the state vector, and it is easy to verify that $G = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$. w_k and $v_{i,k}$ are the zero-mean Gaussian white noises with covariances $R_k = 10^{-2}I$ and $S_{i,k} = 10^{-2}I(i = 1,2,3,4)$. The initial value of the system is chosen as $\bar{x}_0 = [1,1.2,0,0]^T$. In this example, the other parameters are chosen as $L = 10, \lambda_1 = 0.25, \lambda_2 = 0.3, \lambda_3 = 0.2, \lambda_4 = \lambda_5 = \lambda_6 = 0.4, \lambda_7 = 0.2$. The probability distribution of stochastic variables $\rho_{i,k}(i = 1,2,3,4)$ is given in Table 1.

Assume that a constant fault occurs in node 2, i.e.,

$$f_{2,k} = \begin{cases} 0, & \text{otherwise} \\ 0.1, & k \geq 100. \end{cases}$$

The simulation results are shown in Figs. 2-4. From Figs. 2 and 3, it can be found that the fault is detected, and it occurs in the 2-nd node. The faults and their estimations are shown in Figs. 4.

5. CONCLUSIONS

In this paper, the distributed fault diagnosis problem has been investigated for a class of time-varying systems over sensor networks with stochastic protocol. A set of distributed estimators has been presented to detect, isolate and estimate faults signals. By solving two recursive matrix equations, the upper bound of estimation error covariance has been obtained and it has been minimized by designing appropriate estimator gain. Finally, a numerical
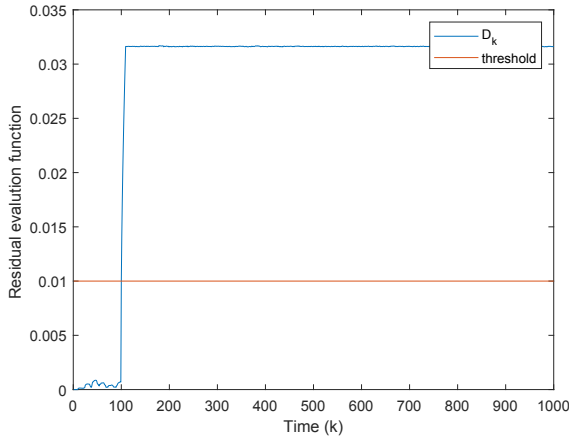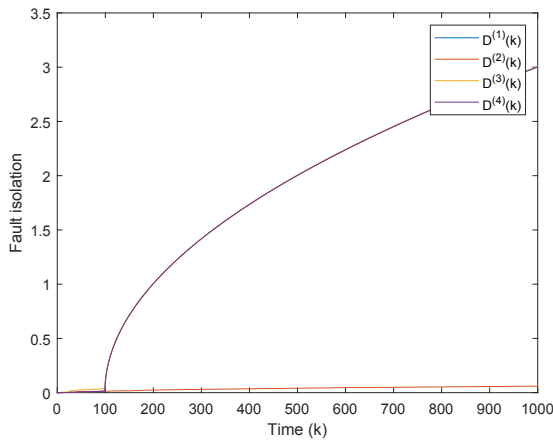
Fig. 2. Fault detection.
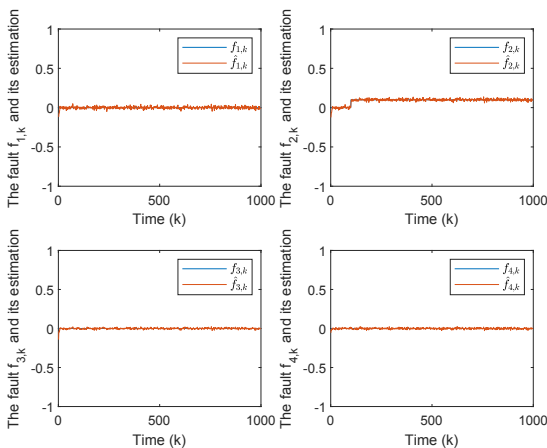

Fig. 3. Fault isolation.

Fig. 4. The actual faults f_k and their estimations $\hat{f}_k$.

example has been provided to illustrate the effectiveness of the proposed design scheme.

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