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Discrete-Time Control Design Based on Symplectic Integration: Linear Systems ⋆

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Abstract: We propose a novel approach to design discrete-time state feedback controllers for sampled control systems with guaranteed stability under arbitrary sampling times that fulfill the Nyquist-Shannon condition. The key idea is borrowed from symplectic integration and backward error analysis of Hamiltonian systems: The closed-loop target system is the discretized version of a continuous-time system with appropriately shaped Hamiltonian, where a symplectic scheme is used for discretization. We adopt this argumentation for a systematic discrete-time design procedure for controllable linear systems based on the implicit midpoint rule. We motivate the approach on the example of two basic linear systems under zero order hold sampling, we show the construction of target systems with desired eigenvalues based on the discrete-time controller canonical form, and we illustrate the quality of the main result on a random sixth order system.

Keywords: Sampled-data systems, discrete-time control, linear SISO systems, symplectic integration, implicit midpoint rule, energy shaping.

1. INTRODUCTION

Continuous-time control design and implementation in *sampled-data* control systems can lead to the deterioration of the closed-loop dynamics or even instability, which is a well-known fact. Therefore, for non-negligible sampling times, a discrete-time control design is mandatory, see e.g. Ogata et al. (1995). Physically inspired state feedback control designs like *Interconnection and Damping Assignment Passivity-Based Control* (IDA-PBC) (Ortega et al., 2002) or *Controlled Lagrangians* (Bloch et al., 2000) are typically split in two steps: While *energy shaping* renders the closed-loop system conservative with the desired equilibrium the minimum of the shaped energy, *damping injection* adds dissipation to enforce asymptotic stability of the closed-loop equilibrium.

To translate this rationale to sampled control systems, the *discrete-time target system* in the energy shaping step must be conservative as well. Appropriate candidate target systems result from *symplectic numerical integration* of continuous-time conservative (e.g. Hamiltonian) systems, see Hairer et al. (2006) or Kotyczka and Lefèvre (2019) for a corresponding definition of discrete-time port-Hamiltonian systems. These discrete-time models lack numerical dissipation and feature conserved quantities like a *modified* Hamiltonian, which are related to the invariants of the underlying continuous-time system, and which can be determined via *backward error analysis*.

In this contribution, we present the systematic assignment of a corresponding "symplectic" discrete-time target system for the case of *sampled controllable linear systems*. We exploit the transformation of the (exact) zero order hold equivalent to discrete-time controller canonical form. Based on two basic examples, which illustrate instability under sampling and its avoidance, as well as discrete-time energy shaping by the *implicit midpoint rule*, we motivate the approach, before generalizing it to linear SISO systems.

Although we do not stress the energy-based perspective in the main result of this paper, the link between eigenvalue assignment and energy shaping for stabilizable linear systems can be easily established via LMIs, see Prajna et al. (2002). Therefore, symplectic integration schemes are an appropriate choice in the considered context.

Recent control design approaches, which aim at a target system defined via geometric integration, are mainly based on *discrete gradient* methods that enforce the *exact* conservation of a desired Hamiltonian. Laila and Astolfi (2005) present a discrete-time IDA-PBC approach for separable systems, which is based on the *forward Euler* discretization of the open-loop mechanical system. Energy-based discrete-time control design using the construction of discrete gradients for the original and the closed-loop mechanical-type system is the topic of Gören Sümer and Yalçin (2008). Tiefensee et al. (2010) propose the sampled-data implementation of a continuous-time IDA-PBC controller based on Taylor series expansion to determine the piecewise constant control input. This way, the value of the closed-loop energy is matched with the continuous-time solution in the sampling instants. In Moreschini et al. (2019),

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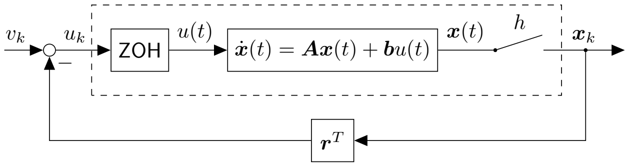

Fig. 1. Continuous-time SISO system, embedded in a sampled control loop.

discrete-time Hamiltonian systems are defined based on higher order discrete gradients.

The remainder of the paper is structured as follows. Section 2 gives two introductory examples, which show the utility of the implicit midpoint rule as a target system structure. In Section 3, we present as the main result the *systematic procedure for discrete-time eigenvalue assignment based on the implicit midpoint rule.* A random sixth-order numerical example illustrates the quality of our approach, in comparison with quasi-continuous state feedback design, in Section 4. We give a summary and concluding remarks in Section 5.

2. INTRODUCTORY EXAMPLES

The two basic examples in this section motivate the use of the implicit midpoint rule to define a discrete-time target system for state feedback control. The simple integrator illustrates the unconditional stabilizability, i.e. independent of the sampling time h. In the second example, we show that the implicit midpoint rule allows to endow the double integrator with a discrete-time conservative behavior in the sense of energy shaping.

2.1 Integrator

We consider the (continuous-time) integrator

$$\dot{y}(t) = u(t), \tag{1}$$

together with a proportional controller

$$u(t) = K(v(t) - y(t)). \tag{2}$$

For positive controller gain $K > 0$, the continuous-time closed-loop system (a lag element)

$$\dot{y}(t) = -Ky(t) + Kv(t) \tag{3}$$

is asymptotically (and BIBO) stable.

Now we put the plant between a zero-order hold element and a sampler with constant sampling time $h > 0$, see Fig. 1. We apply the same control law in discrete time (the *quasi-continuous* implementation),

$$u_k = K(v_k - y_k), \qquad K > 0. \tag{4}$$

The *exact* discrete-time representation of the sampled plant (zero order hold equivalence) is

$$y_{k+1} = y_k + hu_k. \tag{5}$$

With (4), we obtain the discrete-time closed-loop system

$$y_{k+1} = (1 - hK)y_k + hKv_k. \tag{6}$$

Instability of the closed loop occurs for $|1-hK| > 1$, which is the case for sampling times $h > \frac{2}{K}$.

Let us now follow a different approach. We design a controller such that the closed-loop system corresponds to

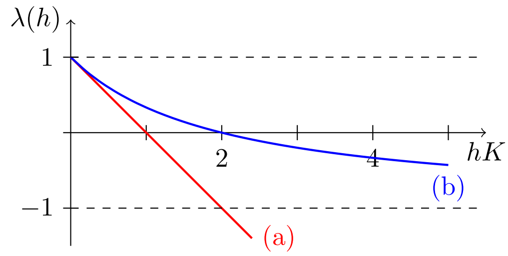

Fig. 2. Eigenvalues for closed-loop system (6) (a, red) and system (10) (b, blue) over hK.

the *symplectic discretization* of the first order lag element (3). We use the *implicit midpoint rule* [1] as one of the simplest symplectic schemes. This gives

$$y_{k+1} = y_k - \frac{hK}{2}(y_k + y_{k+1}) + \frac{hK}{2}(v_k + v_{k+1}). \tag{7}$$

Matching this target dynamics with (5), we obtain

$$u_k = -\frac{K}{2}(y_k + y_{k+1}) + \frac{K}{2}(v_k + v_{k+1}). \tag{8}$$

To render the discrete-time control law *causal* (the future reference value v_{k+1} can be assumed known), we substitute the system dynamics (5) on the right hand side and obtain, after rearrangement,

$$u_k = -\frac{K}{1 + \frac{hK}{2}} y_k + \frac{K}{2 + hK}(v_k + v_{k+1}). \tag{9}$$

For stability analysis, we set $v_k \equiv 0$ for all k, substitute (9) in (5), and obtain

$$(1 + \frac{hK}{2})y_{k+1} = (1 - \frac{hK}{2})y_k, \tag{10}$$

which is exactly the target dynamics (7). The evolution of the eigenvalues for this discrete-time system (the term on the left is invertible for all $hK > 0$) in terms of hK is depicted in Fig. 2, as well as the eigenvalues for (6). Unlike the control law (4) designed in continuous time, for which stability is lost for $h > \frac{2}{K}$, the closed-loop eigenvalue under the controller (9) remains in the unit circle for all $h > 0$. For values $h > \frac{2}{K}$, the eigenvalue moves to the negative real line, which means a stable deterioration of the dynamics in the sense of an alternating system response.

2.2 Double integrator

We now consider the double integrator

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= [1 \ 0]\, \boldsymbol{x}(t). \end{aligned} \tag{11}$$

The continuous-time *energy shaping* feedback control law $u(t) = u_{es}(t)$ with

$$u_{es}(t) = -[1 \ 0]\, \boldsymbol{x}(t) + v(t) \tag{12}$$

renders the system an undamped oscillator [2] (the output remains unchanged)

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t). \tag{13}$$

[1] Which is the Gauss-Legendre scheme with $s = 1$ stage and coincides for linear systems with the trapezoidal rule.

[2] Which can be written as Hamiltonian system with quadratic Hamiltonian $H(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{x}$.

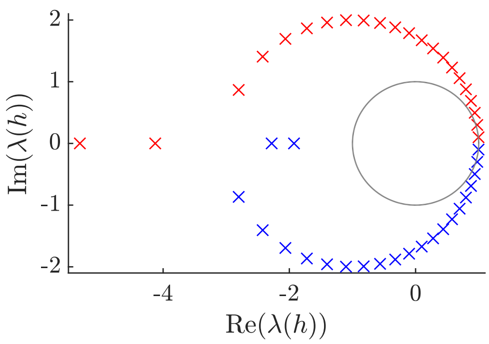

Fig. 3. Locations of the complex conjugate (red and blue) eigenvalues of the state matrix in (17), $h \in \{0.1, 0.3, 0.5, \dots, 4.3\}$.

The additional passive output or *damping injection* feedback $v(t) = v_{di}(t)$ with

$$v_{di}(t) = -r\,[0\ 1]\,\boldsymbol{x}(t), \qquad r > 0 \tag{14}$$

yields the closed-loop state representation of a *damped oscillator*

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & r \end{bmatrix} \boldsymbol{x}(t). \tag{15}$$

As in the previous example, we consider the zero order hold equivalent discrete-time state representation

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} u_k \tag{16}$$

of the double integrator. Substituting the discrete-time evaluation of the energy shaping control law (12) in (16) leads to the closed-loop dynamics

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 1 - \frac{h^2}{2} & h \\ -h & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} v_k. \tag{17}$$

Figure 3 illustrates that for arbitrary $h > 0$, the eigenvalues of the closed-loop system have magnitude greater than 1, i.e. they lie outside the unit circle. The quasi-continuous implementation fails to impose the desired conservative system behavior of a lossless oscillator.

As in the previous section, we define the discretization of (13) using the implicit midpoint rule as the target system:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \frac{h}{2}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\boldsymbol{x}_k + \boldsymbol{x}_{k+1}) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (v_k + v_{k+1}) \right). \tag{18}$$

Matching this desired state representation with the open-loop system (16), we obtain two scalar matching equations.

First matching equation Identifying the first components $x_{1,k+1}$, we get

$$x_{1,k} + \frac{h}{2}(x_{2,k} + x_{2,k+1}) = x_{1,k} + h x_{2,k} + \frac{h^2}{2} u_k. \tag{19}$$

After cancellation of $x_{1,k}$ on both sides and division by $\frac{h}{2}$, what remains is

$$x_{2,k+1} = x_{2,k} + h u_k. \tag{20}$$

This is exactly the second equation of (16), i.e. a part of the discrete-time dynamics, and therefore a valid equation.

Second matching equation Doing accordingly with $x_{2,k+1}$, we obtain

$$x_{2,k} - \frac{h}{2}(x_{1,k} + x_{1,k+1}) + \frac{h}{2}(v_k + v_{k+1}) = x_{2,k} + h u_k. \tag{21}$$

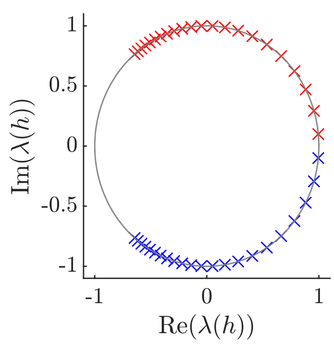

Fig. 4. Eigenvalue locations of the matrix pencil $P(\lambda) = \lambda \boldsymbol{E} - \boldsymbol{A}$ as in (23), $h \in \{0.1, 0.3, 0.5, \dots, 4.3\}$.

Cancellation of $x_{2,k}$ and substitution of $x_{1,k+1}$ according to (16) yields the control law

$$u_k = \frac{1}{1 + \frac{h^2}{4}} \left(-x_{1,k} - \frac{h}{2} x_{2,k} + \frac{1}{2}(v_k + v_{k+1}) \right). \tag{22}$$

Substitution of the control law in the open-loop state representation (16) yields an expression which, by appropriate matrix multiplication, can be brought to the form

$$\begin{bmatrix} 1 & -\frac{h}{2} \\ \frac{h}{2} & 1 \end{bmatrix} \boldsymbol{x}_{k+1} = \begin{bmatrix} 1 & \frac{h}{2} \\ -\frac{h}{2} & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ \frac{h}{2} \end{bmatrix} (v_k + v_{k+1}). \tag{23}$$

This is exactly the descriptor form $(\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{b})$ of the target system (18). The complex conjugate eigenvalues of the matrix pencil $P(\lambda) = \lambda \boldsymbol{E} - \boldsymbol{A}$ are depicted in Fig. 4 (red and blue). We note that *energy shaping* in the sense of endowing the system with conservative discrete-time target dynamics is possible, see the phase portrait in Fig. 5 for illustration.

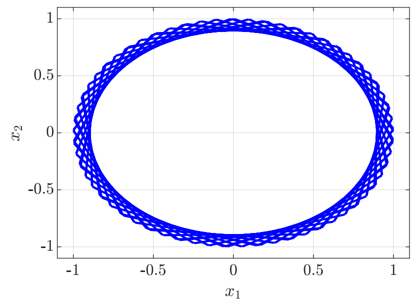

Fig. 5. Phase portrait under the *symplectic* energy shaping controller in Example 2 with sampling time $h = 1$. Simulation until $t = 150$.

Remark 1. In view of the discretization of (15) and (18), it is straightforward to see that $v_k = -r x_{2,k}$ is the appropriate discrete-time damping injection, which corresponds to the implicit midpoint rule, and which could be realized in conjunction with energy shaping in a one-step approach.

3. MAIN RESULT

We present as the main result of the paper a procedure for discrete-time *symplectic* eigenvalue assignment in the sense that the discrete-time target system, based on which the state feedback controller is derived, is the discretization with the *implicit midpoint rule* of a continuous-time target system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_C \boldsymbol{x}(t), \qquad \sigma(\boldsymbol{A}_C) \in \mathbb{C}^-, \tag{24}$$

where $\sigma(\boldsymbol{A}_C)$ denotes the spectrum of the matrix $\boldsymbol{A}_C$.

3.1 Preliminaries and notation

We consider controllable SISO linear time invariant systems[3] $(\boldsymbol{A}_{c0}, \boldsymbol{b}_{c0})$ of order n that are sampled as sketched in Fig. 1. As long as the Nyquist-Shannon sampling condition[4] is satisfied, also the zero order hold sampled system[5] $(\boldsymbol{A}_{d0}, \boldsymbol{b}_{d0}) = (e^{\boldsymbol{A}_{c0}h}, \boldsymbol{A}_{c0}^{-1}(e^{\boldsymbol{A}_{c0}h} - \boldsymbol{I})\boldsymbol{b}_{c0})$ is controllable[6]. Therefore it can be transformed to controller canonical form (CCF)

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u_k, \qquad (25)$$

shortly written $(\boldsymbol{A}_d, \boldsymbol{b}_d)$.

By discrete-time state feedback control

$$u_k = -\boldsymbol{r}_d^T \boldsymbol{x}_k, \qquad (26)$$

we want to impose closed-loop dynamics of the form

$$(\boldsymbol{I} - \frac{h}{2}\boldsymbol{A}_C)\boldsymbol{x}_{k+1} = (\boldsymbol{I} + \frac{h}{2}\boldsymbol{A}_C)\boldsymbol{x}_k, \qquad (27)$$

which is the implicit midpoint rule discretization of (24). With

$$\boldsymbol{A}_D = (\boldsymbol{I} - \frac{h}{2}\boldsymbol{A}_C)^{-1}(\boldsymbol{I} + \frac{h}{2}\boldsymbol{A}_C), \qquad (28)$$

Equation (27), as a feature of the *linear* case, can be written in the *explicit* form[7]

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_D \boldsymbol{x}_k. \qquad (29)$$

All mentioned matrices are summarized in Table 1.

Table 1. Different used state space matrices.

Symbol(s)	Description
$(\boldsymbol{A}_{c0}, \boldsymbol{b}_{c0})$	Continuous-time system
$(\boldsymbol{A}_{d0}, \boldsymbol{b}_{d0})$	Discretized system (zero order hold)
$(\boldsymbol{A}_d, \boldsymbol{b}_d)$	Discretized system in CCF
$(\boldsymbol{A}_c, \boldsymbol{b}_c)$	Corresponding continuous-time system
$\boldsymbol{A}_C$	Target state matrix, continuous-time
$\boldsymbol{A}_D$	Target state matrix, discrete-time in CCF

3.2 Matching condition

We exploit the special structure of $\boldsymbol{D} := \frac{h}{2}\boldsymbol{A}_C$. Note that, unlike $\boldsymbol{A}_d$ and $\boldsymbol{A}_D$, which are in CCF, $\boldsymbol{D}$ has *a priori* no particular form. However, its structure is constrained by the *matching equation*, which results from multiplication of (25) with $(\boldsymbol{I} - \boldsymbol{D})$ and comparison of its right hand side with the one of the target system (27) ($\boldsymbol{b}_d = \boldsymbol{e}_n$, $\boldsymbol{e}_n$ denotes the n-th unit vector):

$$(\boldsymbol{I} - \boldsymbol{D})\boldsymbol{A}_d\boldsymbol{x}_k + (\boldsymbol{I} - \boldsymbol{D})\boldsymbol{e}_n u_k = (\boldsymbol{I} + \boldsymbol{D})\boldsymbol{x}_k. \qquad (30)$$

To characterize the *assignable matrices* $\boldsymbol{D}$, we multiply the whole equation with a full rank left hand annihilator $\boldsymbol{X} \in \mathbb{R}^{(n-1)\times n}$ such that $\boldsymbol{X}(\boldsymbol{I} - \boldsymbol{D})\boldsymbol{e}_n = \boldsymbol{0}$. A possible choice, which follows from $(\boldsymbol{I} - \boldsymbol{D})\boldsymbol{e}_n = [-d_{1,n} \; \dots \; -d_{n-1,n} \; 1 - d_{n,n}]^T$, is

$$\boldsymbol{X} = \begin{bmatrix} 1 & & \frac{d_{1,n}}{1-d_{n,n}} \\ & \ddots & \vdots \\ & 1 & \frac{d_{n-1,n}}{1-d_{n,n}} \end{bmatrix}. \qquad (31)$$

Omitting $\boldsymbol{x}_k$, this yields the *matching condition*

$$\boldsymbol{X}(\boldsymbol{I} - \boldsymbol{D})\boldsymbol{A}_d = \boldsymbol{X}(\boldsymbol{I} + \boldsymbol{D}). \qquad (32)$$

Theorem 1. The matrix

$$\boldsymbol{D} = \boldsymbol{D}_0 - \boldsymbol{g}_D \boldsymbol{k}_D^T \qquad (33)$$

with the upper triangular matrix $\boldsymbol{D}_0 \in \mathbb{R}^{n\times n}$ and the alternating column vector $\boldsymbol{g}_D \in \mathbb{R}^n$

$$\boldsymbol{D}_0 = \begin{bmatrix} -1 & 2 & -2 & \dots & (-1)^n \cdot 2 \\ & -1 & 2 & \dots & (-1)^{n-1} \cdot 2 \\ & & \ddots & \ddots & \vdots \\ & & & -1 & 2 \\ & & & & -1 \end{bmatrix}, \quad \boldsymbol{g}_D = \begin{bmatrix} (-1)^n \\ (-1)^{n-1} \\ \vdots \\ 1 \\ -1 \end{bmatrix} \qquad (34)$$

and an arbitrary row vector $\boldsymbol{k}_D^T \in \mathbb{R}^{1\times n}$ satisfies the matching condition (32).

Proof: Denote by $\boldsymbol{d}_i^T \in \mathbb{R}^{1\times n}$, $i = 1, \dots, n$, the rows of $\boldsymbol{D}$. By inspection of (33), (34), we find that

$$\boldsymbol{d}_n^T = \boldsymbol{k}_D^T + [0 \; \dots \; 0 \; -1], \qquad (35)$$

and for $i = n-1, \dots, 1$,

$$\boldsymbol{d}_i^T + \boldsymbol{d}_{i+1}^T = [0 \; \dots \; 0 \; -1 \; 1 \; 0 \; \dots \; 0], \qquad (36)$$

with -1 and 1 at position i and $i+1$, respectively. The evaluation of the last elements of (35) and (36) yields

$$\begin{aligned} d_{n,n} &= k_{d,n} - 1, \\ d_{n-1,n} &= 1 - d_{n,n}, \\ d_i &= -d_{i+1} \qquad \text{for } i = n-2, \dots, 1. \end{aligned} \qquad (37)$$

With this, the annihilator takes the form

$$\boldsymbol{X} = \left[\boldsymbol{I} \; \boldsymbol{g}_{D[1,\dots,n-1]}\right], \qquad (38)$$

where $\boldsymbol{g}_{D[1,\dots,n-1]} \in \mathbb{R}^{n-1}$ contains the first $n-1$ elements of $\boldsymbol{g}_D$. This matrix can be factorized as $\boldsymbol{X} = \boldsymbol{X}_1\boldsymbol{X}_2$ with $\boldsymbol{X}_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ an upper triangular invertible matrix (diagonals of ± 1 with alternating sign) and

$$\boldsymbol{X}_2 = \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n}. \qquad (39)$$

$\boldsymbol{X}_1$ cancels from the matching condition (32), which can be written in terms of $\boldsymbol{X}_2$ instead of $\boldsymbol{X}$. With

$$\boldsymbol{X}_2\boldsymbol{D} = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n} \qquad (40)$$

according to (36), the matching condition reads

$$\begin{bmatrix} 2 & 0 & & \\ & \ddots & \ddots & \\ & & 2 & 0 \end{bmatrix} \boldsymbol{A}_d = \begin{bmatrix} 0 & 2 & & \\ & \ddots & \ddots & \\ & & 0 & 2 \end{bmatrix}, \qquad (41)$$

and it is obviously satisfied for $\boldsymbol{A}_d$ in CCF. □

[3] We do not fix an output as state feedback control is considered.
[4] See Shannon (1949).
[5] If $\boldsymbol{A}_{c0}$ is not invertible, $\boldsymbol{b}_{d0}$ is well defined by the Taylor series expansion of $e^{\boldsymbol{A}_{c0}h} - \boldsymbol{I}$.
[6] See Eq. (28) in Kalman (1960), which is always true if the sampling condition is respected.
[7] Note that this corresponds to the simplest discrete gradient for linear Hamiltonian systems as given in Moreschini et al. (2019), Proposition 2.

3.3 Eigenvalue assignment

The matrix $\boldsymbol{D}$ is a scaled version of the continuous-time target matrix $\boldsymbol{A}_C$. With the eigenvalues of $\boldsymbol{A}_C$ as design parameters, the corresponding eigenvalues of $\boldsymbol{D}$ are

$$\lambda_i(\boldsymbol{D}) = \frac{h}{2}\lambda_i(\boldsymbol{A}_C), \qquad i = 1, \ldots, n. \tag{42}$$

By inspection of (33) we note that the eigenvalues of $\boldsymbol{D}$ can be assigned arbitrarily by appropriate choice of $\boldsymbol{k}_D^T$ if and only if the pair $(\boldsymbol{D}_0, \boldsymbol{g}_D)$ is controllable.

Lemma 2. The pair $(\boldsymbol{D}_0, \boldsymbol{g}_D)$ is controllable.

Sketch of the proof: Linear independence of the columns of the controllability matrix $\left[\boldsymbol{g}_D \ \boldsymbol{D}_0\boldsymbol{g}_D \ \ldots \ \boldsymbol{D}_0^{n-1}\boldsymbol{g}_D\right]$ follows from the particular structure of $\boldsymbol{D}_0$ and $\boldsymbol{g}_D$. □

Consequently, for an arbitrary desired spectrum $\sigma(\boldsymbol{A}_C)$, we can determine a vector $\boldsymbol{k}_d^T$, which assigns the corresponding scaled eigenvalues to the matrix $\boldsymbol{D}$ according to (33).

3.4 Control law

Having constructed a matrix $\boldsymbol{D}$ with arbitrary assignable eigenvalues, which satisfies the matching condition (32), what remains is to isolate the control law from the matching equation (30). With the structure of the last column of $\boldsymbol{D}$ according to (37), the column vector in front of u_k becomes $(\boldsymbol{I}-\boldsymbol{D})\boldsymbol{e}_n = (d_{n,n}-1)\boldsymbol{g}_D$. Note that the squared Euclidean norm of $\boldsymbol{g}_D$ is $\|\boldsymbol{g}_D\|_2^2 = n$, therefore left multiplication of (30) with $\frac{\boldsymbol{g}_D^T}{(d_{n,n}-1)n}$ yields, after rearrangement,

$$u_k = \frac{\boldsymbol{g}_D^T}{(d_{n,n}-1)n}(\boldsymbol{I}+\boldsymbol{D}-(\boldsymbol{I}-\boldsymbol{D})\boldsymbol{A}_d)\boldsymbol{x}_k. \tag{43}$$

This gives the feedback vector according to (26)

$$\boldsymbol{r}_d^T = \frac{\boldsymbol{g}_D^T}{(1-d_{n,n})n}(\boldsymbol{I}+\boldsymbol{D}-(\boldsymbol{I}-\boldsymbol{D})\boldsymbol{A}_d). \tag{44}$$

Remark 3. A straightforward possibility to determine $\boldsymbol{r}_d^T$ numerically is to solve the matrix equation $(\boldsymbol{I}-\boldsymbol{D})\boldsymbol{A}_D = \boldsymbol{I}+\boldsymbol{D}$ for $\boldsymbol{A}_D$ and to compute $\boldsymbol{r}_d^T = \boldsymbol{e}_n^T\boldsymbol{A}_d - \boldsymbol{e}_n^T\boldsymbol{A}_D$.

3.5 Design procedure

We summarize the approach for a controllable SISO system $(\boldsymbol{A}_{c0}, \boldsymbol{b}_{c0})$.

(1) Compute the zero order hold equivalent system $(\boldsymbol{A}_{d0}, \boldsymbol{b}_{d0})$.
(2) Transform the system to CCF using the controllability matrix $\boldsymbol{Q}_{d0} = \left[\boldsymbol{b}_{d0} \ \boldsymbol{A}_{d0}\boldsymbol{b}_{d0} \ \ldots \ \boldsymbol{A}_{d0}^{n-1}\boldsymbol{b}_{d0}\right]$. Result: $(\boldsymbol{A}_d, \boldsymbol{b}_d)$ according to (25).
(3) Choose a continuous-time target spectrum $\sigma(\boldsymbol{A}_C) \in \mathbb{C}^-$, and obtain $\sigma(\boldsymbol{D})$ by scaling with $\frac{h}{2}$.
(4) Compute $\boldsymbol{k}_D^T$ and obtain $\boldsymbol{D}$ according to (33), (34) by eigenvalue assignment.
(5) Compute the feedback vector $\boldsymbol{r}_d^T$ according to (44) or Remark 3.
(6) Use the inverse transformation from CCF to obtain $\boldsymbol{r}_{d0}^T$ in the modeling coordinates.

4. NUMERICAL EXAMPLE

We illustrate the proposed control design approach on an example of order 6, with system matrices

$$\boldsymbol{A}_{c0} = \begin{bmatrix} 0.11 & 0.93 & 0.98 & 0.13 & 0.47 & 0.35 \\ 0.14 & 0.73 & 0.86 & 0.03 & 0.65 & 0.45 \\ 0.17 & 0.74 & 0.79 & 0.94 & 0.03 & 0.05 \\ 0.62 & 0.06 & 0.51 & 0.30 & 0.84 & 0.18 \\ 0.57 & 0.86 & 0.18 & 0.30 & 0.56 & 0.66 \\ 0.05 & 0.93 & 0.40 & 0.33 & 0.85 & 0.33 \end{bmatrix}, \boldsymbol{b}_{c0} = \begin{bmatrix} 0.90 \\ 0.12 \\ 0.99 \\ 0.54 \\ 0.71 \\ 1.00 \end{bmatrix}, \tag{45}$$

generated by *Matlab's* `rand` command [8]. We compare the effect in the closed-loop sampled system of the discrete-time controller designed according to Subsection 3.5 with a state feedback derived based on continuous-time eigenvalue placement $\boldsymbol{A}_{c0} - \boldsymbol{b}_{c0}\boldsymbol{r}_{c0}^T = \boldsymbol{A}_C$. In both cases, the desired spectrum of the continuous-time target state matrix is chosen as

$$\sigma(\boldsymbol{A}_C) = \{-1 \pm j, -2 \pm 2j, -3 \pm 3j\}. \tag{46}$$

Table 2 shows the obtained feedback vectors. Case (a) represents the continuous-time controller $\boldsymbol{r}_{c,0}^T$, case (b) shows the discrete-time controllers $\boldsymbol{r}_{d,0}^T$ for different sampling times h. The convergence of the latter to $\boldsymbol{r}_{c,0}^T$ for $h \to 0$ is evident.

Table 2. Feedback vectors $\boldsymbol{r}_{c0}^T$ (first row) and $\boldsymbol{r}_{d0}^T$ for different sampling times.

	h	Feedback vector					
(a)	–	[−292.1	− 258.4	− 1026	112.3	612.7	828.4]
(b)	0.01	[−270.2	− 239.5	− 956.0	101.8	572.6	771.2]
	0.05	[−197.7	− 176.4	− 722.7	67.90	437.5	579.9]
	0.15	[−90.69	− 80.97	− 363.1	21.28	226.1	288.2]
	0.3	[−28.85	− 24.02	− 135.9	− 0.4209	89.06	107.3]

The behavior of the closed-loop sampled system as depicted in Fig. 1 is simulated using a continuous-time *Simulink* model, where sample and hold are included as the corresponding blocks, triggered by an external signal with period h. The variable-step solver is automatically chosen, with a tolerance of 10^{-3}.

Figure 6 shows the initial value responses for the closed-loop system with $\boldsymbol{r}_{c,0}^T$ for two sampling times, in particular the instability of the closed loop for $h = 0.15$.

Figure 7 illustrates the expected quality of the presented discrete-time control design. The initial value responses converge asymptotically to the equilibrium $\boldsymbol{x}^* = \boldsymbol{0}$, up to $h = 0.3$. This sampling time, which is very close to the time constant of the largest real eigenvalue of $\boldsymbol{A}_{c0}$ $(0.35 \approx 1/2.85)$, still resolves the system dynamics.

Remark 4. A possible extension of the approach is its application to *stabilizable* linear systems, based on a partial transformation to CCF.

5. CONCLUSIONS

Based on the well-known fact that sufficiently slow sampling can deteriorate and destabilize the closed-loop dynamics of a control system, we proposed a systematic procedure for state feedback design of linear sampled SISO

[8] Eigenvalues: $\sigma(\boldsymbol{A}_{c0}) \approx \{-0.74, -0.26, 0.13 \pm 0.37j, 0.71, 2.85\}$.

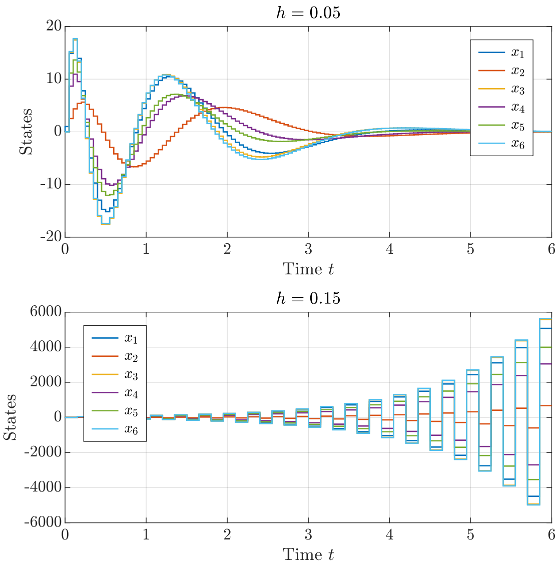

Fig. 6. Closed-loop system responses with the discrete-time implementation of the continuous-time controller $\boldsymbol{r}_{c0}^T$, $\boldsymbol{x}(0) = [1\ 0\ 0\ 0\ 0\ 0]^T$.

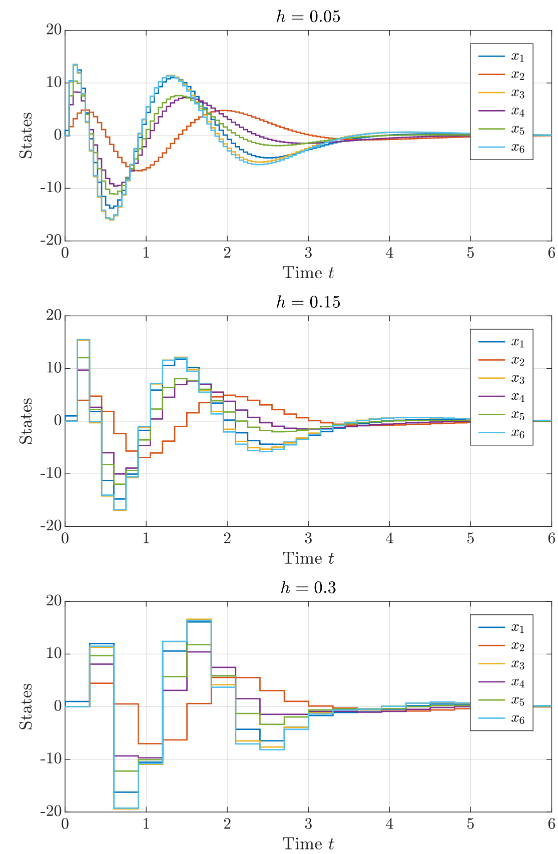

Fig. 7. Closed-loop system responses with the discrete-time controller $\boldsymbol{r}_{d0}^T$, $\boldsymbol{x}(0) = [1\ 0\ 0\ 0\ 0\ 0]^T$.

systems, which utilizes the zero order hold equivalent system representation and *symplectic integration*. The target system is defined as the discretization of an asymptotically stable continuous-time system with the *implicit midpoint rule*, or one-stage Gauss-Legendre collocation. The approach is valid for arbitrary controllable discrete-time systems, which allows for the straightforward inclusion of time delays, which are integer multiples of the sampling time.

We are working on generalizations in several directions, including discrete-time nonlinear passivity-based control with IDA-PBC, where the local linearization helps in the parametrization of the nonlinear control law (Kotyczka, 2013). Other extensions concern higher-order symplectic schemes for the target system and generalized forms of sampling.

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