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Time domain approach toward the calculation of the compliance function of flexible motion systems ⋆

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Abstract: For high-performance distributed parameter motion systems, the dynamics introduced by structural flexibilities need to be considered. Especially at the low frequency region, where most of the energy of the commonly used reference setpoint is concentrated. The contribution of non-rigid body modes at low frequencies is called the compliance function of the system. It is representative for the quasi-static behaviour of the whole non-rigid body modes. This work proposes a new method for the calculation of the compliance function. It is based on employing the differential equation representation for the flexible structure. The approach is validated for a standard damped second order ODE and a one-dimensional flexible model, *i.e.*, the Euler-Bernoulli beam. We show that we get a major reduction in calculation in comparison with the zero frequency response calculation. The extension of this approach to the general PDE's will be the scope of the future works.

Keywords: Compliance function, Partial differential equations, distributed parameter systems, flexible structures, double-integrator system, Euler-Bernoulli beam.

1. INTRODUCTION

High performance motion systems are designed to achieve nano-positioning performance. A typical example of these reference-tracking systems is wafer scanners in semiconductor industry. Wafer scanners are used to produce integrated circuits (ICs). As demanded, the system performs at high acceleration in order to obtain higher throughput, and thus high positioning accuracy becomes challenging.

Perfect tracking of desired trajectories can often be achieved by the use of feedforward control. A comprehensive review of feedforward control design can be found in Lunenburg et al. (2009). Feedforward control can be designed by either model-based or data-based approaches. Since, the major part of system dynamics and the desired output trajectory is known a priori, the model-based feedforward is responsible for 99.7% of the control forces and associated performance (Heertjes et al. (2010)). The feedback controller controls the disturbance of the system.

A common model-based feedforward control design is based on the inversion of the system dynamics as shown in Fig. 1. The feedforward control performance is strictly dependent on the description of system dynamics. The more detailed of the system dynamics are involved, the higher performance of the feedforward controller is achieved. Furthermore, the existence of the system inversion needs to be checked. For example, due to the presence of non-minimum phase (NMP) zeros, the inverse plant may become unstable. Hence, the use of model-based feedforward control can often be obtained in some approximate sense.

The most straightforward model-based feedforward controller is the acceleration/mass feedforward control, which is the inversion of only the rigid body mode of the system. Hence, it cannot compensate for the behaviour of the flexibility of the structural dynamics. Neglecting the flexible structures as well as the time-independent structure, makes this conventional feedforward control often insufficient for a time-varying system to obtain sufficiently high accuracy.

In order to increase tracking accuracy, higher order derivatives of the reference profile is usually added to the acceleration term (Lambrechts et al. (2004); Boerlage et al. (2004)). Snap feedforward control considers the fourth-order approximation of the plant inversion (Boerlage (2006)). It adds a jerk derivative to the traditional acceleration feedforward control structure which compensates for the entire flexible dynamics at low frequencies. The drawback of this feedforward control approach is that it only deals with the linear-time-invariant (LTI) systems since it cannot deal with time-varying dynamics.

Recent approaches deal with linear-time-varying (LTV) feature of flexible positioning systems. Two recent approaches toward feedforward control design for LTV and linear parameter vary-

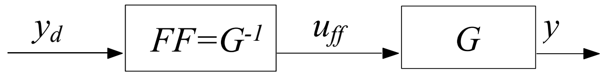

Fig. 1. Model-based feedforward control scheme

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ing (LPV) systems are studied in Kasemsinsup et al. (2016) and Kasemsinsup et al. (2017). Kasemsinsup et al. (2016) provides an algebraic analysis along with conditions to construct an exact model-based feedforward control for LTV/LPV systems. Kasemsinsup et al. (2017) investigates the optimal feedforward control design for LTV/LPV systems. The feedforward control signal is constructed by a linear combination of basis functions which have optimal coefficients computed by a quadratic optimization process.

Different from previous approaches, for an efficient formulation of the flexible dynamics, the term compliance is introduced as a deformation of the system due to the externally applied force (Huston (1981)). In Kontaras et al. (2016), compliance compensation control scheme is discussed as feedforward control technique for time-varying motion systems. This method adds the compliance function of flexible structures as a correction block to the control structure. Mathematically spoken, the compliance function is calculated as zero frequency response of the flexible part of the transfer function of the system. Hence, by adding the compliance feedforward correction block, at low frequencies, the feedforward controller can take the full flexible structure into account.

The main contribution of this work is to derive the compliance function of a dynamic system using a direct time domain approach rather than using the frequency response. The approach significantly facilitates the derivation of compliance function especially for systems described by irrational transfer function. In a later stage, the model is aimed to be extended to 2-D cases, which resembles much better the flexible scanning system.

The outline of this work is as follows, in Section 2 the mathematical representation of the flexible system is introduced and the expected time domain solution of the system is explained. Section 3 presents the class of model to be investigated and its solution. Next, in section 4 two examples are studied as a benchmark of the approach. Two mass-spring-damper system and damped Euler Bernoulli beam are studied as flexible systems with one and infinite number of non-rigid body (NRB) modes, respectively. The derivation of the model solution and correspondingly the compliance function show major reduction in calculation process in comparison with the frequency domain approach. Concluding remarks are given in section 5.

2. SYSTEM DYNAMICS

Flexible systems, in a simplified representation, are double-integrator-based systems including one RB mode and an (in)finite number of NRB modes. The transfer function of these systems is given by

$$G(s) = \frac{1}{ms^2} + G_{flex}(s), \tag{1}$$

where G_{flex} is representative for flexible structures. It is a strictly proper transfer function including the entire flexible structures. G_{flex} can be divided into the compliant and the (damped) resonant dynamics as

$$G_{flex}(s) = G_0 + G_{st}(s), \tag{2}$$

where G_0 is the compliance function of the system which equals the quasi-static behaviour of the entire flexible structures until the occurrence of the first resonance. In other words, G_0 is the zero-frequency response of the structural dynamics, i.e., $G_0 = G_{flex}(0)$. $G_{st}(s)$ is the stable resonant dynamics. Flexible motion systems mostly perform in the low frequency range where the resonant dynamics are less excited. Thus, the plant can be approximated by the low frequency contribution of flexible structures of the system. In other words, the need to incorporate many NRB modes at low frequencies can be relaxed by only taking into account the compliance function of the system.

$$G_{appr}(s) = \frac{1}{ms^2} + G_0. \tag{3}$$

$G_{appr}(s)$ is used in the model-based feedforward control design, where the full system dynamics is taken into account in the accepted frequency content. Thus, once the compliance function G_0 is calculated, the low frequency dynamics of the flexible motion system is fully determined. One way to determine $G_0(s)$ is first to determine the transfer function of the system, next to subtract the rigid-body (RB) mode and finally determining $G_{flex}(0)$. However, this is non-trivial, certainly for complicated models, such those described by partial differential equations. Therefore we take a different route.

If we consider the step function as an actuator input of our mechanical system, then for zero initial conditions the Laplace transform of the output is given by (see (1) and (2))

$$Y(s) = \left(\frac{1}{ms^2} + G_0 + G_{st}(s)\right) U(s), \tag{4}$$

where

$$U(s) = \frac{u_0}{s}. \tag{5}$$

In time domain the output becomes

$$y(t) = \frac{1}{m}\frac{t^2}{2}u_0 + G_0 u_0 + y_{st}(t), \tag{6}$$

where $y_{st}(t)$ is the inverse Laplace transform of $G_{st}(s)\frac{u_0}{s}$. Since G_{st} is stable, and $G_{st}(0) = 0$, this expression has no poles in the closed right-half plane, and so $y_{st}(t)$ converges to zero as $t \to \infty$. In other words, $y_{st}(t)$ is the (asymptotically) stable part of the solution.

So we see that the output in time domain has a clear form in which the compliance can be read of directly. Therefore, the anticipated output format for flexible motion systems is taken in accordance with (6). Hence the following is considered as the expected solution of these systems

$$\omega_{assumed}(t) = \omega_2 \frac{t^2}{2} + \omega_0 + \omega_{st}(t). \tag{7}$$

Here ω_2 is the part of solution that stems from the double-integrator-based part of the system and it equals, for $u_0 = 1$, the inverse of the total mass of the system. The term ω_0 is the compliance function of the dynamic system which is denoted by the G_0 term in the frequency response. It is the static response of the NRB modes at zero frequency. $\omega_{st}(t)$ accounts for the stable resonant dynamics of NRB modes.

3. MODELLING FRAMEWORK

The class of damped equations to be considered in this section is the following:

$$\ddot{\omega}(t) + d\mathscr{A}\dot{\omega}(t) + \mathscr{A}\omega(t) = Bu(t), \tag{8}$$

where $\mathscr{A}$ is $n \times n$ matrix and d is a positive scalar; $d > 0$. The aim is to find the solution, and specifically the unique compliance function ω_0 of these damped systems. To do so, we take $u(t) = u_0, t \geq 0$, and the following initial conditions:

$$\omega(0) = 0, \quad \dot{\omega}(0) = 0. \tag{9}$$

Substituting this and the anticipated solution $\omega_{assumed}$ (7) into (8) gives the following:

$$\begin{aligned}\omega_2 + \ddot{\omega}_{st}(t) + d\mathscr{A}(\omega_2 t + \dot{\omega}_{st}(t)) + \\ \mathscr{A}(\omega_2 \frac{t^2}{2} + \omega_0 + \omega_{st}(t)) = Bu_0.\end{aligned} \tag{10}$$

Collecting the t^2-term, t-term, constant term, and the stable part, equation (10) gives the following dependent equations for ω_2, ω_0 and ω_{st}:

$$\mathscr{A}\omega_2 = 0, \tag{11}$$

$$\mathscr{A}\omega_0 + \omega_2 = Bu_0, \tag{12}$$

$$\ddot{\omega}_{st}(t) + d\mathscr{A}\dot{\omega}_{st}(t) + \mathscr{A}\omega_{st}(t) = 0, \tag{13}$$

$$\omega_{st}(0) = -\omega_0, \quad \dot{\omega}_{st}(0) = 0.$$

Obviously, ω_0 cannot be derived uniquely unless the other terms of $\omega_{assumed}$ is determined. Consequently, each term of the anticipated solution $\omega_{assumed}$ as proposed in (7) needs to be ascertained for these class of systems.

The initial conditions for the stable function $\omega_{st}(t)$ in (13) verifies that the compliance function ω_0 lies in the stable space of the solution. Hence, the ensured stability of the compliance function adds one more extra condition to find out the unique terms of $\omega_{assumed}$. The extra equation is obtained via the following theorem.

Theorem 1. Let the state space X have inner product $\langle x, y\rangle$, and let $\mathscr{A}$ be symmetric and non-negative with respect to this inner product. Assume further that the kernel of $\mathscr{A}$ is one-dimensional. Then X can be written as

$$X = \text{span}\{\varphi_0, \varphi_1, \varphi_2, ..., \varphi_{n-1}\}, \tag{14}$$

where $\mathscr{A}\varphi_k = \lambda_k \varphi_k$, $\lambda_0 = 0$, and $\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_{n-1}$ form an orthonormal basis of the space.

The equations (11) and (12) are uniquely determined by

$$\omega_2 = \alpha\varphi_0, \quad \omega_0 = \sum_{k=1}^{n-1} \beta_k \varphi_k, \tag{15}$$

where

$$\alpha = \langle \varphi_0, Bu_0\rangle, \tag{16}$$

and

$$\beta_k = \frac{\langle \varphi_k, Bu_0\rangle}{\lambda_k}. \tag{17}$$

Proof 1. Since $\mathscr{A}$ is symmetric it has an orthonormal basis of eigenvectors. Furthermore, since the kernel is one-dimensional we may always choose the first eigenvalue to span this space. Thus the orthonormal basis of the space satisfies the following:

$$\mathscr{A}\varphi_k = \lambda_k \varphi_k, \quad \text{with} \begin{cases} \lambda_k > 0 & \text{if } k > 0 \\ \lambda_0 = 0 & , \end{cases} \tag{18}$$

where λ_k are the eigenvalues $\mathscr{A}$.

Equation (11) gives that ω_2 lies in the kernel of $\mathscr{A}$ and by our assumption on this kernel, we find that

$$\omega_2 = \alpha\varphi_0. \tag{19}$$

Next we solve (13). It is not hard to see that the general solution, i.e., disregarding the initial conditions, of this second order differential equation is given by

$$\begin{aligned}\omega_{st}(t) = \sum_{k=1}^{n-1} \left(\beta_{k,1} e^{\mu_{k,1} t} + \beta_{k,2} e^{\mu_{k,2} t}\right)\varphi_k \\ + (\beta_{0,1} + \beta_{0,2} t)\varphi_0,\end{aligned} \tag{20}$$

where

$$\mu_{k,i} = \frac{-d\lambda_k \pm \sqrt{d^2\lambda_k^2 - 4\lambda_k}}{2}, \quad i = 1, 2. \tag{21}$$

Since by assumption ω_{st} must be stable, we find that $\beta_{0,1} = \beta_{0,2} = 0$. In particular, this implies that $\omega_0 = -\omega_{st}(0) \in \text{span}\{\varphi_1, \varphi_2, ..., \varphi_{n-1}\}$. So we have shown the second assertion in (15). Using this, equations (19) and (12) gives the following:

$$\alpha\varphi_0 + \mathscr{A}\sum_{k=1}^{n-1} \beta_k \varphi_k = Bu_0. \tag{22}$$

By taking the inner product of (22) with φ_0 we find

$$\langle \varphi_0, \alpha\varphi_0\rangle + \langle \varphi_0, \mathscr{A}\sum_{k=1}^{n-1} \beta_k \varphi_k\rangle = \langle \varphi_0, Bu_0\rangle, \tag{23}$$

Using the orthogonality of eigenvectors gives the unique coefficient for ω_2 in (16).

Next taking the inner product of (22) with φ_{k_0} gives the following:

$$\langle \varphi_{k_0}, \alpha\varphi_0\rangle + \langle \varphi_{k_0}, \mathscr{A}\sum_{k=1}^{n-1} \beta_k \varphi_k\rangle = \langle \varphi_k, Bu_0\rangle, \tag{24}$$

and hence

$$\lambda_{k_0}\beta_{k_0} = \langle \varphi_{k_0}, Bu_0\rangle. \tag{25}$$

Thus, (25) gives the unique coefficients for finding ω_0 in (17).

As we have seen ω_{st} is given by

$$\omega_{st}(t) = \sum_{k=1}^{n-1} \left(\beta_{k,1} e^{\mu_{k,1} t} + \beta_{k,2} e^{\mu_{k,2} t}\right)\varphi_k. \tag{26}$$

Choosing

$$\begin{aligned}\beta_{k,1} &= \frac{-\langle \omega_0, \varphi_k\rangle \mu_{k,2}}{\mu_{k,2} - \mu_{k,1}} \\ \beta_{k,2} &= \frac{\langle \omega_0, \varphi_k\rangle \mu_{k,1}}{\mu_{k,2} - \mu_{k,1}}\end{aligned} \tag{27}$$

we have that the initial conditions are satisfied. Furthermore, the fact that d and λ_k, $k \geq 1$ are positive, gives that ω_{st} is stable, see also (21), as requested. □

Remark 1. Since the eigenvectors form an orthonormal basis, we have that the condition that $\omega_0 \in \text{span}\{\varphi_1, \cdots, \varphi_{n-1}\}$ is equivalent to the condition that ω_0 is othogonal to φ_0.

Remark 2. It is clear from the proof that if the kernel of $\mathscr{A}$ would only contain the zero element, then the assumed solution (7) (with $\omega_2 \neq 0$) would not exist.

4. EXAMPLES

In this section, our direct compliance calculation approach is investigated by means of two examples. The first example consist a finite order (lumped parameter system) which falls in the class considered in Theorem 1. The second example is an infinite order (distributed parameter system) motion system. Although Theorem 1 does not directly apply, we will see that the ideas used in its proof are easily extendable to this case as well.

4.1 Example I: Mass-spring-damper system

As a first example, we consider a single-input-multiple output system, *i.e.*, a two mass spring damper (MSD) system which is shown in Fig. 2. Actuation occurs via a (constant) force applied

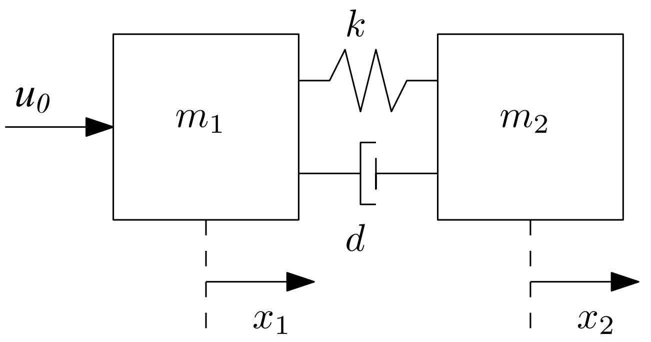

Fig. 2. Two mass spring damper (MSD) system

on the first mass. The equations of the motion of two masses with damping are given by

$$\begin{aligned} m_1\ddot{x}_1 &= d(\dot{x}_2-\dot{x}_1)+k(x_2-x_1)+u_0, \\ m_2\ddot{x}_2 &= d(\dot{x}_1-\dot{x}_2)+k(x_1-x_2), \end{aligned} \tag{28}$$

where m_1, m_2 denotes the masses of the system, k is the stiffness of the spring, d is the damping coefficient and u_0 is the constant force. x_1 and x_2 are the position of each mass. The output of the system can be determined either by considering x_1 or x_2 as the point of interest. Here, the non-collocated response of the system x_2, in which the sensor location is different from the actuator is adopted as an output. Representation of the system similar to (8) is given by

$$\begin{aligned} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\frac{d}{k}}_{d} \underbrace{\begin{bmatrix} \frac{k}{m_1} & \frac{-k}{m_1} \\ \frac{-k}{m_2} & \frac{k}{m_2} \end{bmatrix}}_{\mathscr{A}} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{k}{m_1} & \frac{-k}{m_1} \\ \frac{-k}{m_2} & \frac{k}{m_2} \end{bmatrix}}_{\mathscr{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ = \underbrace{\begin{bmatrix} \frac{1}{m_1} \\ 0 \end{bmatrix}}_{B} u_0. \end{aligned} \tag{29}$$

In this example, we have $\mathbb{R}^2$ as our state space with inner product defined as:

$$\langle a,b\rangle = m_1a_1b_1+m_2a_2b_2. \tag{30}$$

The MSD system lies in the class of damped system in (8). Hence, the assumed solution is applicable to this system in order to find the compliance vector of the MSD system.

$$\omega_{assumed}(t) = \omega_2\frac{t^2}{2}+\omega_0+\omega_{st}(t), \tag{31}$$

where ω_2 and ω_0 are elements of the state space.

Easy calculation gives that $\mathscr{A}$ is symmetric with respect to the inner product (30). The eigenvalues of $\mathscr{A}$ and the corresponding orthonormal basis eigenvectors are as follows:

$$\lambda_0 = 0 \Rightarrow \varphi_0 = \frac{1}{\sqrt{m_1+m_2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{32}$$

$$\lambda_1 = \frac{k}{m_1}+\frac{k}{m_2} \Rightarrow \varphi_1 = \frac{1}{\sqrt{m_1m_2^2+m_2m_1^2}}\begin{bmatrix} m_2 \\ -m_1 \end{bmatrix}. \tag{33}$$

Equation (16) in Theorem 1 gives the coefficient of ω_2 which is based on the zero eigenvalue.

$$\begin{aligned} \alpha &= \langle\varphi_0, Bu_0\rangle = \frac{1}{\sqrt{m_1+m_2}}u_0, \\ \Rightarrow \omega_2 &= \alpha\varphi_0 = \frac{u_0}{m_1+m_2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{34}$$

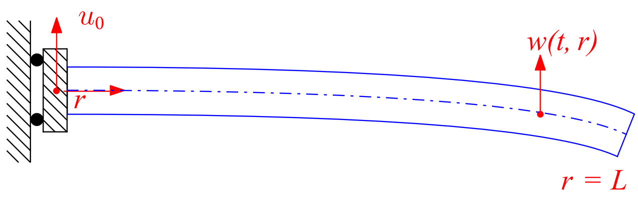

Fig. 3. The damped Euler-Bernoulli beam

Furthermore, (17) gives the unique coefficient of ω_0 as follows:

$$\beta_1 = \frac{\langle\varphi_1, Bu_0\rangle}{\lambda_1} = \frac{u_0m_2}{\lambda_1\sqrt{m_1m_2^2+m_2m_1^2}}. \tag{35}$$

This gives that

$$\begin{aligned} \omega_0 &= \beta_1\varphi_1 = \frac{u_0m_2}{\lambda_1(m_1m_2^2+m_2m_1^2)}\begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} \\ &= \frac{m_2}{k(m_1+m_2)^2}\begin{bmatrix} m_2 \\ -m_1 \end{bmatrix}u_0. \end{aligned} \tag{36}$$

Thus, the compliance vector of the MSD system is calculated by determining the stable subspace of the system which is expected from the definition of the compliance function (see (2)). The first and second element of (36) pertain to the compliance of the first and second mass, respectively. Moreover, if unit force ($u_0 = \mathbf{1}$) is applied, then $\omega(t)$ is given by

$$\omega(t) = \underbrace{\frac{1}{M}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\omega_2}\frac{t^2}{2} + \underbrace{\frac{m_2}{kM^2}\begin{bmatrix} m_2 \\ -m_1 \end{bmatrix}}_{\omega_0} + \omega_{st}(t), \tag{37}$$

where M is the total mass, *i.e.*, $M = m_1+m_2$. In terms of physical interpretation, the compliance function is the inverse of the stiffness of the system, which shows the resistance property of a system in response to an applied force. Consequently, the compliance is dealing with the flexibility of the system.

The compliance vector of the MSD system is validated with the frequency response of the NRB modes at low frequency (Kontaras et al. (2016)).

4.2 Example II: Damped Euler-Bernoulli beam

As a second example, the 1-D Euler-Bernoulli beam is investigated. The analysis of this infinite dimensional system can be considered as a useful step toward calculation of dynamics of flexible motion systems in more complex geometries. In order to better categorize the model similar to (8), damping is included in the model. Different types of damping can be used (Herrmann (2008)). In this example, the most common type of damping referred to Kelvin-Voigt damping is employed in modeling. The partial differential equation (PDE) describing the beam is given by

$$\rho A\frac{\partial^2\omega(t,r)}{\partial t^2}+c_d\frac{\partial^5\omega(t,r)}{\partial t\partial r^4}+EI\frac{\partial^4\omega(t,r)}{\partial r^4} = 0, \tag{38}$$

where ρ, I, A and E are the mass density, second moment of inertia, cross-sectional area, and the Young's modulus, respectively. We denote the spatial point by r and the length by L.

The output of the system is the deflection of the beam $\omega(t,r)$ at position r, and u_0 represents a constant input force applied to the left of the beam. The vertically moving Euler-Bernoulli beam is illustrated in Fig. 3. In order to find the deflection $\omega(t,r)$, four boundary conditions are defined in the following and are visualized in Fig. 3. We consider a roller support at one side of the beam that allows the beam to move vertically. So, the first boundary condition at $r = 0$ is based on the shearing force and is given as

$$EI \frac{\partial^3 \omega(t,r)}{\partial r^3}\bigg|_{r=0} = u_0. \tag{39}$$

Since the beam is clamped at the base, its slope is zero. This gives the second boundary condition.

$$\frac{\partial \omega(t,r)}{\partial r}\bigg|_{r=0} = 0. \tag{40}$$

The other two boundary conditions are located at the end of the beam which is free. Hence, both the bending moment and shearing force equal zero and the third and fourth boundary conditions are given by

$$\frac{\partial^2 \omega(t,r)}{\partial r^2}\bigg|_{r=L} = 0, \tag{41}$$

$$EI \frac{\partial^3 \omega(t,r)}{\partial r^3}\bigg|_{r=L} = 0. \tag{42}$$

As in the previous case, the initial condition of the system is assumed to equal zero.

$$\begin{aligned} \omega(0,r) &= 0, \\ \frac{\partial \omega}{\partial t}(0,r) &= 0. \end{aligned} \tag{43}$$

The PDE representing this example is similar to the damped class of equations in (8). It is formally given by

$$\ddot{\omega}(t) + \underbrace{\frac{c_d}{EI}}_{d} \mathscr{A}\dot{\omega}(t) + \mathscr{A}\omega(t) = 0, \tag{44}$$

where the dot denotes the derivative with respect to time, and $\mathscr{A}$ is the fourth order spatial derivative operator given by

$$\mathscr{A} = \frac{EI}{\rho A}\frac{\partial^4}{\partial r^4}. \tag{45}$$

Similar to the MSD system, the assumed solution for this mechanical system corresponds to the direct solution of PDE, and it is of the form, see also (7),

$$\omega_{assumed}(t,r) = \omega_2(r)\frac{t^2}{2} + \omega_0(r) + \omega_{st}(t,r).$$

Assuming that this is a (classical) solution of the PDE, we find equations for the ω_2, ω_0 and ω_{st} similar to those formulated in (11)–(13). However, next to differential equations which they should satisfy, also boundary conditions need to hold. For instance, for ω_2 we find the (differential) equation

$$\mathscr{A}\omega_2(r) = 0, \tag{46}$$

with boundary conditions

$$\begin{aligned} \frac{d\omega_2(0)}{dr} &= 0, \quad \frac{d^2\omega_2(L)}{dr^2} = 0, \\ \frac{d^3\omega_2(L)}{dr^3} &= 0, \quad \frac{d^3\omega_2(0)}{dr^3} = 0. \end{aligned} \tag{47}$$

Hence ω_2 is constant, *i.e.*,

$$\omega_2(r) = \omega_{20}. \tag{48}$$

Secondly, the following equation with three homogeneous and one non-homogeneous boundary condition needs to be solved for ω_0.

$$\mathscr{A}\omega_0(r) + \omega_2(r) = 0, \tag{49}$$

with

$$\begin{aligned} \frac{d\omega_0(0)}{dr} &= 0, \quad \frac{d^2\omega_0(L)}{dr^2} = 0, \\ \frac{d^3\omega_0(L)}{dr^3} &= 0, \quad EI\frac{d^3\omega_0(0)}{dr^3} = u_0. \end{aligned} \tag{50}$$

Using (48) and integrating (49) from 0 to L, gives the following:

$$\frac{EI}{\rho A}\left[\frac{d^3\omega_0(L)}{dr^3} - \frac{d^3\omega_0(0)}{dr^3}\right] + \omega_{20}L = 0. \tag{51}$$

Using the boundary conditions, we see that

$$\omega_2(r) = \omega_{20} = \frac{1}{\rho A L}u_0. \tag{52}$$

Next, ω_0 is calculated with (49) and non-homogeneous boundary conditions as

$$\omega_0(r) = \frac{-u_0}{24EI \cdot L}r^4 + \frac{u_0}{6EI}r^3 - \frac{u_0 L}{4EI}r^2 + C_4. \tag{53}$$

Obviously, this equation does not yield to the unique compliance function of the damped beam ω_0. Therefore one more condition is needed. After substitution of $\omega_{assumed}$ in the main equation (44), the p.d.e. derived associated with the stable part of the solution, ω_{st}, is given by

$$\ddot{\omega}_{st}(r,t) + d\mathscr{A}\dot{\omega}_{st}(r,t) + \mathscr{A}\omega_{st}(r,t) = 0 \tag{54}$$

with homogeneous boundary conditions. Combining this with the fact that ω_{st} is a stable solution, it implies that

$$\int_0^L \omega_{st}(r,t)dr = 0 \text{ for all } t \geq 0. \tag{55}$$

As mentioned, the initial condition of the assumed solution is zero. This leads to the following initial condition for ω_{st} (see also (13))

$$\omega_{st}(0,r) = -\omega_0(r). \tag{56}$$

Hence ω_0 must lie in the stable subspace and satisfies (55) which provides the extra condition for structuring the compliance function ω_0.

$$\int_0^L \omega_0(r)dr = 0 \Rightarrow C_4 = \frac{L^3}{20EI}. \tag{57}$$

Thus, the position dependent compliance of the beam $\omega_0(r)$ for unit force is the following:

$$\omega_0(r) = -\frac{1}{24EIL}r^4 + \frac{1}{6EI}r^3 - \frac{L}{4EI}r^2 + \frac{L^3}{20EI}. \tag{58}$$

Next we show that we could have mimicked the approach we took for the finite-dimensional state space. We take as state space $L^2(0,L)$ with inner product

$$\langle f, g\rangle = \int_0^L f(r)g(r)dr. \tag{59}$$

It is easy to see that the operator $\mathscr{A}$ defined in (45) with boundary conditions (47) is symmetric with respect to this inner product. Thus ω_0 lies in the stable subspace if and only if it is orthogonal to the unstable subspace, see Remark 1. Since we have found that the unstable subspace contains only constant functions, the extra condition emerges from the orthogonality of ω_0 and constant functions in the unstable subspace. This is equivalent to (57) in the PDE approach.

In the above example we have shown how the compliance function can be calculated for a PDE model. It is clear that we follow (at least in spirit) the same approach as for an ODE model. However, the PDE presentation of the beam (44) does not include the input term exactly like in (8) and so it does not comply with it exactly. Using the language of Boundary Control

Systems, see *e.g.* Curtain & Zwart (1995), a boundary control PDE can be rewritten as an abstract differential equation like in (8). However, this technique introduces the derivative of the input. Furthermore, the weak formulation of the PDE of the beam introduces the derivative of the input as well, and since our input is a step, this will cause additional difficulties.

The compliance function of the damped Euler-Bernoulli beam $\omega_0(r)$ is plotted for a numerical example with properties specified in Fig. 4. It is a fourth order function with respect to the position at the beam which shows transversal deflection of the beam. The calculated compliance function is validated with the frequency response of the NRB modes at low frequency (Kontaras et al. (2016)).

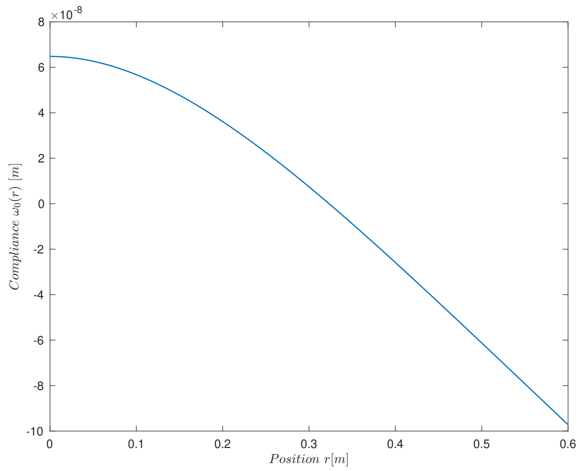

Fig. 4. The compliance function of the damped Euler Bernoulli beam; Length $L = 0.6$ [m], Cross-sectional area $A = h^2 = 10^{-4}$ [m^2], Mass density $\rho = 7.75 \times 10^3$ [kg/m^3], Young's modulus $E = 2 \times 10^{10}$ [kg/(m$\cdot$s^2)], Second moment of inertia $I = h^4/12 = 10^{-4}/12$ [m^4], and Kelvin-Voigt damping $c_d = 0.4625$.

5. CONCLUSIONS

The compliance function of the flexible dynamical systems is defined as the low frequency approximation of the flexible modes of the mechanical system. In this work, this function is calculated by a direct time domain approach which proposes a general solution for the mechanical systems.

The proposed method has been applied to the motion systems with one structural mode (mass-spring-damper system) and with infinite number of modes (Damped Euler-Bernoulli beam). Extension of this time domain approach to a 2D flexible motion system is the scope of future works. This will notably ease the calculation in comparison to the frequency response calculation of the compliance function.

Furthermore, the PDE counterpart of Theorem 1 needs to be formulated and proved and is also part of future research.

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REFERENCES

Boerlage, M., Tousain, R., and Steinbuch, M. (June 2004). Jerk derivative feedforward control for motion systems, *In Proceedings of the 2004 American Control Conference*, Boston, USA, pp. 4843-4848.

Boerlage, M. (June 2006). MIMO jerk derivative feedforward for motion systems, *American Control Conference (ACC)*, Minneapolis, USA, pp. 3892-3897.

Curtain, R.F., and Zwart, H.J. (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer Verlag, New York, 1995.

Heertjes, M.F., Hennekens, D., and Steinbuch, M. (2010). Mimo feedforward design in wafer scanners using a gradient approximation-based algorithm. *Control Engineering Practice*, vol. 18, no. 5, pp. 495-506.

Herrmann, L. (2008). Vibration of the Euler-Bernoulli Beam with Allowance for Dampings. *Lecture Notes in Engineering and computer science*, vol. 2078, pp. 901-904.

Huston, R.L. (1981). Multi-body dynamics including the effects of flexibility and compliance, *Computers and structures*, No. 5-6, vol. 14, pp. 443-451.

Kontaras, N., Heertjes, M.F., and Zwart, H.J. (2016). Continuous compliance compensation of position dependent flexible structures. *IFAC-PapersOnLine*, vol. 49, no. 13 , pp. 76-81.

Kasemsinsup, Y. Heertjes, M.F., Butler, H., and Weiland, S. (June 2016). Exact plant inversion of flexible motion systems with a time-varying state-to-output map, *European Control Conference, (ECC)*, Aalborg, Denmark, pp. 2483-2488.

Kasemsinsup, Y., Romagnoli, R., Heertjes, M.F., Weiland, S., and Butler, H. (May 2017). Reference-tracking feedforward control design for linear dynamical systems through signal decomposition, *American Control Conference (ACC)*, Seattle, Washington, pp. 2387-2392.

Lambrechts, P., Boerlage, M., and Steinbuch, M. (June 2004). Trajectory planning and feedforward design for high performance motion systems, *In Proceedings of the 2004 American Control Conference*, Boston, USA, 4637-4642.

Lunenburg, J., Bosgra, O., Oomen, T. (2009). Inversion-based feedforward design for beyond rigid body systems: A literature survey, DCT report, Eindhoven University of Technology.