Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

On Real Stable Pole Placement for Structured Systems Using Sturm and Sturm-Habicht Sequences

Klaus Röbenack * **Rick Voßwinkel** **

* *Institute of Control Theory, Technische Universität Dresden, 01062 Dresden, Germany, (e-mail: klaus.roebenack@tu-dresden.de)*
** *Development Center Chemnitz/Stollberg, IAV GmbH, 09366 Stollberg, Germany, (e-mail: Rick.Vosswinkel@iav.de).*

Abstract: During the last decades, many approaches for controller design of linear time-invariant systems have been developed. However, if a prescribed controller structure is desired, controller design may become more complicated. Typical examples include PID controllers and static output feedback. We propose a method for purely real pole or eigenvalue placement. Our approach is based on the closed-loop characteristic polynomial whose coefficients are polynomials in the controller parameters. We employ quantifier elimination to verify the existing conditions and to compute the controller gain.

Keywords: Quantifier elimination, Sturm sequence, Sturm Habicht sequence

1. INTRODUCTION

For finite-dimensional linear time-invariant systems, there are several systematic approaches to controller design. In particular, if the plant is modeled by a rational transfer function with a coprime numerator and a polynomial denominator, the Youla parametrization describes the set of all linear stabilizing controllers (Doyle et al., 1990). For a stabilizable and detectable state-space system, the combination of state feedback with an observer results in a dynamic output feedback (Hautus, 1970).

The controller design problem may become significantly more complicated if a certain controller structure is desired. Although there are several tuning rules for PI and PID controller design that work in practice, the systematic design is much more difficult (O'Dwyer, 2009; Datta et al., 2000; Munro, 2001). In the case of a state-space system, the conditions of eigenvalue assignability and stabilizability for state feedback and observer design have been known for many decades (Kalman, 1960; Hautus, 1970). However, the conditions for static output feedback controller design are much more complicated (Syrmos et al., 1997; Rosenthal and Willems, 1999; Franke, 2014).

In some applications, oscillations or overshoots are not desired. In such cases, one would aim for a real eigenvalue assignment in controller design. A typical example of this is the distance control of a convoy of vehicles. A purely real eigenvalue placement can also be desirable in electrical networks (Galeani et al., 2014).

Various questions in control engineering lead to decision problems. In the system analysis, for example, this concerns the question of the stability of a system, in the controller design the question of the stabilizability. The mentioned decision problems can be tested for several system classes using special conditions, e.g. stability with the Routh or Hurwitz test, or stabilizability by the Hautus condition (Hautus, 1970). In general, decision problems can be formulated as expressions with quantifiers such as the existence quantifier $\exists$ or the universal quantifier $\forall$. Decision problems with polynomial expressions can be solved based on Tarski's Theorem (Tarski, 1948). The associated computation methods are known as *quantifier elimination* (Caviness and Johnson, 1998).

To the authors' knowledge, the first application of quantifier elimination in control theory was the stabilization of a state-space system by static output feedback (Anderson et al., 1975). This is still a topic of active research (Syrmos et al., 1997; Röbenack et al., 2018a,b). Further application concern robust and nonlinear control (Jirstrand, 1997; Dorato et al., 1997, 1999; Anai and Hara, 1999, 2000; Tong and Bajcinca, 2017; Voßwinkel et al., 2018; Röbenack and Voßwinkel, 2020). A recent overview is given in (Röbenack and Voßwinkel, 2019).

In this contribution, we describe the stabilization of structured systems by purely real eigenvalue adjustment. Controller design of fixed structured systems employing quantifier elimination has been discussed in (Anai and Hara, 2000; Anai et al., 2004). Although the authors used real roots counting based on Sturm-Habicht sequences to characterize stability regions (such as the left half plane or a shifted left half plane for robustness), the real stabilization problem was not explicitly discussed in these papers. We want to exploit this approach in the present paper.

The paper is structured as follows. In Section 2. Quantifier elimination is introduced in Section 3 and applied to the above-mentioned controller design problem. Our approach is illustrated on some example systems in Section 4. Finally, we will draw some conclusions in Section 5.

2. NUMBER OF REAL ZEROS

2.1 Euclidean Algorithm

Consider real polynomials $P_0, P_1 \in \mathbb{R}[s]$ with the degrees $\deg P_0 \geq \deg P_1 > 0$. We apply the polynomial division gradually until the division yields a zero remainder:

$$\begin{aligned} P_0 &= Q_1 P_1 + P_2 \\ P_1 &= Q_2 P_2 + P_3 \\ P_2 &= Q_3 P_3 + P_4 \\ &\vdots \\ P_{k-1} &= Q_k P_k + 0 \quad \text{(end)}. \end{aligned}$$

In these equations, $Q_1, \ldots, Q_k \in \mathbb{R}[s]$ are the quotient polynomials. The last denominator is the greatest common divisor (gcd) of the startinng polynomials P_0 and P_1:

$$P_k = \gcd(P_0, P_1).$$

Without loss of generality, we assume the gcd is a monic polynomial, where the leading coefficient is normalized to one.

Let $f \in \mathbb{R}[s]$ be a real polynomial. We apply the euclidean algorithm to $P_0 := f$ and $P_1 := f'$. The following result is well-known (Dunaway, 1974):

Corollary 1. The polynomial $f \in \mathbb{R}[s]$ has only simple roots if and only if $\gcd(f, f') = 1$.

2.2 Sturm Sequence

Sturm's theorem delivers a statement about the number of real zeros in an interval. For this, the Sturm sequence is needed (Gantmacher, 1959).

Definition 2. Let $f \in \mathbb{R}[s]$ be a real polynomial. A *Sturm sequence* is a finite sequence $(P_0, \ldots, P_n)$ of polynomials with decreasing degree with $P_0 := f$, $P_1 := f'$ and

$$\begin{aligned} P_0 &= Q_1 P_1 - P_2 \\ P_1 &= Q_2 P_2 - P_3 \\ &\vdots \\ P_{k-1} &= Q_k P_k - P_{k+1} \\ &\vdots \\ P_n &= \text{const.} \neq 0. \end{aligned} \tag{1}$$

Let SC denote the number of sign changes (ignoring the zeros) in a finite sequence of real numbers. Furthermore, we denote the number of sign changes of a Sturm sequence at a point $s \in \mathbb{R}$ by

$$V(s) = \mathrm{SC}(P_0(s), P_1(s), \ldots, P_n(s)). \tag{2}$$

Theorem 3. (Sturm's Theorem, cf. Gantmacher (1959)). Let $f \in \mathbb{R}[s]$ be a real polynomial with simple zeros. The number r of real zeros in the interval (a, b) with $a < b$, $f(a) \neq 0$, $f(b) \neq 0$ is given by the difference

$$r = V(a) - V(b). \tag{3}$$

Remark 4. If the polynomial f has multiple zeros, the procedure (1) will terminate prematurely, i.e., we have an integer $i < n$ with $P_i \not\equiv 0$ and $P_{i+1} \equiv 0$. In this case we replace the first zero polynomial by the derivative of the last non-zero polynomial $P_{i+1}(s) := P_i'(s)$ and continue with the procedure (1) as above (Gantmacher, 1959).

2.3 Real Stable Zeros in a Finite Interval

Consider a characteristic polynomial

$$f(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \tag{4}$$

In particular, we assume that the polynomial is monic, i.e, the highest coefficient is normalized to one. In this way, we avoid unnecessary case distinctions. A polynomial (4) is called *stable* or *Hurwitz polynomial*, if all zeros have negative real part. The polynomial is called *real stable* if all zeros are real and negative.

Theorem 5. Consider the real poylnomial (4) with the Sturm sequence $(P_0, \ldots, P_n)$. All zeros of the polynomial (4) are simple and lie in the real interval (a, b) with $a < b < 0$ if and only if

$$\bigwedge_{i=0}^{n} P_i(b) > 0 \wedge \bigwedge_{i=0}^{n-1} P_i(a)(-1)^{n-i} > 0. \tag{5}$$

Proof. First, we want to derive conditions for all zeros lying in the interval (a, b) with $a < b < 0$. Theorem 3 implies

$$V(a) - V(b) = n. \tag{6}$$

Since $V(s) \in \{0, \ldots, n\}$, Eq. (6) is equivalent to

$$V(a) = n \wedge V(b) = 0. \tag{7}$$

We have to show that (7) and (5) are equivalent.

First, assume (5) is fulfilled. The conditions on $P_i(b)$ in (5) directly imply $V(b) = 0$, see (2). Since P_n is constant we have $P_n(a) = P_n(b) > 0$. The second conditions in (5) imply alternating signs $P_{n-1}(a) < 0, P_{n-2}(a) > 0, \ldots$ in the Sturm sequence at the point a, i.e., $V(a) = n$. Therefore, condition (7) is fulfilled.

Now, assume (7) is fulfilled. Then, all zeros lie in the real interval (a, b) with $a < b < 0$. Hence, f is Hurwitz, i.e., all coefficients must have the same sign. Because f is monic we have $a_0 > 0$ and therefore $a_0 = f(0) = P_0(0) > 0$. Since there is no real root in $(b, 0)$ we have $V(b) = V(0) = 0$. Then, the conditions on $P_i(b)$ in (5) are fulfilled. Again, P_n is constant, i.e., $P_n(a) > 0$. Then, $V(a) = n$ implies the conditions on $P_i(a)$ in (5). □

2.4 Real Stable Zeros in an Infinite Interval

We want to discuss the condition, under which all zeros of the polynomial (4) are real and stable. This corresponds to the situation analyzed in Section 2.3 with $a = -\infty$ and $b = 0$. From (7) we obtain

$$V(-\infty) = n \wedge V(0) = 0. \tag{8}$$

In order to formulate conditions equivalent to (8) we consider a polynomial

$$g(s) = g_k s^k + \cdots + g_1 s + g_0$$

with $g_0 \neq 0$ and $g_k \neq 0$. We denote the leading and the trailing coefficient by

$$\begin{aligned} \mathrm{lcf}(g) &= g_k \quad \text{(leading coefficient)}, \\ \mathrm{tcf}(g) &= g_0 \quad \text{(trailing coefficient)}. \end{aligned}$$

Based on these notations we are able to formulate the conditions for real stability as follows:

Theorem 6. Consider the real polynomial (4) with the Sturm sequence $(P_0, \ldots, P_n)$. All zeros of the polyno-

Table 1. Sturm sequence and sign changes of example polynomial from Remark 7

	$\operatorname{sign}(P_i(0))$	$\operatorname{sign}(P_i(+\infty))$
$P_0(s) = s^2 - 2s + 2$	$+$	$+$
$P_1(s) = 2s - 2$	$-$	$+$
$P_2(s) = -1$	$-$	$-$
sign changes:	$V(0) = 1$	$V(+\infty) = 1$

mial (4) are simple and lie in the real interval $(-\infty, 0)$ if and only if

$$\bigwedge_{i=0}^{n} \operatorname{tcf}(P_i) > 0 \wedge \bigwedge_{i=0}^{n-1} \operatorname{lcf}(P_i) > 0. \tag{9}$$

Proof. We have to show that conditions (8) and (9) are equivalent. For a Sturm sequence $(P_0, \ldots, P_n)$ we clearly have

$$P_i(0) = \operatorname{tcf}(P_i), \quad i = 0, \ldots, n.$$

Similarly, we have

$$\operatorname{sign}\left(\lim_{s \to +\infty} P_i(s)\right) = \operatorname{sign}\left(\operatorname{lcf}(P_i)\right),$$

since $\operatorname{sign}\left(\lim_{s \to +\infty} s^i\right) = +1$ for $i = 0, \ldots, n$. This imples $V(+\infty) = \mathrm{SC}(\operatorname{lcf}(P_0), \operatorname{lcf}(P_1), \ldots, \operatorname{lcf}(P_{n-1}), \operatorname{lcf}(P_n))$. We also have $\operatorname{sign}\left(\lim_{s \to -\infty} s^i\right) = (-1)^i$, which imlies $V(-\infty) = \mathrm{SC}((-1)^n \operatorname{lcf}(P_0), \ldots - \operatorname{lcf}(P_{n-1}), \operatorname{lcf}(P_n))$. Hence, we have

$$V(-\infty) + V(+\infty) = n. \tag{10}$$

Therefore, condition (8) can equivalently be written as

$$V(0) = 0 \wedge V(\infty) = 0 \tag{11}$$

if no sign changes occur at zero and at infinity, respectively. Since the polynomial (4) is monic, the trailing and leading coefficients must be positive as stated in (9). □

Remark 7. Note that condition (11) is *not* equivalent to

$$V(0) - V(+\infty) = 0 \tag{12}$$

corresponding to the sign definite condition (SDC) of f. To illustrate this, consider the polynomial $f(s) = s^2 - 2s + 2$. Table 1 shows the Sturm sequence and the number of sign changes. Clearly, condition (12) is fulfilled, but not condition (11). In particular, the polynomial has a complex conjugated pair of zeros at $s = 1 \pm j$. Therefore, the polynomial is neither stable nor real stable.

2.5 Parameter Dependency and Sturm-Habicht Sequence

We assume that the coefficients of the characteristic polynomial (4) depend polynomially on some controller parameters $k_1, \ldots, k_m$, i.e., $a_0, \ldots, a_{n-1} \in \mathbb{R}[k_1, \ldots, k_m]$. The Sturm sequence is computed using polynomial division with remainder. As a consequence, although the coefficients of the first two entries are polynomial, we will generally get polynomials with rational coefficients. In order to check the sign, we can transform the rational entry as follows:

$$\frac{x}{y} > 0 \iff x \cdot y > 0 \tag{13}$$

$$\iff (x > 0 \wedge y > 0) \vee (x < 0 \wedge y < 0). \tag{14}$$

The formulation (14) yields conditions with polynomials of lower degree compared to (13), which is advantages from a computational point of view (see Section 3).

Alternatively, we could use a *Sturm-Habicht sequence*, which is a generalization of the Sturm sequence (Habicht, 1948). The Sturm-Habicht sequence has the same sign properties as the Sturm sequence but is computed based on subdeterminants. As a consequence, the coefficients of the involved polynomials are also polynomial w.r.t. the controller parameters. Computational issues are discussed in (Gonzalez-Vega et al., 1989; Abdeljaoued et al., 2009).

3. QUANTIFIER ELIMINATION

3.1 Mathematical Preliminaries

From a control-theoretic point of view, we are interested in proper controller parameterization. Here, proper means controller parameters that lead to a stable closed-loop system with real roots of the resulting characteristic polynomial. The existence of a suitable parameterization can be formulated as a decision problem

$$\exists k_1, \ldots, k_m : \quad F(k_1, \ldots, k_m). \tag{15}$$

The expression $F(k_1, \ldots, k_m)$ is called *quantifier-free formula* and results from a Boolean combination of *atomic formulas*

$$\varphi(k_1, \ldots, k_m) \, \tau \, 0,$$

with $\tau \in \{=, <\}$ and $\varphi(k_1, \ldots, k_m) \in \mathbb{Q}[k_1, \ldots, k_m]$, where $\mathbb{Q}[k_1, \ldots, k_m]$ denotes the set of all polynomials with rational coefficients. Following this terminology we call (15) *prenex formula*. These prenex formulas are given by

$$PF(V, U) := Q_1 u_1, \ldots, Q_l u_l F(U, V) \tag{16}$$

with $Q_i \in \{\exists, \forall\}$ and the quantifier-free expression $F(U, V)$. The variables connected to quantifiers (U) are called *quantified* and *free* otherwise (V). Using known theorems and conditions, we are often able to formulate such prenex formulas. However, they are not suitable for a concrete application, such as controller design like in our case. We are more interested in a set of proper controller parameters. These sets can be described by quantifier-free formulas. So the question arises if there always exists a quantifier-free equivalent to a given prenex formula and how can we compute them? This leads us to the concept of *quantifier elimination (QE)*. The first question is addressed by the following theorem, which is a direct consequence of the Tarski-Seidenberg-Theorem (Tarski, 1948; Seidenberg, 1954).

Theorem 8. (Quantifier Elimination). For every real prenex formula $PF(V, U)$ exists an equivalent quantifier-free formula $H(V)$.

3.2 Algorithms and Software

There are several approaches to tackle the second question. The historically first approach was developed by Tarski himself. Unfortunately, the computational effort of that approach can not be bounded by any stack of exponentials due to its inherent constructive basis. The first practical relevant algorithm is the *cylindrical algebraic decomposition (CAD)* (Collins, 1974). This approach consists of a sign-based decomposition in the $\mathbb{R}^n$ which leads to semialgebraic sets, called *cells*. These cells are successively projected from $\mathbb{R}^n$ to $\mathbb{R}^1$. This leads to semialgebraic sets in $\mathbb{R}^1$ as well. Afterward, these sets are evaluated and the results are lifted back to the $\mathbb{R}^n$. This approach is applicable to arbitrary prenex formulas, but the computational complexity might increase doubly exponential in the number of variables (Davenport and Heintz, 1988). However, nowadays applications other algorithms dominate. *Virtual substitution (VS)* (Weispfenning, 1994) and *Real Root Classification (RRC)* based approaches (Gonzalez-Vega et al., 1989; Iwane et al., 2013) are to be mentioned here. These algorithms have much better computational properties.

The implementation of the aforementioned approaches is a non-trivial task. Fortunately exists a bunch of tools for handling QE problems. For common proprietary computer algebra systems such as Mathematica and Maple, specialized toolboxes are available (Chen and Maza, 2016; Yanami and Anai, 2007). Additionally, there exist open-source tools like QEPCAD (Quantifier Elimination by Partial Cylindrical Algebraic Decomposition) (Collins and Hong, 1991). and QEPCAD B (Brown, 2003).

In this paper, we used the Reduce package Redlog for quantifier elimination (Dolzmann and Sturm, 1997). The calculation we carried out on a PC with a quad-core Intel® Core™ with 3.40 GHz and 32 GiB RAM under Fedora Linux 29.

3.3 Application to the Real Stabilization Problem

Now, we will apply quantifier elimination to the real stabilization problem. Let be (4) the closed-loop characteristic polynomial, where the coefficients depend controller parameters $k_1, \ldots, k_m$. To carry out an exact computation we assume that these coefficients themselves are polynomials over the field of rational numbers, i.e., $a_0, \ldots, a_{n-1} \in \mathbb{Q}[k_1, \ldots, k_m]$. Based on Theorem 6, the real stabilizability of the system using the controller parameters $k_1, \ldots, k_m$ can be formulated as a prenex formula:

$$\exists k_1 \cdots \exists k_m : \quad \bigwedge_{i=0}^{n} \mathrm{tcf}(P_i) > 0 \wedge \bigwedge_{i=0}^{n-1} \mathrm{lcf}(P_i) > 0. \tag{17}$$

If quantifier elimination applied to the decision problem (17) yields `true` the controller parameters $k_1, \ldots, k_m$ can be computed as follows:

(1) Remove the quantifier for a variable k_i. Then, k_i becomes a free variable.
(2) Calculate the admissible set via elimination of the remaining quantifiers.
(3) Select a value for k_i from this set.
(4) Continue with the next variable.

4. EXAMPLES

4.1 PID Control of a Second Order System

We consider a second order system

$$G(s) = \frac{K}{T^2 s^2 + 2dTs + 1}$$

with the transfer function G together with a PID controller

$$R(s) = K_P + \frac{K_I}{s} + K_D s \tag{18}$$

with the transfer function R. The stability of the closed-loop system is discussed in Schrödel et al. (2015) using parameter space methods. As in this paper we set $K = K_P = T = 1$, $d = -0.5$ resulting in the third-order closed-loop characteristic polynomial

$$\begin{aligned} f(s) &= s^3 + a_2 s^2 + a_1 s + a_0 \\ &= s^3 + (K_D - 1)s^2 + 2s + K_I. \end{aligned} \tag{19}$$

The stability conditions $a_0 > 0 \wedge a_2 > 0 \wedge a_1 a_2 - a_0 > 0$ result in

$$K_I > 0 \wedge 2K_D - K_I - 2 > 0.$$

These linear inequalities describe a cone in the parameter space.

Next, we want to compute the conditions for real stabilization. The Sturm sequence resulting from (19) is shown in Tab. 2. Clearly, the last element P_3 is not polynomial but in rational in the controller parameters K_I and K_D.

Alternatively, we compute the Sturm-Habicht sequence

$$\begin{aligned} P_0(s) &= s^3 + K_D\, s^2 - s^2 + 2s + K_I \\ P_1(s) &= 3s^2 + (2K_D - 2)\, s + 2 \\ P_2(s) &= 2\left(K_D^2 - 2K_D - 5\right) s\; - 9K_I + 2K_D - 2 \\ P_3(s) &= -27K_I^2 - 4K_D^3 K_I + 12K_D^2 K_I + 24K_D K_I \\ &\quad -32K_I + 4K_D^2 - 8K_D - 28, \end{aligned}$$

where all elements are polynomial in the controller parameters. The real stability condition (9) from Theorem 6 can be written as

$$\begin{aligned} &K_I > 0 \wedge 2\left(K_D^2 - 2K_D - 5\right) > 0 \wedge \\ &-9K_I + 2K_D - 2 > 0 \wedge \\ &-27K_I^2 - 4K_D^3 K_I + 12K_D^2 K_I + 24K_D K_I \\ &-32K_I + 4K_D^2 - 8K_D - 28 > 0, \end{aligned} \tag{20}$$

where we omitted trivial conditions resulting from constants. To (20) we can apply quantifier elimination in different scenarios. Quantifying both parameters K_d, K_I yields a decision problem

$$\exists K_D, K_I : \;\; \text{Cond. (20)} \iff \texttt{true},$$

where QE confirms the solvability. To find bounds on the parameter K_I we only quantify K_D and use K_I as a free variable:

$$\begin{aligned} \exists K_D : \text{Cond. (20)} &\iff K_I > 0 \wedge 27K_I^2 - 8 < 0 \\ &\iff 0 < K_I < \sqrt{8/28} \approx 0.5443. \end{aligned}$$

Similarly, we calculate the bound on K_D:

$$\begin{aligned} \exists K_I : \text{Cond. (20)} &\iff K_D > 0 \wedge K_D^2 - 2K_D - 5 > 0 \\ &\iff K_D > 1 + \sqrt{6} \approx 3.4495. \end{aligned}$$

To verify these results, we calculated the eigenvalues numerically and plotted the stability regions in Fig. 1. In particular, the limits on the region of real stability match the above calculations.

4.2 Proportional & PID Control of a Third Order System

Consider the third order system

$$G(s) = \frac{6}{(s+1)(s+2)(s+3)} \tag{21}$$

discussed in (Dutoon et al., 1997, pp. 272). The design of a proportional controller with $R(s) = K_P$ can be carried out using the root locus method (see Fig. 2).

For practical reasons, we assume $K_P > 0$. In Scilab, the limit feedback gain for stabilization can be computed with the function `kpure`, where we obtain $K_P = 10$. Similarly,

Table 2. Sturm sequence of polynomial (19) from Section 4.1

$P_0(s)$	$=$	$s^3 + K_D\, s^2 - s^2 + 2s + K_I$
$P_1(s)$	$=$	$3s^2 + (2K_D - 2)\, s + 2$
$P_2(s)$	$=$	$\frac{(2K_D^2 - 4K_D - 10)s - 9K_I + 2K_D - 2}{9}$
$P_3(s)$	$=$	$\frac{243K_I^2 + (36K_D^3 - 108K_D^2 - 216K_D + 288)\, K_I - 36K_D^2 + 72K_D + 252}{4K_D^4 - 16K_D^3 - 24K_D^2 + 80K_D + 100}$

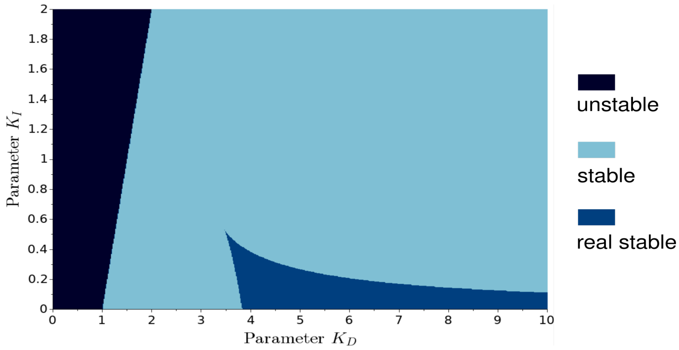

Fig. 1. Stability regions in the parameter space (K_d, K_I)

the limit feedback gain for real stabilization can be calculated with the function `krac2`. In this case, we obtain $K_P \approx 0.06415$. We want to very this numerical result using our approach. For the characteristic polynomial

$$f(s) = s^3 + 6s^2 + 11s + 6K_P + 6$$

of the closed-loop system under proportional control we obtain the Sturm-Habicht sequence

$$\begin{aligned} P_0(s) &= s^3 + 6s^2 + 11s + 6K_P + 6, \\ P_1(s) &= 3s^2 + 12s + 11, \\ P_2(s) &= 6s - 54K_P + 12, \\ P_3(s) &= 4 - 972K_P^2. \end{aligned}$$

The real stability condition (9) yields

$$K_P < 3^{-5/2} \approx 0.06415002990995844, \qquad (22)$$

which confirms the numerical calculation.

In addition, we want to investigate the control of (21) using a PID controller (18). The closed-loop system has the fourth order characteristic polynomial

$$f(s) = s^4 + 6s^3 + 6K_D s^2 + 11s^2 + 6K_P s + 6s + 6K_I.$$

The associated Sturm-Habicht sequence is too long to be shown here. We want to compute the admissible range for the proportional gain K_P such that real stabilization is possible for suitable values of K_D and K_I. This problem can be formulated as follows

$$\exists K_D, K_I : \;\; \text{Cond. (17)},$$

where K_P is a free variable. QE yields the condition $K_P + 1 > 0 \wedge 4K_P - 5 < 0$. Again, we assume $K_P > 0$ for practical reasons. The possibility of an appropriate adjustment of the additional controller parameters K_D and K_I results in a significantly larger limit gain

$$K_P < \frac{5}{4} = 1.25$$

for the proportional part compared to (22).

4.3 Proportional Control of a Further Third Order System

We consider the transfer function

$$G(s) = \frac{5s}{s(s+5)(s+1)+5} = \frac{5s}{s^3 + 6s^2 + 5s + 5} \qquad (23)$$

arising in the modelling of an antenna positioner (Dutoon et al., 1997, pp. 294). Proportional control with $R(s) = K_V$ corresponds to velocity feedback in the considered system. With the Scilab function `krac2` we obtain two values $K_V \approx 0.9849196$ and $K_V \approx 1.1528683$ as limit feedback gains for real stabilization. These limits can also be obtained from the root locus shown in Fig. 3.

Again, we want to verify these results with our approach. From (9) we obtain the conditions

$$K_V > 0 \wedge 100K_V^3 + 120K_V^2 - 600K_V + 379 < 0.$$

The admissible range of K_V for real stabilization given by

$$0.9849196 \lessapprox K_V \lessapprox 1.1528683$$

is consistent with the above mentioned results.

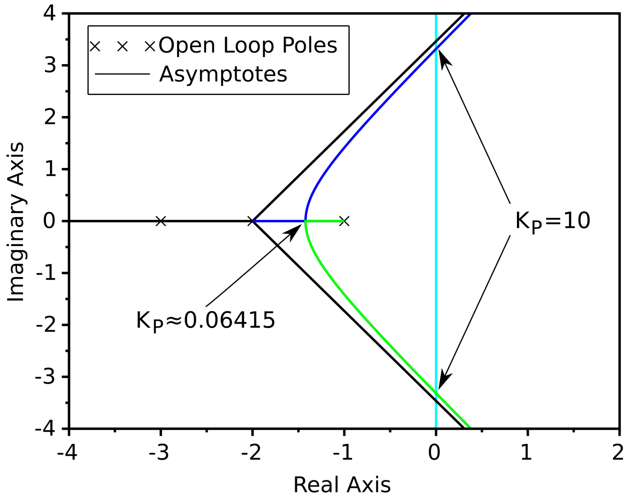

Fig. 2. Root locus of system (21)

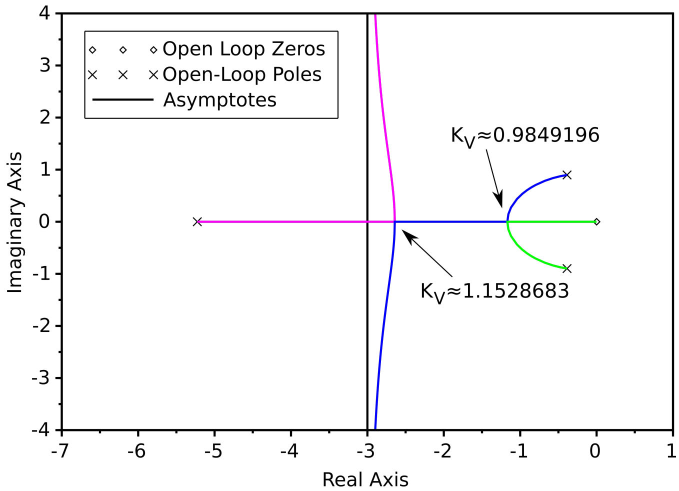

Fig. 3. Root locus of system (23)

4.4 Static Output Feedback Control

The following example was the first application of quantifier elimination to the static output feedback stabilization problem (Anderson et al., 1975). We consider a linear time-invariant state-space system with the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -5 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}. \tag{24}$$

We want to exploit real stabilizability. The closed-loop system has the following characteristic polynomial

$$\begin{aligned} f(s) &= \det(sI - (A + BKC)) \\ &= a_0 + a_1 s + a_2 s^2 + s^3 \\ &= k_2 + (k_2 - 5k_1 - 13)s + k_1 s^2 + s^3. \end{aligned} \tag{25}$$

From (25) we calculate the Sturm-Habicht sequence

$$\begin{aligned} P_0(s) &= s^3 + k_1\, s^2 + k_2 s - 5k_1 s - 13s + k_2 \\ P_1(s) &= 3s^2 + 2k_1 s + k_2 - 5k_1 - 13 \\ P_2(s) &= (-6k_2 + 2k_1^2 + 30k_1 + 78)s \\ &\quad + k_1\, k_2 - 9k_2 - 5k_1^2 - 13k_1 \\ P_3(s) &= -4k_2^3 + k_1^2\, k_2^2 + 78k_1\, k_2^2 + 129k_2^2 - 14k_1^3\, k_2 \\ &\quad -416k_1^2\, k_2 - 1794k_1\, k_2 - 2028k_2 + 25k_1^4 \\ &\quad +630k_1^3 + 4069k_1^2 + 10140k_1 + 8788 \end{aligned}$$

The real stability condition (9) from Theorem 6 can be written as

$$\begin{aligned} & -5k_1 s - 13s + k_2 > 0 \wedge k_2 - 5k_1 - 13 > 0 \wedge \\ & -6k_2 + 2k_1^2 + 30k_1 + 78 > 0 \wedge \\ & k_1\, k_2 - 9k_2 - 5k_1^2 - 13k_1 > 0 \wedge \\ & -4k_2^3 + k_1^2\, k_2^2 + 78k_1\, k_2^2 + 129k_2^2 - 14k_1^3\, k_2 \\ & -416k_1^2\, k_2 - 1794k_1\, k_2 - 2028k_2 + 25k_1^4 \\ & +630k_1^3 + 4069k_1^2 + 10140k_1 + 8788 > 0. \end{aligned} \tag{26}$$

Quantifying both gain entries k_1, k_2 results in the decision problem

$$\exists k_1, k_2 : \quad \text{Cond. (26)} \iff \texttt{true},$$

Quantifying only one variableat a time results in

$$\begin{aligned} & \exists k_1 : \quad \text{Cond. (26)} \\ \iff & k_2^3 - 201k_2^2 - 1113k_2 - 2197 > 0 \\ \iff & k_2 > 10 \cdot 18^{2/3} + 27 \cdot 18^{1/3} + 67 \approx 206.44 \end{aligned}$$

and

$$\begin{aligned} & \exists k_2 : \quad \text{Cond. (26)} \\ \iff & k_1^3 - 9k_1^2 - 135k_1 - 351 > 0 \\ \iff & k_1 > 18^{2/3} + 3 \cdot 18^{1/3} + 3 \approx 17.73\,. \end{aligned}$$

5. CONCLUSIONS

In this paper, we have shown that the purely real stabilization problem for fixed-structure controllers can be solved by quantifier elimination. This approach can be seen as an addition to existing parameter space methods, see e.g. Hohenbichler and Abel (2006); Voßwinkel et al. (2019).

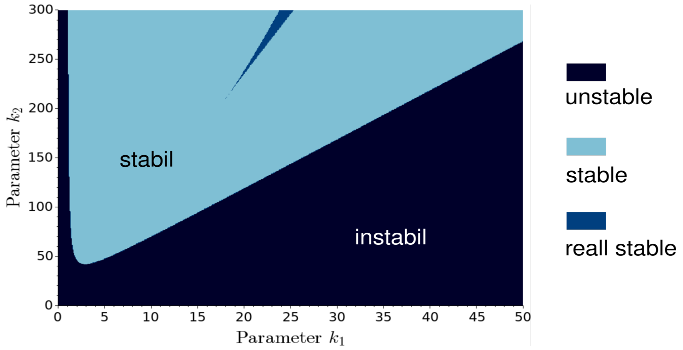

Fig. 4. Stability regions in the parameter space (k_1, k_2)

REFERENCES

Abdeljaoued, J., Diaz-Toca, G.M., and Gonzalez-Vega, L. (2009). Bezout matrices, subresultant polynomials and parameters. *Applied Mathematics and Computation*, 214(2), 588–594.

Anai, H. and Hara, S. (1999). A robust control system design by a special quantifier elimination methods using a Sturm-Habicht sequence. *Proc. IMACS-ACA'99.*

Anai, H. and Hara, S. (2000). Fixed-structure robust controller synthesis based on sign definite condition by a special quantifier elimination. In *Proc. American Control Conference (ACC)*, volume 2, 1312–1316. doi: 10.1109/ACC.2000.876713.

Anai, H., Yanami, H., Hara, S., and Sakabe, K. (2004). Fixed-structure robust controller synthesis based on symbolic-numeric computation: design algorithms with a CACSD toolbox. In *Proc. Int. Conf. on Control Applications*, volume 2, 1540–1545. doi: 10.1109/CCA.2004.1387594.

Anderson, B., Bose, N., and Jury, E. (1975). Output feedback stabilization and related problems-solution via decision methods. *IEEE Trans. on Automatic Control*, 20(1), 53–66. doi:10.1109/TAC.1975.1100846.

Brown, C.W. (2003). QEPCAD B: A program for computing with semi-algebraic sets using CADs. *ACM SIGSAM Bulletin*, 37(4), 97–108. doi: https://doi.org/10.1145/968708.968710.

Caviness, B.F. and Johnson, J.R. (eds.) (1998). *Quantifier Elimination and Cylindical Algebraic Decomposition.* Springer, Wien.

Chen, C. and Maza, M.M. (2016). Quantifier elimination by cylindrical algebraic decomposition based on regular chains. *Journal of Symbolic Computation*, 75, 74–93.

Collins, G.E. (1974). Quantifier elimination for real closed fields by cylindrical algebraic decomposition – preliminary report. *ACM SIGSAM Bulletin*, 8(3), 80–90.

Collins, G.E. and Hong, H. (1991). Partial cylindrical algebraic decomposition for quantifier elimination. *Journal of Symbolic Computation*, 12(3), 299–328. doi: https://doi.org/10.1016/S0747-7171(08)80152-6.

Datta, A., Ho, M.T., and Bhattacharyya, S.P. (2000). *Structure and synthesis of PID controllers.* Springer-Verlag, London.

Davenport, J.H. and Heintz, J. (1988). Real quantifier elimination is doubly exponential. *Journal of Symbolic Computation*, 5(1), 29–35. doi: https://doi.org/10.1016/S0747-7171(88)80004-X.

Dolzmann, A. and Sturm, T. (1997). Redlog: Computer algebra meets computer logic. *ACM SIGSAM Bulletin*, 31(2), 2–9. doi:https://doi.org/10.1145/261320.261324.

Dorato, P., Famularo, D., Abdallah, C.T., and Yang, W. (1999). Robust nonlinear feedback design via quantifier elimination theory. *International Journal of Robust and Nonlinear Control*, 9(11), 817–822.

Dorato, P., Yang, W., and Famularo, D. (1997). Robust nonlinear feedback design via quantifier elimination theory. {*IFAC*} *Proceedings Volumes*, 30(16), 51 – 54. doi:https://doi.org/10.1016/S1474-6670(17)42580-8. {IFAC} Symposium on Robust Control Design (ROCOND'97), Budapest, Hungary, 25-27 June 1997.

Doyle, J., Francis, B., and Tannenbaum, A. (1990). *Feedback Control Theory*. Macmillan Publishing Co.

Dunaway, D.K. (1974). Calculation of zeros of a real polynomial through factorization using Euclid's algorithm. *SIAM Journal on Numerical Analysis*, 11(6), 1087–1104.

Dutoon, K., Thompson, S., and Barraclough, B. (1997). *The Art of Control Engineering*. Prentice Hall.

Franke, M. (2014). Eigenvalue assignment by static output feedback – on a new solvability condition and the computation of low gain feedback matrices. *Int. J. Control*, 87(1), 64–75. doi: https://doi.org/10.1080/00207179.2013.822102.

Galeani, S., Henrion, D., Jacquemard, A., and Zaccarian, L. (2014). Design of Marx generators as a structured eigenvalue assignment. *Automatica*, 50(10), 2709–2717.

Gantmacher, F.R. (1959). *Theory of Matrices*, volume II. Chelsea, New York.

Gonzalez-Vega, L., Lombardi, H., Recio, T., and Roy, M.F. (1989). Sturm-Habicht sequence. In *Proc. of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation*, 136–146. Portland, Oregon, USA. doi:https://doi.org/10.1145/74540.74558.

Habicht, W. (1948). Eine Verallgemeinerung des Sturmschen Wurzelzählverfahrens. *Commentarii Mathematici Helvetici*, 21(1), 99–116. doi: https://doi.org/10.1007/BF02568028.

Hautus, M.L.J. (1970). Stabilization, controllability, and observability for linear autonomous systems. *Ned. Akad. Wetenschappen, Proc. Ser. A*, 73, 448–455.

Hohenbichler, N. and Abel, D. (2006). Robust PID-Controller Design Meeting Pole Location and Gain/Phase Margin Requirements for Time Delay Systems. *Automatisierungstechnik*, 54(10), 495.

Iwane, H., Yanami, H., Anai, H., and Yokoyama, K. (2013). An effective implementation of symbolic-numeric cylindrical algebraic decomposition for quantifier elimination. *Theoretical Computer Science*, 479, 43–69. doi:https://doi.org/10.1016/j.tcs.2012.10.020.

Jirstrand, M. (1997). Nonlinear control system design by quantifier elimination. *Journal of Symbolic Computation*, 24(2), 137–152. doi: https://doi.org/10.1006/jsco.1997.0119.

Kalman, R.E. (1960). On the general theory of control systems. In *First International Congress of the International Federation of Automatic Control (IFAC)*. Moscow.

Munro, N. (2001). The systematic design of pid controllers using nyquist stabiliy. In *European Control Conference (ECC)*, 528–533.

O'Dwyer, A. (2009). *Handbook of PI and PID controller tuning rules*. Imperial College Press, London.

Röbenack, K. and Voßwinkel, R. (2019). Lösung regelungstechnischer Problemstellungen mittels Quantorenelimination. *Automatisierungstechnik*, 67(9), 714–726. doi:https://doi.org/10.1515/auto-2019-0045.

Röbenack, K., Voßwinkel, R., and Franke, M. (2018a). On the eigenvalue placement by static output feedback via quantifier elimination. In *Mediterranean Conference on Control and Automation (MED'18)*, 133–138. Zadar, Croatia. doi:https://doi.org/10.1109/MED.2018.8442817.

Röbenack, K., Voßwinkel, R., Franke, M., and Franke, M. (2018b). Stabilization by static output feedback: A quantifier elimination approach. In *International Conference on System Theory, Control and Computing (ICSTCC 2018)*, 715–721. Sinaia, Romania. doi: https://doi.org/10.1109/ICSTCC.2018.8540735.

Röbenack, K. and Voßwinkel, R. (2020). Eigenvalue placement by quantifier elimination — the static output feedback problem. *Acta Cybernetica*, 24(3), 409–427. doi:10.14232/actacyb.24.3.2020.8.

Rosenthal, J. and Willems, J. (1999). Open problems in the area of pole placement. In V.D. Blondel, E.D. Sontag, M. Vidyasagar, and J.C. Willems (eds.), *Open Problems in Mathematical Systems and Control Theory*, Communication and Control Engineering Series, 181–191. Springer–Verlag, London.

Schrödel, F., Almodaresi, E., Stump, A., Bajcinca, N., and Abel, D. (2015). Lyapunov stability bounds in the controller parameter space. In *Proc. IEEE Conference on Decision and Control (CDC)*, 4632–4636. doi: 10.1109/CDC.2015.7402941.

Scilab (2020). Scilab: Free and Open Source Software for Numerical Computation. `http://www.scilab.org/`.

Seidenberg, A. (1954). A new decision method for elementary algebra. *Annals of Mathematics*, 60(2), 365–374. doi:10.2307/1969640.

Syrmos, V.L., Abdallah, C.T., Dorato, P., and Grigoriadis, K. (1997). Static output feedback — A survey. *Automatica*, 33(2), 125–137. doi:https://doi.org/10.1016/S0005-1098(96)00141-0.

Tarski, A. (1948). *A Decision Method for a Elementary Algebra and Geometry*. Project rand. Rand Corporation.

Tong, J. and Bajcinca, N. (2017). Computation of feasible parametric regions for Lyapunov functions. In *Asian Control Conference (ASCC)*, 2453–2458. doi: https://doi.org/10.1109/ASCC.2017.8287559.

Voßwinkel, R., Pyta, L., Schrödel, F., Mutlu, İ., Mihailescu-Stoica, D., and Bajcinca, N. (2019). Performance boundary mapping for continuous and discrete time linear systems. *Automatica*, 107, 272–280. doi: https://doi.org/10.1016/j.automatica.2019.05.055.

Voßwinkel, R., Röbenack, K., and Bajcinca, N. (2018). Input-to-state stability mapping for nonlinear control systems using quantifier elimination. In *European Control Conference (ECC)*, 906–911. Limassol, Cyprus.

Weispfenning, V. (1994). Quantifier elimination for real algebra – the cubic case. In *Proc. of the Int. Symp. on Symbolic and Algebraic Computation (ISSAC)*, 258–263. doi:https://doi.org/10.1145/190347.190425.

Yanami, H. and Anai, H. (2007). The Maple package SyNRAC and its application to robust control design. *Future Generation Computer Systems*, 23(5), 721–726. doi:https://doi.org/10.1016/j.future.2006.10.009.