

# Passivity Conditions for Plug-and-Play Operation of Nonlinear Static AC Loads

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Abstract: The complexity arising in AC microgrids from multiple interacting *distributed generation units* (DGUs) with intermittent supply behavior requires local *voltage-source inverters* (VSIs) to be controlled in a distributed or decentralized manner at primary level. In (Strehle et al., 2019a), we use passivity theory to design decentralized, plug-and-play voltage and frequency controllers for such VSIs. However, the stability analysis of the closed-loop system requires a load-connected topology, in contrast to real grids where loads are arbitrarily located. In this paper, we expand our former approach by considering the more realistic and general case of nonlinear static AC loads (ZIP and exponential) at arbitrary locations within an AC microgrid. Investigating the monotonicity of differentiable mappings, we derive sufficient inequality conditions for the strict passivity of these nonlinear static AC loads. Together with our plug-and-play VSI controller, this allows us to use passivity arguments to infer asymptotic voltage and frequency stability for AC microgrids with arbitrary topologies. An illustrative simulation validating our theoretical findings concludes our work.

*Keywords:* Plug-and-Play Control, AC Microgrids, Passivity, Voltage Stability, AC loads

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## 1. INTRODUCTION

Inverter-based AC microgrids comprise electrical distribution networks linking a multitude of loads and various renewable energy sources, referred to as *distributed generation units* (DGUs), which also include storage devices (Lasseter, 2001)(Schiffer et al., 2016)(Olivares et al., 2014). When compared to classical power systems, voltage and frequency control in such AC microgrids represent a far more complex challenge which is the subject of much recent research activity.

Particularly the heavily increased number of interacting components (loads, DGUs) together with their intermittent supply/demand behavior requires scalability of the frequency and voltage control methods for the local VSIs at primary level (Lasseter, 2001)(Guerrero et al., 2013). Decentralized control methods enable such scalability as they only rely on local DGU information and measurements for the corresponding local VSI control design. This allows for the addition or removal of DGUs in a *plug-and-play* fashion without requiring changes to any existing local controllers. An extensive framework for modular modeling and decentralized control methods of complex large-scale systems is passivity theory (van der Schaft, 2017)(Duindam et al., 2009)(Fiaz et al., 2013)(van der Schaft and Stgink, 2016)(Strehle et al., 2018). Thus, recent publications focusing on decentralized plug-and-play voltage and frequency controllers of VSIs in AC microgrids, use passivity-based approaches (Nahata and Ferrari-Trecate, 2019)(Strehle et al., 2019a). Exploiting the compositional property of passive systems, such approaches provide a

modular analysis of microgrid-wide voltage and frequency stability via the passivity of all subsystems within an AC microgrid and their passivity-preserving interconnection.

However, these former results are either (i) obtained under a load-connected AC microgrid topology in which all loads are connected to the controllable VSI terminals of DGUs (Strehle et al., 2019a), or (ii) necessitate solving linear matrix inequalities after each DGU plug-in/out operation (Nahata and Ferrari-Trecate, 2019). While the load-connected topology in (i) distorts the original topology and thus hampers insight into operationally relevant phenomena, e.g. line congestion or voltage transients at nodes that are reduced, the numerical optimization in (ii) might be infeasible and prevent the plug-in/out of the respective DGU. Furthermore, both approaches use reduced forms of the nonlinear static *constant impedance*, *constant current*, *constant power* (ZIP) load model (ZP in Strehle et al. (2019a); ZI in Nahata and Ferrari-Trecate (2019)). However, prevalent nonlinear static AC load models comprise full ZIP and exponential load models (Machowski et al., 2008, pp. 111-112)(Cañizares et al., 2018, pp. 33-34).

In this paper, we address these issues by considering the more general case of nonlinear static ZIP and exponential AC loads at arbitrary locations within an AC microgrid. Note that load-connected topologies are still included in that consideration as a special case. In a first step, we set up a port-Hamiltonian model of uncontrollable, lone-standing load nodes and include it in the modular, passivity-based voltage and frequency stability analysis from Strehle et al. (2019a). In so doing, we expand our

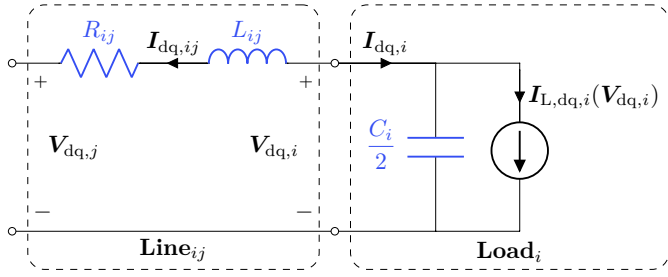


Figure 1. Circuit diagram of a three-phase  $\pi$ -line (in blue) connected to a nonlinear static AC load

former results to arbitrary AC microgrid topologies comprising electrical lines, controllable DGU nodes, with optional uncontrollable local loads, and uncontrollable lone-standing load nodes. In order to investigate the passivity of the load nodes, we then establish a helpful reformulation of the load models as voltage-dependent current functions in  $dq$  coordinates. To the best of the authors' knowledge, such a representation is not directly available in the pertinent literature, where loads are commonly described by their consumed active and reactive powers (see for example (Machowski et al., 2008, pp. 111-112)(Cañizares et al., 2018, pp. 33-34)). With this, we finally derive sufficient inequality conditions for the strict passivity of nonlinear static ZIP and exponential AC load models by investigating the monotonicity of the load current-voltage relation. Additionally, we give an interesting perspective on how our results are related to incremental passivity of memoryless functions.

In summary, our main contributions comprise (i) the extension of our decentralized plug-and-play approach from Strehle et al. (2019a) by uncontrollable lone-standing load nodes to guarantee asymptotic voltage and frequency stability of AC microgrids with arbitrary topologies; (ii) the proposition of sufficient inequality conditions for the strict passivity of the prevalent nonlinear static ZIP and exponential AC load models.

## 2. PROBLEM FORMULATION

The overarching problem we investigate in this work is the asymptotic voltage and frequency stability of AC microgrids with arbitrary topologies. The AC microgrid is modeled as in Strehle et al. (2019a) with electrical lines, controllable DGUs (with optional uncontrollable local loads), and uncontrollable lone-standing loads in the  $dq$  reference frame rotating at  $\omega_0 = 2\pi 50$  Hz. The zero-sequence is neglected by assuming a balanced network. As we are considering nonlinear static AC load models, we model the lone-standing loads as voltage-dependent current sources connected via  $\pi$ -model electrical lines to the remaining AC microgrid (see Figure 1). Consequently, an AC microgrid can be represented by a bipartite graph as in Figure 2. Without loss of generality, we consider only the connection of the load with a single line in the subsequent sections. The generalization to multiple connecting lines can readily be obtained by replacing  $C_i$  with the sum of all parallel-connected capacitances  $C_{ij}$  of these lines and  $I_{dq,i}$  with the negative sum of all outgoing  $I_{dq,ij}$ .

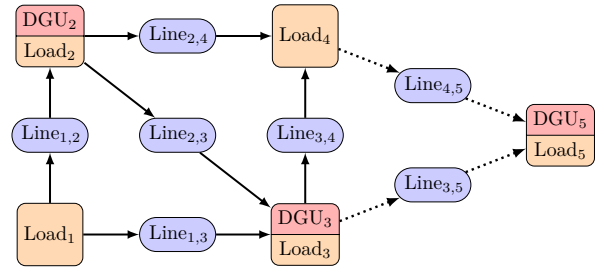


Figure 2. Bipartite graph representation of an AC microgrid with electrical lines interconnecting controllable DGU nodes (with optional local loads) and uncontrollable load nodes; dotted lines indicate the plug-and-play nature of the microgrid

### 2.1 Preliminaries

For the remaining part of this contribution, we establish the following assumption which holds under normal grid conditions:

*Assumption 1.* Any voltage amplitudes (node, reference, nominal) are strictly positive, i.e.

$$V(t) = \sqrt{V_d(t)^2 + V_q(t)^2} > 0, \quad \forall t \geq 0 \quad (1)$$

with  $V_d(t), V_q(t) \in \mathcal{U} \subset \mathbb{R}^2$ . The reference frequency is strictly positive  $\omega_0 > 0$ . All load model parameters are positive, i.e. real numbers greater or equal to zero.

*Remark 1.* Note that due to Assumption 1, we only use the term asymptotic voltage and frequency stability in this work and drop the denotation global. Although, from a practical perspective, the obtained stability result is global in the sense that it holds for the complete operationally relevant area of  $V(t) > 0$  and  $\omega_0 > 0$  for all  $t \geq 0$ .

Furthermore we notice that variables in a  $dq$  frame (rotating at frequency  $\omega_0$ ) can be considered Cartesian representations of complex vectors with the equivalent polar representation  $V(t)\angle\theta(t)$ . Asymptotically stable  $dq$  systems imply  $\lim_{t \rightarrow \infty} \theta(t) = const.$ , and are therefore also asymptotically stable as phasors with the frequency of the rotating phasors  $\omega = \dot{\theta} + \omega_0 = \omega_0$ , since  $\dot{\theta} = 0$ . Asymptotic frequency stability is therefore implicitly present in an asymptotically stable  $dq$  system. This means that microgrid-wide asymptotic voltage stability in  $dq$  coordinates implies microgrid-wide asymptotic frequency stability, if the various local  $dq$  frames at each node are synchronized. Thus, we establish the following Assumption similar to (Cucuzzella et al., 2018, Assumption 2) (Nahata and Ferrari-Trecate, 2019, Remark 1):

*Assumption 2.* The various  $dq$  reference frames in the local VSI controllers and at load nodes are synchronized.

However, note that the  $dq$  reference frames at load nodes are only used as means to facilitate the analysis of the balanced, three-phase AC signals. There are no actual controller clocks to be synchronized at such nodes.

### 2.2 Voltage and Frequency Stability

Investigating the asymptotic voltage and frequency stability of AC microgrids with arbitrary topologies means that

in addition to the asymptotic stability of controllable DGU equilibria (cf. (10) in Strehle et al. (2019a))

$$\mathbf{x}_i^* = [L_{ti}I_{d,i}^*, L_{ti}I_{q,i}^*, C_{ti}V_{d,i}^*, C_{ti}V_{q,i}^*]^\top, \quad (2)$$

we now also investigate the asymptotic stability of uncontrollable load equilibria. For this, we model the load node  $i$  in Figure 1 as a port-Hamiltonian system with a nonlinear resistive structure (cf. (van der Schaft, 2017, p. 114))

$$\begin{aligned} \begin{bmatrix} C_i \dot{V}_{d,i} \\ C_i \dot{V}_{q,i} \end{bmatrix} &= \mathbf{J}_{L,i} \begin{bmatrix} V_{d,i} \\ V_{q,i} \end{bmatrix} - \mathbf{I}_{L,dq,i}(\mathbf{V}_{dq,i}) + \mathbf{K}_{L,i} \begin{bmatrix} I_{d,i} \\ I_{q,i} \end{bmatrix} \\ \mathbf{z}_i &= \mathbf{K}_{L,i}^\top \begin{bmatrix} V_{d,i} \\ V_{q,i} \end{bmatrix} \end{aligned} \quad (3a)$$

$$H_{L,i}(\mathbf{x}_{L,i}) = \frac{1}{2} \mathbf{x}_{L,i}^\top \text{Diag} \left[ \frac{1}{C_i}, \frac{1}{C_i} \right] \mathbf{x}_{L,i}$$

with states  $\mathbf{x}_{L,i} = [C_i V_{d,i}, C_i V_{q,i}]^\top$ , uncontrollable input  $\mathbf{d}_i = [I_{d,i}, I_{q,i}]^\top$ , uncontrollable output  $\mathbf{z}_i = [V_{d,i}, V_{q,i}]^\top$ , nonlinear resistive structure

$$\mathcal{R}(\mathbf{x}_{L,i}) = \mathbf{I}_{L,dq,i}(\mathbf{V}_{dq,i}), \quad (3b)$$

and interconnection and uncontrollable input matrices

$$\mathbf{J}_{L,i} = \begin{bmatrix} 0 & \omega_0 C_i \\ -\omega_0 C_i & 0 \end{bmatrix}, \quad \mathbf{K}_{L,i} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (3c)$$

As the load node voltages  $\mathbf{V}_{dq,i}$  are not directly controllable, the equilibrium

$$\mathbf{x}_{L,i}^* = C_i \mathbf{V}_{dq,i}^*, \quad (4)$$

of (3) is determined by the exchange currents  $\mathbf{d}_i = \mathbf{I}_{dq,i}$  resulting from the voltages  $\mathbf{V}_{dq,j}$  at neighboring nodes (see Figure 1). For the stability analysis of (4), we define the error variables

$$\boldsymbol{\epsilon}_{V_{dq,i}} := \mathbf{V}_{dq,i} - \mathbf{V}_{dq,i}^*, \quad \boldsymbol{\epsilon}_{V_{dq,i}} \in \mathbb{R}^2, \quad (5)$$

$$\boldsymbol{\xi}_{L,i} := \mathbf{x}_{L,i} - \mathbf{x}_{L,i}^*, \quad \boldsymbol{\xi}_{L,i} \in \mathbb{R}^2, \quad (6)$$

and set up the error system of the load model (3) as

$$\begin{aligned} C_i \dot{\boldsymbol{\epsilon}}_{V_{dq,i}} &= \mathbf{J}_{L,i} \boldsymbol{\epsilon}_{V_{dq,i}} - (\mathbf{I}_{L,dq,i}(\mathbf{V}_{dq,i}) - \mathbf{I}_{L,dq,i}(\mathbf{V}_{dq,i}^*)) \\ &\quad \cdots + \mathbf{K}_{L,i} (\mathbf{d}_i - \mathbf{d}_i^*), \end{aligned} \quad (7a)$$

$$\mathbf{z}_i - \mathbf{z}_i^* = \mathbf{K}_{L,i}^\top \boldsymbol{\epsilon}_{V_{dq,i}}, \quad (7b)$$

$$H_{L,i}(\boldsymbol{\xi}_{L,i}) = \frac{1}{2} \boldsymbol{\xi}_{L,i}^\top \text{Diag} \left[ \frac{1}{C_i}, \frac{1}{C_i} \right] \boldsymbol{\xi}_{L,i} \quad (7c)$$

*Remark 2.* Note that the current references  $L_{ti}I_{d,i}^*, L_{ti}I_{q,i}^*$  in (2) as well as the line equilibria (cf. (13) in Strehle et al. (2019a))

$$\mathbf{x}_{ij}^* = [L_{ij}I_{d,ij}^*, L_{ij}I_{q,ij}^*]^\top \quad (8)$$

are not explicitly specified and follow as a consequence of the load demand and node voltages.

With (7), we can formulate the following extended version of our former Proposition 7 from Strehle et al. (2019a):

*Proposition 1.* An AC microgrid with an arbitrary topology represented by a bipartite graph as in Figure 2 consisting of  $\pi$ -model lines, strictly passive DGU PHSs, and strictly passive nonlinear static ZIP and exponential loads described by (7) is itself strictly passive. Its asymptotically stable equilibrium  $\mathbf{x}_M^*$  is given by the combined equilibria  $\mathbf{x}_{ij}^*$ ,  $\mathbf{x}_i^*$ , and  $\mathbf{x}_{L,i}^*$  of the individual subsystems.

**Proof.** From Strehle et al. (2019a) we know that  $\pi$ -model lines and DGUs with our plug-and-play controller both

are strictly passive systems. The minima of their storage functions (Hamiltonians) are  $\mathbf{x}_{ij}^*$  and  $\mathbf{x}_i^*$ , respectively. The equilibria of the load nodes are the minima of (7c) which is  $\boldsymbol{\xi}_{L,i} = 0$  implying (4) and thus the unknown-steady voltages  $\mathbf{V}_{dq,i}^*$ . If the load model error system (7) is strictly passive for ZIP and exponential loads, respectively, then an AC microgrid with arbitrary topology comprises only strictly passive subsystems. The remainder of the proof follows according to Strehle et al. (2019a) via the interconnection of passive systems and Lyapunov's direct method. ■

In light of Proposition 1 and our former results, the voltage and frequency stability analysis of AC microgrids with arbitrary topologies reduces to investigating under which conditions the error system (7) of nonlinear static ZIP and exponential AC loads is strictly passive. For this, we first establish the static load current-voltage relation  $\mathbf{I}_{L,dq,i}(\mathbf{V}_{dq,i})$ .

### 3. MODELING OF NONLINEAR STATIC AC LOADS

In order to obtain  $\mathbf{I}_{L,dq,i}(\mathbf{V}_{dq,i})$ , we introduce standard AC load models described by the consumed active and reactive powers (Section 3.1). Then, we derive the desired current equations in the  $dq$  frame (Section 3.2).

#### 3.1 ZIP and Exponential AC Loads

The most common nonlinear static AC load models are polynomial and exponential models described by the active and reactive powers  $P_L(V)$  and  $Q_L(V)$  as voltage-dependent functions (Machowski et al., 2008, pp. 111-112)(Cañizares et al., 2018, pp. 33-34)(van Cutsem and Vournas, 1998, pp. 95ff). Polynomial models comprise constant impedance ( $a_Z$ ), constant current ( $a_I$ ) and constant power ( $a_P$ ) coefficients, leading to the ZIP load equations

$$P_L(V) = P_0 \left[ a_{Z,P} \left( \frac{V}{V_0} \right)^2 + a_{I,P} \left( \frac{V}{V_0} \right) + a_{P,P} \right], \quad (9a)$$

$$Q_L(V) = Q_0 \left[ a_{Z,Q} \left( \frac{V}{V_0} \right)^2 + a_{I,Q} \left( \frac{V}{V_0} \right) + a_{P,Q} \right], \quad (9b)$$

where  $V_0$  is the nominal phase-to-phase RMS value (e.g. 400 V), and  $P_0$  and  $Q_0$  are the nominal active and reactive powers. By grouping the coefficients and nominal values in (9) into the model parameters

$$Y_P = \frac{a_{Z,P}P_0}{V_0^2}, \quad I_P = \frac{a_{I,P}P_0}{V_0}, \quad P_P = a_{P,P}P_0, \quad (10a)$$

$$Y_Q = \frac{a_{Z,Q}Q_0}{V_0^2}, \quad I_Q = \frac{a_{I,Q}Q_0}{V_0}, \quad P_Q = a_{P,Q}Q_0, \quad (10b)$$

we obtain the simplified equations

$$P_L(V) = Y_P V^2 + I_P V + P_P, \quad (11a)$$

$$Q_L(V) = Y_Q V^2 + I_Q V + P_Q. \quad (11b)$$

Note that the constant impedances (Z) are expressed as admittances (Y). Exponential load models, on the other hand, are given by

$$P_L(V) = P_0 \left( \frac{V}{V_0} \right)^{n_P}, \quad (12a)$$

$$Q_L(V) = Q_0 \left( \frac{V}{V_0} \right)^{n_Q}, \quad (12b)$$

where the model parameters are the voltage indexes  $n_P$  and  $n_Q$  of the active and reactive power, respectively.

*Remark 3.* As per (Machowski et al., 2008, pp. 110-112), the models in (9), (11), and (12) are only accurate above  $0.7V_0$ . Below  $0.7V_0$ , real loads typically exhibit a rapid power drop and are only modeled as constant impedances ( $Z$ ) or rather admittances ( $Y$ )

$$P_L(V) = P_0 \left[ a_{Z,P} \left( \frac{V}{V_0} \right)^2 \right] = Y_P V^2, \quad (13a)$$

$$Q_L(V) = Q_0 \left[ a_{Z,Q} \left( \frac{V}{V_0} \right)^2 \right] = Y_Q V^2. \quad (13b)$$

The combination of (13) for  $V < 0.7V_0$  and (11) and (12), respectively, for  $V \geq 0.7V_0$  is referred to as two-tier load model. However, (13) is a special case of (11) for  $I_P, I_Q, P_P, P_Q = 0$ . Thus, for the sake of brevity, we focus on (11) and (12) in the sequel.

### 3.2 Power Conjugated Voltage-Current Functions of Loads

Relating the power-conjugated input and output as  $\mathbf{I}_{L,dq,i}(\mathbf{V}_{dq,i})$  is achieved by considering the voltage-dependent powers  $P_L$  and  $Q_L$  of the loads.

*Lemma 1.* (Current equations of a load). A balanced, nonlinear, static three-phase AC load described by voltage-dependent active and reactive power equations,  $P_L(V)$  and  $Q_L(V)$ , respectively, can be described by the  $dq$  current equations

$$\mathbf{I}_{L,dq}(\mathbf{V}_{dq}) = \frac{1}{\sqrt{2}} \begin{bmatrix} P_L(V) & Q_L(V) \\ -Q_L(V) & P_L(V) \end{bmatrix} \begin{bmatrix} V_d \\ V_q \end{bmatrix}, \quad (14)$$

where  $V$  is the amplitude (2-norm) of the  $dq$  load voltage vector  $\mathbf{V}_{dq}$  (cf. (1)), i.e.

$$V^2 = V_d^2 + V_q^2. \quad (15)$$

**Proof.** Let the instantaneous complex power of a load  $S_L$  be expressed in terms of  $dq$  voltages  $\mathbf{V}_{dq}$  and currents  $\mathbf{I}_{L,dq}$  as in Schiffer et al. (2016) (van Cutsem and Vournas, 1998, p. 57)

$$\begin{aligned} S_L &= P_L + iQ_L \\ &= (V_d I_{L,d} + V_q I_{L,q}) + i(V_q I_{L,d} - V_d I_{L,q}). \end{aligned} \quad (16)$$

By equating the real and imaginary parts of (16), we obtain the linear system

$$\begin{bmatrix} P_L \\ Q_L \end{bmatrix} = \begin{bmatrix} V_d & V_q \\ V_q & -V_d \end{bmatrix} \begin{bmatrix} I_{L,d} \\ I_{L,q} \end{bmatrix}. \quad (17)$$

We then solve (17) for  $\mathbf{I}_{L,dq}$ . This yields

$$I_{L,d}(\mathbf{V}_{dq}) = \frac{V_d P_L(V) + V_q Q_L(V)}{V_d^2 + V_q^2}, \quad (18a)$$

$$I_{L,q}(\mathbf{V}_{dq}) = \frac{V_q P_L(V) - V_d Q_L(V)}{V_d^2 + V_q^2}, \quad (18b)$$

which is equivalent to (14). ■

By applying Lemma 1, we calculate the power-conjugated voltage-current functions for the ZIP load in (11) as

$$\mathbf{I}_{L,dq}(\mathbf{V}_{dq}) = \begin{bmatrix} \frac{P_P V_d + P_Q V_q}{V^2} + \frac{I_P V_d + I_Q V_q}{V} \\ + Y_P V_d + Y_Q V_q \\ \hline \frac{P_P V_q - P_Q V_d}{V^2} + \frac{I_P V_q - I_Q V_d}{V} \\ + Y_P V_q - Y_Q V_d \end{bmatrix}, \quad (19)$$

and for the exponential load in (12) as

$$\mathbf{I}_{L,dq}(\mathbf{V}_{dq}) = \begin{bmatrix} \frac{P_0 V^{n_P-2}}{V_0^{n_P}} V_d + Q_0 \frac{V^{n_Q-2}}{V_0^{n_Q}} V_q \\ \hline \frac{P_0 V^{n_P-2}}{V_0^{n_P}} V_q - Q_0 \frac{V^{n_Q-2}}{V_0^{n_Q}} V_d \end{bmatrix}. \quad (20)$$

## 4. PASSIVITY OF NONLINEAR STATIC AC LOADS

In this section, we establish sufficient conditions for the strict passivity of the ZIP and exponential load model error systems (7) with (19) and (12), respectively. During the passivity analysis, we use the following established mathematical results, which we repeat here for consistency:

*Lemma 2.* (Monotonicity of differentiable mappings) A differentiable mapping  $\mathbf{f}: \mathbb{R}^m \rightarrow \mathbb{R}^m$  is monotone if and only if its Jacobian  $\nabla \mathbf{f}(\mathbf{u})$  is positive semi-definite for all  $\mathbf{u}$ . Furthermore, it is strictly monotone if  $\nabla \mathbf{f}(\mathbf{u})$  is positive definite for all  $\mathbf{u}$ . (Rockafellar and Wets, 1998, Prop. 12.3)

*Lemma 3.* (Positive definiteness of real quadratic matrices) Any real quadratic matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ , not necessarily symmetric, is positive (semi)-definite, if and only if its symmetric part  $\mathbf{A}_s = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$  is positive (semi)-definite. (Rockafellar and Wets, 1998, pp. 533-534)

*Remark 4.* For clarity in the subsequent analysis, the subscript  $i$  is dropped from all variables and parameters in this section, i.e.  $\mathbf{V}_{dq} := \mathbf{V}_{dq,i}$  etc.

### 4.1 Passivity conditions

*Proposition 2.* The load model error system (7) is strictly passive, if

- for ZIP load models with (19) it holds that

$$Y_P + \frac{I_P}{2V} > 0, \quad (21a)$$

$$\begin{aligned} Y_P^2 V^4 + Y_P I_P V^3 &> \frac{1}{4} I_Q V^2 + (I_P P_P + I_Q P_Q) V \dots \\ &\dots + (P_P^2 + P_Q^2). \end{aligned} \quad (21b)$$

- for exponential load models with (20) it holds that

$$n_P P_0 > 0, \quad (22a)$$

$$4(n_P - 1) P_0^2 \left( \frac{V}{V_0} \right)^{2n_P} > (n_Q - 2)^2 Q_0^2 \left( \frac{V}{V_0} \right)^{2n_Q} \quad (22b)$$

**Proof.** The load model error system (7) is strictly passive w.r.t. the supply rate  $(\mathbf{z} - \mathbf{z}^*)^T (\mathbf{d} - \mathbf{d}^*)$  and the positive definite storage function (7c), i.e.  $H_L(\boldsymbol{\xi}_L) : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ ,  $H_L(\mathbf{0}) = 0$ ,  $H_L(\boldsymbol{\xi}_L) > 0 \boldsymbol{\xi}_L \neq \mathbf{0}$ , if

$$\begin{aligned} \dot{H}_L(\boldsymbol{\xi}_L) &= \boldsymbol{\epsilon}_{V_{dq}}^T (\mathbf{J}_L \boldsymbol{\epsilon}_{V_{dq}} - (\mathbf{I}_{L,dq}(\mathbf{V}_{dq}) - \mathbf{I}_{L,dq}(\mathbf{V}_{dq}^*))) \\ &+ \boldsymbol{\epsilon}_{V_{dq}}^T \mathbf{K}_L (\mathbf{d} - \mathbf{d}^*) < (\mathbf{z} - \mathbf{z}^*)^T (\mathbf{d} - \mathbf{d}^*) \end{aligned} \quad (23)$$

holds. This is given if

$$-\epsilon_{\mathbf{V}_{dq}}^T (\mathbf{I}_{L,dq}(\mathbf{V}_{dq}) - \mathbf{I}_{L,dq}(\mathbf{V}_{dq}^*)) < 0, \quad (24)$$

which, by substituting (5), results in

$$(\mathbf{V}_{dq} - \mathbf{V}_{dq}^*)^T (\mathbf{I}_{L,dq}(\mathbf{V}_{dq}) - \mathbf{I}_{L,dq}(\mathbf{V}_{dq}^*)) > 0. \quad (25)$$

In order to derive conditions under which (25) is fulfilled, we consider the load current function  $\mathbf{I}_{L,dq}(\mathbf{V}_{dq})$  as a differentiable mapping  $\mathbf{I}_{L,dq} : \mathcal{U} \subset \mathbb{R}^2 \rightarrow \mathcal{Y} \subset \mathbb{R}^2$ . According to (Rockafellar and Wets, 1998, Def. 12.1), such a mapping  $\mathbf{I}_{L,dq}(\mathbf{V}_{dq})$  is called strictly monotone, if (25) is fulfilled. Thus, investigating the strict monotonicity of  $\mathbf{I}_{L,dq}(\mathbf{V}_{dq})$  serves as proxy for investigating the strict passivity of (7). To determine whether  $\mathbf{I}_{L,dq}(\mathbf{V}_{dq})$  represents a strictly monotone mapping, we use Lemma 2 and Lemma 3.

Consequently, we first derive the Jacobian

$$\nabla \mathbf{I}_{L,dq}(\mathbf{V}_{dq}) = \frac{\partial \mathbf{I}_{L,dq}(\mathbf{V}_{dq})}{\partial \mathbf{V}_{dq}} = \begin{bmatrix} \frac{\partial I_{L,d}}{\partial V_d} & \frac{\partial I_{L,d}}{\partial V_q} \\ \frac{\partial I_{L,q}}{\partial V_d} & \frac{\partial I_{L,q}}{\partial V_q} \end{bmatrix}, \quad (26)$$

and extract its symmetric part

$$\frac{\nabla \mathbf{I}_{L,dq}(\mathbf{V}_{dq}) + \nabla \mathbf{I}_{L,dq}^T(\mathbf{V}_{dq})}{2} =: \begin{bmatrix} a & b \\ b & c \end{bmatrix}. \quad (27)$$

Note that the  $\mathbf{V}_{dq}$  dependence of  $V$  (see (15)) must be taken into account when calculating the partial derivatives in (26).

For the ZIP current function in (19), (27) results in

$$a = Y_P + \frac{I_P V_q^2 - I_Q V_d V_q}{V^3} - \frac{P_P (V_d^2 - V_q^2) + 2P_Q V_d V_q}{V^4} \quad (28a)$$

$$b = \frac{I_Q V_d^2 - I_P V_d V_q}{V^3} + \frac{P_Q (V_d^2 - V_q^2) - 2P_P V_d V_q}{V^4} \quad (28b)$$

$$c = Y_P + \frac{I_P V_d^2 + I_Q V_d V_q}{V^3} + \frac{P_P (V_d^2 - V_q^2) + 2P_Q V_d V_q}{V^4} \quad (28c)$$

Similarly, for the exponential current function in (20), (27) is given by

$$a = \frac{P_0 [(n_P - 1)V_d^2 + V_q^2]}{V_0^{n_P} V^{4-n_P}} + \frac{(n_Q - 2)Q_0 V_d V_q}{V_0^{n_Q} V^{4-n_Q}} \quad (29a)$$

$$b = \frac{(n_P - 2)P_0 V_d V_q}{V_0^{n_P} V^{4-n_P}} + \frac{(n_Q - 2)Q_0 [V_d^2 - V_q^2]}{2V_0^{n_Q} V^{4-n_Q}} \quad (29b)$$

$$c = \frac{P_0 [V_d^2 + (n_P - 1)V_q^2]}{V_0^{n_P} V^{4-n_P}} - \frac{(n_Q - 2)Q_0 V_d V_q}{V_0^{n_Q} V^{4-n_Q}} \quad (29c)$$

Then, we find the eigenvalues  $\lambda_{1,2}$  of (27) with (28) and (29) using computer algebra software. For ZIP loads with (27) and (28), this results in

$$\lambda_{1,2} = Y_P + \frac{I_P}{2V} \pm \frac{1}{V^2} \left[ \frac{1}{4}(I_P^2 + I_Q^2)V^2 \cdots + (I_P P_P + I_Q P_Q)V + (P_P^2 + P_Q^2) \right]^{\frac{1}{2}}, \quad (30)$$

and for exponential loads (27) and (29) in

$$\lambda_{1,2} = \frac{1}{2V^2} \left( n_P P_0 \left( \frac{V}{V_0} \right)^{n_P} \pm \left[ (n_P - 2)^2 P_0^2 \left( \frac{V}{V_0} \right)^{2n_P} \cdots + (n_Q - 2)^2 Q_0^2 \left( \frac{V}{V_0} \right)^{2n_Q} \right]^{\frac{1}{2}} \right). \quad (31)$$

Finally, we evaluate the conditions under which (30) and (31) are positive to infer positive definiteness of (27) with (28) and (29), respectively. As we are considering eigenvalues of symmetric matrices, their values are always real (Rugh, 1996, p. 8). Consequently, the  $\lambda_2$ , where the square root terms are subtracted, are critical when investigating the positiveness of (30) and (31). With this preliminary consideration and Assumption 1, we obtain (21a) and

$$Y_P V^2 + \frac{I_P}{2} V > \left[ \frac{1}{4}(I_P^2 + I_Q^2)V^2 + (I_P P_P + I_Q P_Q)V \cdots \cdots + (P_P^2 + P_Q^2) \right]^{\frac{1}{2}} \quad (32)$$

from (30), and (22a) and

$$n_P P_0 \left( \frac{V}{V_0} \right)^{n_P} > \left[ (n_P - 2)^2 P_0^2 \left( \frac{V}{V_0} \right)^{2n_P} \cdots \cdots + (n_Q - 2)^2 Q_0^2 \left( \frac{V}{V_0} \right)^{2n_Q} \right]^{\frac{1}{2}} \quad (33)$$

from (31). Since both sides in (32) and (33) are positive, by squaring them, we arrive at (21b) and (22b), respectively. ■

*Remark 5.* For  $V < 0.7V_0$ , we insert  $I_P, I_Q, P_P, P_Q = 0$  (cf. Remark 3) in (21) to obtain

$$Y_P > 0 \quad (34)$$

as sufficient condition for the strict passivity of (7).

*Remark 6.* Equation (25) requires that the product of the power-conjugated input (voltage difference) and output (current difference) be positive. This corresponds to the incremental passivity, or more specifically to the shifted passivity (van der Schaft, 2017, p. 136) also referred to as equilibrium-independent passivity (Arcak et al., 2016, p. 24), of static systems, i.e. memoryless systems with zero storage functions (cf. (Khalil, 2002, p. 228) for passivity of memoryless functions). This in turn comes full circle to the monotonicity of the resistive relation specified by the voltage-dependent load current function  $\mathbf{I}_{L,dq,i}(\mathbf{V}_{dq,i})$  (van der Schaft and Jeltsema, 2014, p. 92). When considering passivity as a special case of dissipativity with specific quadratic supply rate, a similar link can be made to equilibrium-independent dissipativity of nonlinearities (cf. (10) in Simpson-Porco (2019))(Arcak et al., 2016, p. 24).

*Remark 7.* Note that the DC case is a special case of the presented AC case where  $Q_L(V)$  and all  $q$  components of  $dq$  quantities are set to zero. Defining the voltage amplitude  $V =: V_d$ , we can rewrite (21) into the single DC load current equation

$$I_L(V) = \frac{P_L(V)}{V}. \quad (35)$$

The passivity analysis then follows analogously to the monotonicity considerations of Section 4. For the ZIP load (11a), (35) results in

$$I_L(V) = Y_P V + I_P + \frac{P_P}{V}. \quad (36)$$

and the restriction

$$\frac{dI_L(V)}{dV} = Y_P - \frac{P_P}{V^2} > 0, \quad (37)$$

which is a well-established result in pertinent literature Nahata et al. (2020) Cucuzzella et al. (2019) Strehle et al. (2019b).

## 5. SIMULATION

In this section, we evaluate the results in Section 4 by simulating a load node in MATLAB/SIMULINK under various parameter configurations. We show that voltage stability (implying frequency stability under Assumption 2) is not guaranteed a priori as violating our sufficient conditions can lead to instability. For this, we consider a voltage source at node  $j$  connected via an electrical line to a load node  $i$  as in Figure 1. In contrast to the direct parallel connection of the load node  $i$  with a voltage source, this creates a non-stiff system in which the load node can exhibit instability independent of the source supplying it. The voltage source is set to a constant amplitude of  $V_j = 400$  V for the  $dq$  voltage vector  $\mathbf{V}_{dq,j}$ . Any stabilizing or destabilizing behavior thus arises from within the system, i.e. the load behavior. The system is simulated with both a ZIP load and an exponential load, as per (19) and (20), respectively. The line and load parameters are described in Table 1. The line parameters are derived from Strehle et al. (2019a) for a 1 km line. The loads parameters are changed during the course of the simulation to alternate between satisfying and violating the restrictions (32) and (33), respectively.

The results of the simulation are shown in Figures 3 and 4, respectively. In each figure, the voltage amplitude as per (15) is shown along with the changes made to the load parameters. Furthermore, sufficient voltage limits for strict passivity, which we calculated by solving (32) and (33) for the respective load parameters, are also indicated in the figures. Strict passivity and thus asymptotic voltage stability (implying frequency stability under Assumption 2) is only guaranteed when the operating voltage amplitude  $V_i$  of the load node, induced by the input voltage  $\mathbf{V}_{dq,j}$ , falls within this range. If these conditions are not met, the load could reverse the polarity of its dissipation, potentially resulting in instability. Note that due to the non-zero line impedance, changes to the ZIP load parameters, which result in differing current magnitudes, yield different operating points for  $V_i$  as in Figure 3. However, since the exponents of exponential loads change the nature of the loads without the total power drawn at the nominal

Table 1. Simulation Parameter Values

Parameter	Value	Change 1		Change 2	
		$(t = 0.5 \text{ s})$	$(t = 1.0 \text{ s})$	$(t = 0.6 \text{ s})$	$(t = 1.1 \text{ s})$
ZIP Load	$Y_P$ (1/ $\Omega$ )	0.15	(0.1)	(0.2)	
	$Y_Q$ (1/ $\Omega$ )	0.05			
	$I_P$ (A)	2			
	$I_Q$ (A)	9			
	$P_P$ (kW)	4.5			
Exponential Load	$P_Q$ (kVAR)	19	[11]	(24)	
	$P_0$ (kW)	5.5			
	$Q_0$ (kVAR)	3.7			
Line	$n_P$	1.7	(1.1)	(1.3)	
	$n_Q$	0.7	[1.9]	(0.45)	
	$R_{ij}$ ( $\Omega$ )	0.01273			
	$L_{ij}$ (mH)	0.9337			
	$C_{ij}$ (nF)	12.74			

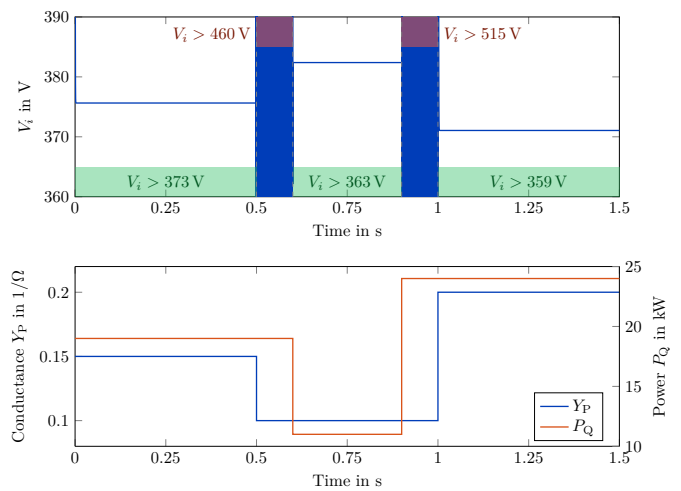


Figure 3. Stability for various ZIP load configurations

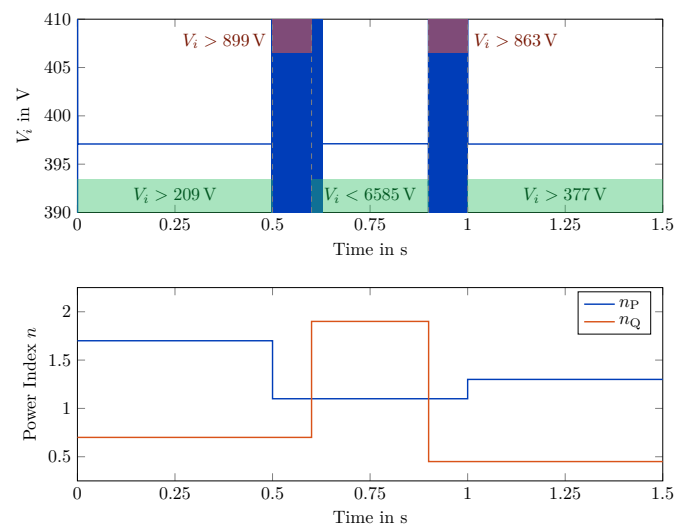


Figure 4. Stability for various exponential load configurations

voltage, the variations in the points of operation are not seen in Figure 4.

## 6. CONCLUSION

In this work, we presented sufficient inequality conditions for the strict passivity of the prevalent nonlinear static ZIP and exponential AC load models. Together with our former results from Strehle et al. (2019a), this allows us to infer asymptotic voltage and frequency stability of AC microgrids with arbitrary topologies in a plug-and-play manner via passivity arguments. Future research will include a more extensive simulative validation of our complete plug-and-play framework and investigate possibilities to broaden the feasible voltage operating area by providing necessary and sufficient inequality conditions.

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