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# One Port Impedance Quantization For a Class of Annihilation Operator Linear Quantum Systems ⋆

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**Abstract:** This paper provides a procedure for building a one port impedance quantization involving annihilation operators only for a class of linear quantum systems having a positive real impedance transfer function matrix. Based on the positive real properties of these quantum systems, it is shown that it is possible to use the Brune algorithm in order to find an electrical circuit that can physically implement these quantum systems. This theory, illustrated for one-port circuits may be useful for the implementation of superconducting microwave circuits used in quantum filters found in the field of quantum computing.

*Keywords:* Quantum control, Brune Algorithm, superconducting microwave circuit, impedance quantization.

## 1. INTRODUCTION

Super-conducting circuits are used in many hardware implementations of a quantum computer. In fact, some super-conducting quantum processors have been already developed using 3D microwave cavities and coplanar waveguide circuits. For further information, please refer to Blais et al. (2004), Paik et al. (2011), Rigetti et al. (2012). These processors are typically made using Josephson junctions that are directly connected to cavity resonators. These resonators are in turn connected to microwave circuits that are responsible for reading data and the operation of quantum gates. It is important to mention here that Josephson junctions are lossless nonlinear elements. They create a quantum energy spectrum where it is possible to get two low levels of energy that could be used to construct a qubit. On the other hand, super-conducting quantum processors are microwave systems with many resonant modes. Therefore, it is essential to build models of these systems with the aim of increasing the quality factor of the qubit as well the cavity modes. Some previous attempts have failed to provide accurate models of multi-mode super-conducting qubit systems and that is due mainly to the absence of accurate estimation of the loss rates Bourassa et al. (2011).

In this paper, the modeling of a system involving a one port and a Josephson junction is illustrated. In that case, the microwave circuit is modelled by its impedance $Z(s)$ as illustrated in Figure 1, which also shows a voltage source as well as a shunting resistor.

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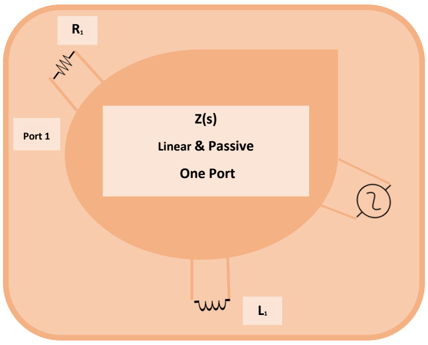


Fig. 1. The quantum circuit Impedance.

## 2. STATE-SPACE FORMALISM FOR A TYPE OF ANNIHILATION OPERATOR LINEAR QUANTUM SYSTEM

An important type of quantum system can be described by using the time evolution of creation and annihilation operators in the Heisenberg picture for harmonic oscillators that are coupled to optical fields; e.g., see Wiseman and Milburn (2010), Walls and Milburn (2008) and Gardiner and Zoller (2000). A specific type of these quantum systems involves quantum Wiener processes (e.g.; see James et al. (2008)). In that case, the question arises as to whether the quantum system in question can be represented by a quantum harmonic oscillator. This is related to the physical realizability conditions that are developed in James et al. (2008). Moreover, in Maalouf and Petersen (2009), Maalouf and Petersen (2011b) and Maalouf and Petersen (2011c), the lossless bounded real property of annihilation-operator quantum systems has been connected to physical realizability. The type of an-

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nihilation operator quantum systems considered in this paper can be described by using quantum probability theory Bouten et al. (2007) as in Maalouf and Petersen (2011b) and Maalouf and Petersen (2011a). In that case, the quantum differential equations (QSDEs) describing the systems under consideration are of the form

$$\begin{aligned} da(t) &= Fa(t)dt + Gdw(t); \qquad a(0) = a_0 \\ dy(t) &= Ha(t)dt + Jdw(t) \end{aligned} \tag{1}$$

where $J \in \mathcal{C}^{n_y \times n_w}$, $H \in \mathcal{C}^{n_y \times n}$, $F \in \mathcal{C}^{n \times n}$, $G \in \mathcal{C}^{n \times n_w}$. Also, ($n_y$, $n_w$, $n$ are positive integers).

In addition, the vector of annihilation operators $a(t)$ is given by $a(t) = [a_1(t) \cdots a_n(t)]^T$. In that case, $w$ represents the input fields and has the following partition:

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t).$$

Here, $\beta_w(t)$ and $w(t)$ are a self-adjoint adapted vector process and the quantum noise signal respectively (Please refer to Bouten et al. (2007), K.R.Parthasarathy (1992) and Hudson and Parthasarathy (1984)). The Ito table of the quantum noise $\tilde{w}(t)$ is

$$d\tilde{w}(t)d\tilde{w}^\dagger(t) = F_{\tilde{w}}dt \tag{2}$$

(see Belavkin (1992) and K.R.Parthasarathy (1992)) where $F_{\tilde{w}}$ is a Hermitian positive definite matrix. In that case, the notation $^\dagger$ refers to the adjoint transpose vector of operators. Also, the quantum noise components satisfy the following commutation relations:

$$[d\tilde{w}(t), d\tilde{w}^*(t)] = d\tilde{w}(t)d\tilde{w}^\dagger(t) - (d\tilde{w}^*(t)d\tilde{w}^T(t))^T = T_w dt \tag{3}$$

Here, $T_w$ is a Hermitian complex matrix. The signals involving noises are operators on a Fock space (e.g; see Belavkin (1992) and K.R.Parthasarathy (1992)). The process $\beta_w(t)$ represents the variables of fields interacting with the system (1). Hence, $\beta_w(0)$ should be an operator on a Hilbert space an operator that is different from that of $a_0$ and the noises. The assumption is made that $\beta_w(t)$ and $a(t)$ commute together for any $t \geq 0$. In addition, being an adapted field, $\beta_w(t)$ and $d\tilde{w}(t)$ commutes together for all $t \geq 0$. The following assumption is made on the system: (1), $n_w = n_y$.

Equation (1) is an annihilation-operator quantum differential equation where the integration is considered to be quantum Ito integration with respect to $dw(t)$. Note that $a(t)$ is adapted, and the commutator of $d\tilde{w}(t)$ with $a(t)$ is zero. If $\beta_w(t)$ represents the currents and $y(t)$ represent the output voltages of a quantum network in question then $n_w = n_y$ and the resulting impedance transfer function matrix is

$$Z_a(s) = J + H(sI - F)^{-1}G. \tag{4}$$

It is important to mention here that complex realizations are considered such that the matrices $J, H, G, F$ are all complex.

## 3. MINIMAL REALIZATIONS FOR THE IMPEDANCE

In classical control theory, the set of matrices $D_1, C_1, B_1, A_1$ is considered to be a minimal realization for $Z_{1_r}(s)$ if $[A_1, C_1]$ is completely observable and $[A_1, B_1]$ is completely controllable. More explicitly, the pair $[A_1, B_1]$ is said to be completely controllable, if there exists a control signal $\beta_{w_1}$ such that if at time $t_0$, the system is at state $x_1(t_0)$, then the system can be led to reach the state of zero ($x_1(t_1) = 0$) at time $t_1$. The pair $[A_1, C_1]$ is said to be completely observable over the interval $[t_0, t_1]$, if for a given input and ouput functions $\beta_{w_1}$ and $y_1(t)$ respectively, then it is possible to determine $x_1(t_0)$ uniquely. For further information, the reader is referred to Anderson and Vongpanitlerd (2006) where the properties of minimal realizations and state-space realizations are discussed explicitly.

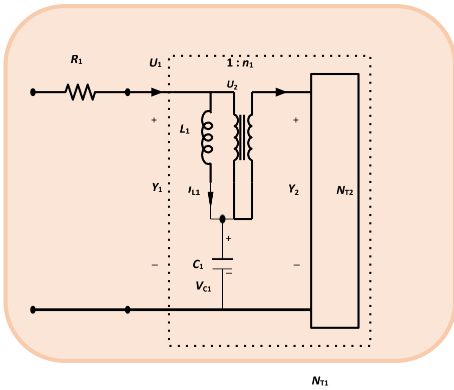


Fig. 2. State Space extraction in the Brune Algorithm.

## 4. THE STATE SPACE REPRESENTATION OF BRUNE'S ALGORITHM-ONE PORT CASE

In this section, $(F, G, H, J)$ is assumed to be a minimal realization of the lossless positive real impedance transfer function $Z_a(s)$ for the case of a one-port network. It is important to mention here that **J** is a scalar in this case. As indicated in Figure 2, to execute Brune's algorithm in state space, we have to start by extracting the series resistance $R_1$.

Using $Z_a(s) = J + H(sI - F)^{-1}G$, it is possible to determine the real part of the impedance as follows:

$$\begin{aligned} Re[Z_a(j\omega)] &= \frac{1}{2}(Z_a(j\omega) + Z_a(-j\omega)); \\ &= J + \frac{1}{2}H(j\omega I - F)^{-1}G \\ &\quad + \frac{1}{2}H(-j\omega I - F)^{-1}G; \\ &= J - HF(\omega^2 I + A^2)^{-1}G \end{aligned} \tag{5}$$

By using the equivalence between the annihilation operator quantum system (1) and its corresponding real quantum system as given by system (4) on page 787 in Maalouf and Petersen (2011a) along with the conditions (8) satisfied (see page 787 in Maalouf and Petersen (2011a)), it is possible to get $Z_{1_r}(s) = D_1 + C_1(sI - A_1)^{-1}B_1$. Also, by using the equivalence between real quantum systems of the form (4) on page 787 in Maalouf and Petersen (2011a) and the corresponding stochastic systems established in James et al. (2008), then $Z_{1_r}(s)$ would correspond to the impedance of a stochastic system. On the other hand, by using Brune's algorithm, with $Z(s) = D + C(sI - A)^{-1}B$ representing the impedance at the terminals of the network $N_{T_1}$, then the extracted resistance $R_1$ is given by

$$R_1 = \min_{\omega} Re[Z_1(j\omega)] \tag{6}$$

where for some frequency $\omega_0$

$$Re[Z_1(j\omega_0)] = R_1. \tag{7}$$

Let the network $N_{T_2}$ in Figure 2 be described by the state-space equations

$$\dot{x}_2 = A_2x_2 + B_2u_2$$
$$y_2 = C_2x_2 + D_2u_2;$$

so that the realization $A_2, B_2, C_2, D_2$ corresponds to the impedance $Z_{2_r}(s) = D_2 + C_2(sI - A_2)^{-1}B_2$ at the terminals of the network $N_{T_2}$ ($D_2$ is a scalar). Thus, the network $N_{T_1}$ has the following state-space equations:

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_{C_1} \\ \dot{x}_{L_1} \end{bmatrix} = \begin{bmatrix} A_2 & 0 & \frac{-B_2}{n_1\sqrt{L_1}} \\ 0 & 0 & \frac{1}{n_1\sqrt{L_1C_1}} \\ \frac{C_2}{n_1\sqrt{L_1}} & -\frac{1}{n_1\sqrt{L_1C_1}} & -\frac{D_2}{n_1^2L_1} \end{bmatrix} \begin{bmatrix} x_2 \\ x_{C_1} \\ x_{L_1} \end{bmatrix} +$$

$$\begin{bmatrix} \frac{B_2}{n_1} \\ \frac{1-\frac{1}{n_1}}{\sqrt{C_1}} \\ \frac{D_2}{n_1^2\sqrt{L_1}} \end{bmatrix} u_1;$$

$$y_1 = \begin{bmatrix} \frac{C_2}{n_1} & \frac{1-\frac{1}{n_1}}{\sqrt{C_1}} & -\frac{D_2}{n_1^2L_1} \end{bmatrix} \begin{bmatrix} x_2 \\ x_{C_1} \\ x_{L_1} \end{bmatrix} + \frac{D_2}{n_1^2}u_1;$$

where $x_{C_1} = \sqrt{C_1}v_{C_1}$ and $x_{L_1} = \sqrt{L_1}i_{L_1}$. Consequently, the state-space equations of the network $N_{T_1}$ are

$$\dot{x} = Ax + Bu;$$
$$y = Cx + Du; \tag{8}$$
$$x = \begin{bmatrix} x_2^T \\ x_{C_1} \\ x_{L_1} \end{bmatrix}; \tag{9}$$

$$A = \begin{bmatrix} A_2 & 0 & \frac{-B_2}{n_1\sqrt{L_1}} \\ 0 & 0 & \frac{1}{n_1\sqrt{L_1C_1}} \\ \frac{C_2}{n_1\sqrt{L_1}} & -\frac{1}{n_1\sqrt{L_1C_1}} & -\frac{D_2}{n_1^2L_1} \end{bmatrix}; \tag{10}$$

$$B = \begin{bmatrix} \frac{B_2}{n_1} \\ \frac{1-\frac{1}{n_1}}{\sqrt{C_1}} \\ \frac{D_2}{n_1^2\sqrt{L_1}} \end{bmatrix}; \tag{11}$$

$$C = \begin{bmatrix} \frac{C_2}{n_1} & \frac{1-\frac{1}{n_1}}{\sqrt{C_1}} & -\frac{D_2}{n_1^2L_1} \end{bmatrix}; \tag{12}$$

$$D = \frac{D_2}{n_1^2}. \tag{13}$$

Then $A, B, C, D$ corresponds to the impedance $Z(s) = D + C(sI - A)^{-1}B$ at the network terminals $N_{T_1}$ ($D$ is a scalar) which is connected to $Z_{1_r}(s)$ according to the following equation

$$Z_{1_r}(s) = Z(s) - R_1. \tag{14}$$

It is shown in Anderson and Moylan (1975) that if $Z(s)$ (a positive-real impedance function) has $A, B, C, D$ as a minimal realization with the condition $Z(j\omega_0) + Z(-j\omega_0) = 0$ for some $\omega_0 > 0$ then a transformation $T_1$ exists which would yield an equivalent state-space realization $A_1, B_1, C_1, D_1$ for the impedance $Z_{1_r}(s)$ in the form given in (8-13) with

$$A = T_1A_1T_1^{-1};$$
$$B = T_1B_1;$$
$$C = C_1T_1^{-1};$$
$$D = D_1.$$

### *4.1 Lemma 1 (one-port case)*

The following lemma provides a procedure that can be used to construct the transformation matrix $T_1$.

*Lemma 1.* Let $A_1, B_1, C_1, D_1$ be a minimal realization of $Z_{1_r}(s)$ (positive-real impedance) satisfying $Z_{1_r}(j\omega_0) + Z_{1_r}(-j\omega_0) = 0$ for some frequency $\omega_0$ ($A_1$ does not have $j\omega_0$ as an eigenvalue). It is possible then to obtain a transformation matrix $T_1$ such that $A = T_1A_1T_1^{-1}$, $B = T_1B_1$, $C = C_1T_1^{-1}$ and $D = D_1$ a re of the form given in (8-13).

The matrix $T_1$ could be constructed as follows: 1) Find a nonsingular matrix $T_{1_a}$ with $(\omega_0^2I + A_1^2)^{-1}B_1$ and $-A_1(\omega_0^2I + A_1^2)^{-1}B_1$ being the last two columns of $T_1^{-1}$. 2) Set $A_b = T_{1_a}A_1T_{1_a}^{-1}$, $B_b = T_{1_a}B_1$ and $C_b = C_1T_{1_a}^{-1}$ and compute

$$\begin{bmatrix} C_b(\omega_0^2I + A_b^2)^{-1} \\ C_b(\omega_0^2I + A_b^2)^{-1}A_b \end{bmatrix} = [\ R_{12}\ \ R_{22}]$$

where $R_{22}$ is a $2 \times 2$ matrix

$$T1_b = \begin{bmatrix} I & 0 \\ R_{22}^{-1}R_{12} & I \end{bmatrix}.$$

2) Set $A_c = T_{1_b}A_bT_{1_b}^{-1}$; $B_c = T_{1_b}B_b$ and $C = C_bT_{1_b}^{-1}$. Then

$$\begin{bmatrix} C_c(\omega_0^2I + A_c^2)^{-1} \\ C_c(\omega_0^2I + A_c^2)^{-1}A_c \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \lambda^2 & 0 \\ 0 & \cdots & 0 & 0 & \psi^2 \end{bmatrix}$$

for non-zero $\lambda, \psi$. Define

$$T_{1_c} = \begin{bmatrix} I & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \psi \end{bmatrix}.$$

Then $T_1 = T_{1_c}T_{1_b}T_{1_a}$. By constructing $T_1$ according to the previous lemma and using the equivalence between the system (1) and the system (4) on page 787 in Maalouf and Petersen (2011a), it is possible to associate the one-port electric circuit in Figure 2 to the annihilation operator linear quantum system given by (1).

## 5. QUANTIZATION OF THE BRUNE CIRCUIT-ONE PORT MODEL

The circuit obtained by using the Brune algorithm described previously is referred to as 'the state-space Brune circuit'. Figure 3 shows a Brune circuit with M stages. In that case, the inductors shunting ideal transformers in Figure 3 are used to replace the inductors at each stage of the Brune circuit in Figure 1 which is justified by the equivalence shown in Figure 4.

Previous circuit-quantization analysis methods did not use ideal transformers Burkard et al. (2004); Burkard (2005);

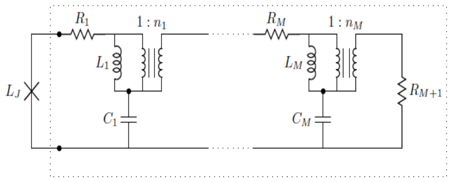


Fig. 3. Brune circuit.

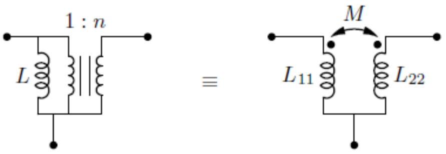


Fig. 4. The equivalent inductive circuit resulting from the Brune algorithm.

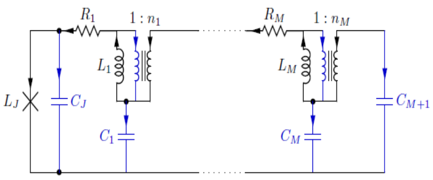


Fig. 5. Modified Brune circuit.


Fig. 6. The inductor-ideal transformer for a turns ratio equal to one.

Devoret (1997), treats them via a new technique which is introduced eliminating matrices involving turns ratios as their entries.

The augmented Brune circuit is shown in Figure 5. $R_{M+1}$ is replaced with a capacitor $C_{M+1}$ through the substitution $\frac{C_{M+1}}{i\omega R_{M+1}}$.

The lossless part of the Brune circuit is illustrated in Figure 5. As shown in Figure 6, this circuit is one of the lossless Foster forms in Foster (1924) for the special case of a unity turns ratio. A capacitance $C_J$ shunting the Josephson junction is also added in order to ensure a non-singular capacitance matrix (case of no degenerate stages).

Next, the loop analysis in Burkard (2005) is extended to ideal transformers. Kirchhoff's laws are given by Equations (4-5) in Burkard (2005)

$$\begin{aligned} FI_{ch} &= -I_{tr}; \\ F^T V_{tr} &= V_{ch}; \end{aligned} \tag{15}$$

where the assumption of having no external fluxes in the circuit loops is made. Here, $F$ is being a matrix with entries being 0, 1 or $-1$ (see Burkard (2005)). After the effective Kirchhoff analysis done below, $F$ will be replaced by the effective matrix $F_{eff}$ with real-valued entries. $I_{tr}$ and $I_{ch}$ are the tree and chord branch current vectors respectively arranged as follows:

$$\begin{aligned} I_{tr} &= (I_J, I_L, I_Z, I_T^{tr}); \\ I_{ch} &= I_{ch}(I_C, I_T^{ch}). \end{aligned}$$

Here the labels $J, L, Z, C, T$ correspond to Josephson junction, inductor, resistor, capacitor and ideal transformer branches, respectively. $T$ represennts the current vectors for ideal transformer branches in the tree and chords respectively. We also partition the loop matrix $F$ according to the partitioning of current vectors

$$\begin{bmatrix} F_{JC} & F_{JT} \\ F_{LC} & F_{LT} \\ F_{ZC} & F_{ZT} \\ F_{TC} & F_{TT} \end{bmatrix}.$$

We will eliminate ideal transformer branches from Kirchhoff laws in equation (15) to obtain loop matrices $F_{eff}$ and $(F^T)_{eff}$ such that we have a new set of effective Kirchhoff relations

$$\begin{aligned} F_{eff I_{ch_{eff}}} &= -I_{tr_{eff}}; \\ (F^T)_{eff V_{tr_{eff}}} &= V_{ch_{eff}} \end{aligned} \tag{16}$$

where

$$\begin{aligned} I_{tr_{eff}} &= (I_J, I_L, I_Z); \\ I_{ch_{eff}} &= I_C \end{aligned}$$

and

$$F_{eff} = \begin{bmatrix} F_{eff_{JC}} \\ F_{eff_{LC}} \\ F_{eff_{ZC}} \end{bmatrix}, \tag{17}$$

$$(F^T)_{eff} = \begin{bmatrix} (F_{JC}^T)_{eff} \\ (F_{LC}^T)_{eff} \\ (F_{ZC}^T)_{eff} \end{bmatrix}.$$

We note here that the entries of the matrix $F^{eff}$ are real numbers, being functions of the ideal transformer turn ratios as we will see below.

In this section, we will derive only the effective Kirchhoff current law in Equation (16) by computing $F_{eff}$ .

Now, we claim that $F_{eff}$ in Equation (17) is given by

$$F_{JC_{eff}} = [(\ 1\ 1 \cdots 1\ 1)] \tag{18}$$

$$F_{LC_{eff}} = \begin{bmatrix} 1 & (1-n_1) & \cdots & (1-n_1) & (1-n_1) \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & (1-n_{M-1}) & (1-n_{M-1}) \\ \text{\huge 0} & & & 1 & (1-n_M) \end{bmatrix}; \tag{19}$$

$$F_{ZC_{eff}} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ \text{\huge 0} & & & 1 & 1 \end{bmatrix} \tag{20}$$

where $F_{JCeff}$ is a row vector of length $(M+1)$, $F_{LCeff}$ and $F_{ZCeff}$ are $M \times (M+1)$ matrices. Note that

$$I_T^{(tr)} = -F_{TC} I_C \tag{21}$$

with

$$F_{TC} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ & 0 & 1 & \cdots & 1 \\ & & \ddots & \ddots & \vdots \\ \text{\Large 0} & & & 0 & 1 \end{bmatrix} \tag{22}$$

where $F_{TC}$ is a $M \times (M+1)$ matrix. We note that $F_{TC}$ does not involve any turns ratios. Using the ideal transformer relations $I_T^{(ch)} = -N I_T^{(tr)}$ where $N$ is the diagonal matrix of turns ratios

$$\begin{bmatrix} n_1 & & 0 \\ & \ddots & \\ 0 & & n_M \end{bmatrix} \tag{23}$$

and Equation (21) we get

$$I_T^{(ch)} = N F_{TC} I_C. \tag{24}$$

The inductor currents are given by

$$I_L = -F_{LC} I_C - F_{LT} I_T^{ch} \tag{25}$$

where

$$F_{LC} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ \text{\Large 0} & & & 1 & 1 \end{bmatrix} \tag{26}$$

and $F_{LT} = -I$. Using equations (24) and (25) we get

$$I_L = -(F_{LC} - N F_{TC}) I_C \tag{27}$$

which gives the loop matrix $F_{LC_{eff}}$,

$$\begin{aligned} F_{LC_{eff}} &= F_{LC} - N F_{TC} \\ &= \begin{bmatrix} 1 & (1-n_1) & \cdots & (1-n_1) & (1-n_1) \\ & \ddots & \ddots & \cdots & \cdots \\ & & 1 & (1-n_{M-1}) & (1-n_{M-1}) \\ \text{\Large 0} & & & 1 & (1-n_M) \end{bmatrix}. \end{aligned}$$

We note that $F_{LC_{eff}}$ is no longer a binary matrix as we have turns ratios appearing in its entries. $F_{JC_{eff}}$ is simply given by

$$\begin{aligned} F_{JC_{eff}} &= F_{JC} \\ &= [1 \ 1 \ \cdots \ 1 \ 1]. \end{aligned}$$

Since the current through the Josephson junction is a function only of the chord capacitor currents

$$I_J = -F_{JC} I_C. \tag{28}$$

Note that $F_{JC_{eff}}$ does not depend on the turns ratios. Similarly the currents through the resistors $R_j$ for $1 \leq j \leq M$ depend only on the chord capacitor currents,

$$I_Z = -F_{ZC} I_C. \tag{29}$$

Hence,

$$\begin{aligned} F_{ZC_{eff}} &= F_{ZC} \\ &= \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ \text{\Large 0} & & & 1 & 1 \end{bmatrix}. \end{aligned}$$

Now one can write an equation of motion for the one-port Brune circuit in Figure 3 in the form of Equation (29) in Burkard (2005).

$$(C + C_Z) * \ddot{\Phi} = -\frac{-\partial U}{\partial \Phi} \tag{30}$$

However, the matrices derived above have to replace the original matrices when computing the quantities in (30) such as $C$ and $C_Z$, as it is shown next.

$C_0$ is computed for the Brune circuit in Figure 3 using Equation (22) of Burkard (2005) with the matrix $\mathcal{F}_{C^{eff}}$

$$\mathcal{C}_0 = \begin{bmatrix} C_J & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{F}_C^{eff} C (\mathcal{F}_{C_{eff}})^T \tag{31}$$

where $C$ is the diagonal matrix of capacitances

$$C = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_{M+1} \end{bmatrix} \tag{32}$$

and

$$\mathcal{F}_{C_{eff}} = \begin{bmatrix} \mathcal{F}_{JC_{eff}} \\ \mathcal{F}_{LC_{eff}} \end{bmatrix}. \tag{33}$$

$L_t$ in Equation (15) of Burkard (2005) is a diagonal matrix of inductances

$$L_t = \begin{bmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_M \end{bmatrix} \tag{34}$$

. with

$$\mathcal{G} = \begin{bmatrix} 0 \\ 1_{(M \times M)} \end{bmatrix} \tag{35}$$

since there are no chord inductors. It follows using Equation (31) of Burkard (2005) that

$$\begin{aligned} M_0 &= \mathcal{G} L_t^{-1} \mathcal{G}^t \\ &= \begin{bmatrix} 0 & 0 \\ 0 & L_t^{-1} \end{bmatrix}. \end{aligned}$$

A transformation matrix $T$ is determined in order to make the Langrangian description (i.e., both $C_0$ and $M_0$) of the system band-diagonal:

$$T = \begin{bmatrix} \begin{matrix} 1 \\ \frac{-1}{(1-n_1)} \end{matrix} & \frac{-1}{(1-n_1)} & & \\ & \frac{1}{(1-n_2)} & \frac{1}{(1-n_2)} & \text{\Large 0} \\ & & \ddots & \ddots \\ & \text{\Large 0} & & \frac{(-1)^M}{(1-n_M)} & \frac{(-1)^M}{(1-n_M)} \end{bmatrix}.$$

Applying $T$ to $C_0$ and $M_0$ we get

$$\mathcal{C} = T^t \mathcal{C}_0 T \begin{bmatrix} C_J + n_1^2 C_1' & n_1 C_1' & & & \\ n_1 C_1' & C_1' + n_2^2 C_2' & \ddots & \text{\Large 0} & \\ & \ddots & \ddots & & \\ & \text{\Large 0} & & C_{M-1}' + n_M^2 C_M' & n_M C_M' \\ & & & n_M C_M' & C_M' + C_{M+1} \end{bmatrix};$$

$$\mathcal{M}_0 = T^t \mathcal{M}_0 T \begin{bmatrix} \frac{1}{L_1'} & \frac{1}{L_1'} & & & \\ \frac{1}{L_1'} & \frac{1}{L_1'} + \frac{1}{L_2'} & \frac{1}{L_2'} & & 0 \\ & \frac{1}{L_2'} & \frac{1}{L_2'} + \frac{1}{L_3'} & \ddots & \\ & & \ddots & \ddots & \\ & 0 & & \frac{1}{L_{M-1}'} + \frac{1}{L_M'} & \frac{1}{L_M'} \\ & & & \frac{1}{L_M'} & \frac{1}{L_M'} \end{bmatrix}$$

where $C_j' = C_j/(1-n_j)^2$, $L_j' = L_j/(1-n_j)^2$. A Lagrangian $\mathcal{L}_0$ (and equivalently a Hamiltonian $\mathcal{H}_S$) is given by

$$\mathcal{L}_0 = \frac{1}{2}\dot{\Phi}^T \mathcal{C} \dot{\Phi} - U(\Phi), \mathcal{H}_S = \frac{1}{2} Q^T \mathcal{C}^{-1} Q + U(\Phi). \quad (36)$$

Here, $\Phi$ is the vector representing the transformed coordinates of length $(M+1)$ and $\Phi_1 = (\frac{\Phi_0}{2\pi})\varphi_J$. It is important to mention here that the transformation $T$ in Equation (36) introduces a relationship between $\Phi$ and $\Phi'_{L_j}$ across the inductors $L_j's$ for $1 \leq j \leq M$ in the Brune circuit in Figure 4. By this means, it follows that the flux $\Phi_{L_j}$ is a superposition of two consecutive coordinates $\Phi_j$ and $\Phi_{j+1}$ given by the relation $\Phi_L = T\Phi$ which gives

$$\Phi_{L_j} = \frac{(-1)^j}{(1-n_j)}(\Phi_j + \Phi_{j+1}) \quad (37)$$

for $1 \leq j \leq M$. $\Phi_L$ is the vector representing the fluxes flowing in the inductors of the Brune circuit in Figure 3 such that

$$\Phi_L = (\Phi_J \, \Phi_{L_1} \cdots \Phi_{LM})^T \quad (38)$$

with $\Phi_J = (\frac{\Phi_0}{2\pi})\varphi_J$ being the flux across the Josephson junction.

## 6. CONCLUSION

In this paper, a one port impedance quantization theory for a class of annihilation operator linear quantum systems with positive real impedance properties has been developed using the Brune algorithm. This theory, illustrated for one-port circuits may be useful for the implementation of superconducting microwave circuits related to quantum filters found in the field of quantum computing.

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