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Control of Dynamical Systems with Given Restrictions on Output Signal with Application to Linear Systems ⋆

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Abstract: We propose a novel method for control of dynamical systems that ensures the belonging of an output signal to the given set at any time. The method is based on a special change of coordinates such that the initial problem with given restrictions on an output variable can be performed as the problem of the input-to-state stability analysis of a new extended system without restrictions. This method is used for control of linear plants.

Keywords: Dynamical system, nonlinear transformation, input-to-state stability, control.

1. INTRODUCTION

The paper Miller and Davison (1991) proposes the adaptive control method which ensures belonging of output signal to given sets. These sets may be different for transient and steady state modes. The sets are performed by a sequence of rectangles. The height of each rectangle corresponds to the desired maximum deviation of the output variable from the equilibrium position. The length of the rectangle corresponds to the desired time when the output variable belongs to the corresponding rectangle. However, the rectangular areas in Miller and Davison (1991) are rather rough and the algorithm is applicable only for plants with scalar input and output signals.

Differently from Miller and Davison (1991), in the paper Bechlioulis and Rovithakis (2008) a control method with the guarantee of belonging the output signal to a given set for plants with vector input and vector output is proposed. However, the implementation of this method requires knowledge of the sign and knowledge of the set of initial conditions. Moreover, obtained upper and lower bounds for transients are rather rough because these bounds are determined by the same function with different signs. Additionally, the upper and lower bounds asymptotically converge to some constants.

In the present paper, we propose a new control method with providing an output signal to a given set. Differently from Bechlioulis and Rovithakis (2008), the given set can be described by functions that independent on the sign of plant initial conditions. Only knowledge of the set of initial values is required. Also, unlike Miller and Davison (1991); Bechlioulis and Rovithakis (2008), the configuration of the given set can be described by arbitrary continuously differentiable functions for which asymptotic convergence is not required. As a result, the obtained method significantly expands the class of tasks compared with Miller and Davison (1991); Bechlioulis and Rovithakis (2008).

The paper is organized as follows. In Section 2 the control problem is formulated. Section 3 describes the main result, where a special change of coordinate is proposed. As a result, the initial problem with restrictions can be performed as the problem of the input-to-state stability analysis of a new extended dynamical system without restrictions. Also in Section 3 examples of coordinate change are given. Sections 4 and 5 propose a state feedback control algorithms for linear plants with known parameters and unknown external bounded disturbances. Also, in Sections 4-?? the simulations illustrate confirmation of theoretical results and show the effectiveness of the proposed method in the presence of parametric uncertainty and external disturbances.

Notations. Throughout the paper the superscript T stands for matrix transposition; $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$; $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices; I is the identity matrix of corresponding order; A^* is the adjugate of the matrix A.

2. PROBLEM FORMULATION

Consider a dynamical system in the form

$$\begin{aligned} \dot{x} &= F(x,u,t), \\ y &= h(x), \end{aligned} \tag{1}$$

where $t \geq 0$, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control signal, $y = col\{y_1, ..., y_v\}$ is the output signal.

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The vector function F is defined for all x, u, t and it is a piecewise continuous and bounded function in t. The function $h(x)$ is continuously differentiable w.r.t. x. Plant (1) is controllable and observable for all $x \in \mathbb{R}^n$.

Our objective is to design a control law that ensures the input-to-state stability (ISS) of the closed-loop system and the signal $y(t)$ belongs to the following set

$$\mathcal{Y} = \left\{ y \in \mathbb{R}^v : \ \underline{g}_i(t) < y_i(t) < \overline{g}_i(t), \ i = 1, ..., v \right\} \quad (2)$$

for all $t \geq 0$. Here $\underline{g}_i(t)$ and $\overline{g}_i(t)$ are bounded functions with their first time derivatives. These functions are chosen by the designer.

Differently from Bechlioulis and Rovithakis (2008), goal (2) is independed on the sign of plant initial conditions. Also, unlike Miller and Davison (1991); Bechlioulis and Rovithakis (2008), the set $\mathcal{Y}$ in (2) can be described by arbitrary continuously differentiable functions for which asymptotic convergence is not required.

3. CONDITIONS OF CONTROL OF DYNAMICAL SYSTEMS WITH GIVEN RESTRICTIONS ON OUTPUT SIGNAL

Let us consider a change of the output variable $y(t)$ in the form

$$y(t) = \Phi(\varepsilon(t), t), \quad (3)$$

where $\varepsilon(t) \in \mathbb{R}^v$ is the continuously differentiable vector function w.r.t. t, the function $\Phi(\varepsilon, t) = col\{\Phi_1(\varepsilon, t), ..., \Phi_v(\varepsilon, t)\}$ satisfies the following conditions:

(a) $\underline{g}_i(t) < \Phi_i(\varepsilon, t) < \overline{g}_i(t)$, $i = 1, ..., v$ for all $t \geq 0$ and $\varepsilon \in \mathbb{R}^v$;
(b) there exists the inverse function $\varepsilon = \Phi^{-1}(y, t)$ for all $y \in \mathcal{Y}$ and $t \geq 0$;
(c) the function $\Phi(\varepsilon, t)$ is continuously differentiable in ε and t as well as $\det\left(\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon}\right) \neq 0$ for all $t \geq 0$ and $\varepsilon \in \mathbb{R}^v$;
(d) the function $\frac{\partial \Phi(\varepsilon,t)}{\partial t}$ is bounded on $t \geq 0$ for all $\varepsilon \in \mathbb{R}^v$.

Consider several examples of the function $\Phi(\varepsilon, t)$.

Example 1. In Example 1 introduce the function $S(\varepsilon)$ in the form $S(\varepsilon) = \frac{\overline{r}e^{\varepsilon} + \underline{r}}{e^{\varepsilon}+1}$, where $0 < \underline{r} < \overline{r}$. Then the inverse function $\varepsilon = \ln \frac{\underline{r}gy}{y - \overline{r}g}$ is valid for $\underline{r}g(t) < y(t) < \overline{r}g(t)$ and $g(t) > 0$ or for $\overline{r}g(t) < y(t) < \underline{r}g(t)$ and $g(t) < 0$.

Example 2. Let $\Phi(\varepsilon, t) = \frac{\overline{g}(t)e^{\varepsilon} + \underline{g}(t)}{e^{\varepsilon}+1}$, where $\Phi(\varepsilon, t) \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, the functions $\overline{g}(t)$, $\underline{g}(t)$, $\dot{\overline{g}}(t)$ and $\dot{\underline{g}}(t)$ are bounded for all t and $\underline{g}(t) < \overline{g}(t)$. Taking into account (3), the inverse function $\varepsilon = \ln \frac{\underline{g} - y}{y - \overline{g}}$ is performed for $\underline{g}(t) < y(t) < \overline{g}(t)$ for all t

Now we define the dynamics of the variable $\varepsilon(t)$ for the ISS analysis of the closed-loop system. Take the derivative of (3) w.r.t. t and rewrite result as $\dot{y} = \frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon}\dot{\varepsilon} + \frac{\partial \Phi(\varepsilon,t)}{\partial t}$. Since $\det\left(\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon}\right) \neq 0$ (see condition (c)), rewrite the dynamics of $\varepsilon(t)$ in the form

$$\dot{\varepsilon} = \left(\frac{\partial \Phi(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \left(\dot{y} - \frac{\partial \Phi(\varepsilon, t)}{\partial t}\right). \quad (4)$$

Theorem 1. Let conditions (a)-(d) hold for (3). If there exists the control law u such that the solutions of (1) and (4) are bounded, then $y(t) \in \mathcal{Y}_\alpha \subset \mathcal{Y}$. If the solutions of (4) are unbounded, then $y(t) \in \mathcal{Y}_\beta \subseteq \mathcal{Y}$.

Proof 1. Let the control law u be chosen such that the solutions of (4) are bounded. Then $|\varepsilon(t)| < N$ for all t, where $N > 0$. According to (3), $y \in \mathcal{Y}_\alpha = \{y \in \mathbb{R}^v : \underline{M}_i(t) \leq y_i(t) \leq \overline{M}_i(t), \ i = 1, ..., v\}$ for all t, where $\underline{M}_i(t) = \inf_{|\varepsilon| \leq N}\{\Phi_i(\varepsilon, t)\}$ and $\overline{M}_i(t) = \sup_{|\varepsilon| \leq N}\{\Phi_i(\varepsilon, t)\}$. Since (3) is a bijective function, $\overline{M}_i(t) < \overline{g}_i(t)$ and $\underline{M}_i(t) > \underline{g}_i(t)$ for all t.

If the control law does not provide the boundedness of the solution of (4), then $y \in \mathcal{Y}_\beta = \{y \in \mathbb{R}^v : \underline{S}_i(t) < y_i(t) < \overline{S}_i(t), \ i = 1, ..., v\}$, where $\underline{S}_i(t) = \inf_{\varepsilon \in \mathbb{R}^v}\{\Phi_i(\varepsilon, t)\}$ and $\overline{S}_i(t) = \sup_{\varepsilon \in \mathbb{R}^v}\{\Phi_i(\varepsilon, t)\}$ for all t. Since (3) is a bijective function, $\overline{S}_i(t) \leq \overline{g}_i(t)$ and $\underline{S}_i(t) \geq \underline{g}_i(t)$ for all t . Theorem 1 is proved.

In the next sections we will demonstrate the proposed method for some plants.

4. STATE FEEDBACK CONTROL

Let the plant be described by the following linear differential equation

$$\begin{aligned} \dot{x} &= Ax + Bu + Df, \\ y &= Lx. \end{aligned} \quad (5)$$

The signals $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$ are measured, $f \in \mathbb{R}^l$ is the unknown bounded disturbance, the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$ and $L \in \mathbb{R}^{1 \times n}$ are known, the matrix D is unknown. The pair (A, B) is controllable and the pair (L, A) is observable.

We formulate a result that contains the "simplest" control law in the sense of the "convenience" stability analysis of the closed-loop system.

Theorem 2. Let conditions (a)-(d) hold for transformation (3), $\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon} > 0$ for all ε and t, and there exists the vector $T \in \mathbb{R}^n$ such that the matrix $(I - (LB)^{-1}BL)A - TL$ is Hurwitz. Given $\alpha > 0$ and $K > 0$ there exists $\beta > 0$ such that the linear matrix inequality (LMI)

$$\begin{bmatrix} \alpha - K & 0.5 \\ 0.5 & -\beta \end{bmatrix} \leq 0 \quad (6)$$

holds. Then the control law

$$u = -(LB)^{-1}\left[LAx + K\varepsilon\right] \quad (7)$$

ensures goal (2).

Proof 2. Taking into account (3) and (5), rewrite expression (4) in the form

$$\dot{\varepsilon} = \left(\frac{\partial \Phi(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} (LAx + LBu + \varphi), \quad (8)$$

where $\varphi = LDf - \frac{\partial \Phi(\varepsilon,t)}{\partial t}$ is the bounded function w.r.t. ε and t. Substituting the control law (7) into the first equation of (5) and (8), we get

$$\dot{x} = (A - B(LB)^{-1}LA - TL)x \qquad (9)$$
$$- KB(LB)^{-1}\varepsilon + Df + T\Phi(\varepsilon, t), \qquad (10)$$
$$\dot{\varepsilon} = \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}[-K\varepsilon + \varphi]. \qquad (11)$$

Analyze equation (11) on the ISS. To this end, choose Lyapunov function of the form $V = 0.5\varepsilon^2$. Substituting (11) into the condition $\dot{V} + 2\alpha V \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} - \beta\varphi^2 \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} \leq 0$, where $\alpha > 0$ and $\beta > 0$, we get $-(K-\alpha)\varepsilon^2 + \varepsilon\varphi - \beta\varphi^2 \leq 0$. If LMI (6) holds, then the last inequality is satisfied and system (11) is stable. Consequently, the signal $\varepsilon(t)$ is bounded. If the matrix $A - B(LB)^{-1}LA - TL$ is Hurwitz, then the boundedness of the signal $x(t)$ follows from the boundedness of the signals $\varepsilon(t)$, $\Phi(\varepsilon, t)$ and $f(t)$. Therefore, the control law $u(t)$ given by (7) is bounded. Taking into account Theorem 1, goal (3) is satisfied. Theorem 2 is proved.

Example 5. Let in (5) parameters are given in the forms

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad L = [1 \ 2],$$
$$x(0) = [2 \ 1]^{\mathrm{T}}, \quad f(t) = 0.1 + \sin(3t) + sat\left(\frac{d(t)}{0.3}\right), \qquad (12)$$

where $sat(\cdot)$ is the saturation function, the signal $d(t)$ is simulated in Matlab Simulink by using the "Band-Limited White Noise" block with a noise power of 0.1 and a sampling time of 0.1. It is required to ensure that the output signal $y(t)$ belongs to the set $\underline{r}g(t) < y(t) < \overline{r}g(t)$, where $\underline{r} = 0.8$ and $\overline{r} = 1$, and the function $g(t)$ will be given below.

The matrix $A - B(LB)^{-1}LA - TL$ is Hurwitz, for example, for all $T = [T_1 \ T_2]^{\mathrm{T}}$, where $T_1 > 0$ and $T_2 > 0$. Choose $K = 1$ in (7). Define the function $\Phi(\varepsilon, t)$ as in Example 2, where g is given by

$$g(t) = (g_0 - g_\infty)e^{-kt} + g_\infty. \qquad (13)$$

Here $g_0 = y(0) + 0.01$, $g_\infty = 0.1$ and $k = 0.5$. Fig. 1 shows the transients in $y(t)$, $u(t)$ and $f(t)$. The oscillations of the control signal in Fig. 1,*b* are caused by the presence of the disturbance f. Moreover, it follows from Fig. 1,*b* that after third second the magnitude of the control signal is comparable with the magnitude of the disturbance. Fig. 2 presents the simulations under $f = 0$. Thus, the plant can be stabilized in a given set by a not large value of the control signal.

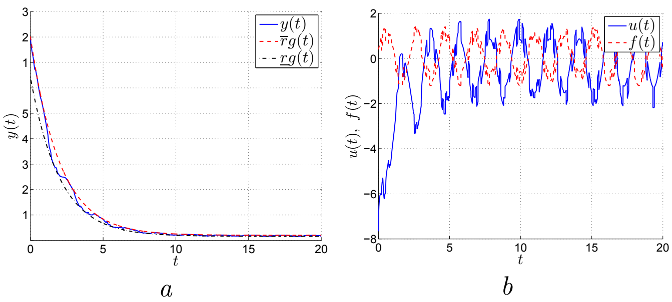

Fig. 1. The transients in $y(t)$ (a), $u(t)$ $f(t)$ (b) for $g(t)$ given by (13).

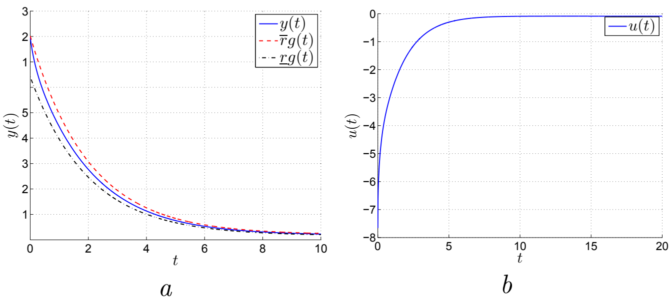

Fig. 2. The transients in $y(t)$ (a) $u(t)$ (b) for $g(t)$ given by (14) for $f = 0$.

Fig. 3 shows the simulations for $y(t)$ and $u(t)$ for the set $0.8g(t) < y(t) < g(t)$, where the function $g(t)$ is given by

$$g(t) = g_0 \sin(kt) + g_0 + g_\infty. \qquad (14)$$

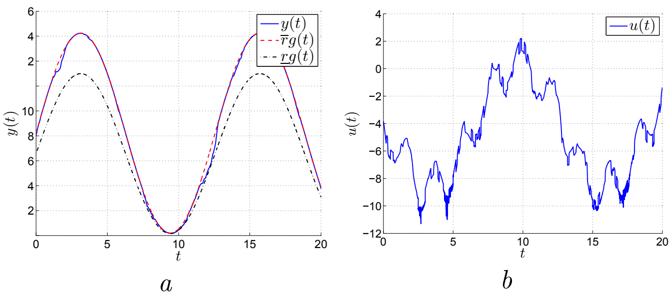

Fig. 3. The transients in $y(t)$ (a) $u(t)$ (b) for $g(t)$ given by (14).

5. OUTPUT FEEDBACK CONTROL

Consider a plant model in the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Df, \\ y &= Lx. \end{aligned} \qquad (15)$$

Here the state vector $x \in \mathbb{R}^n$ is unmeasured, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^v$ are measured signals, the disturbance $f \in \mathbb{R}^l$ is bounded signal. The matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $L \in \mathbb{R}^{v \times n}$ are known and the matrix $D \in \mathbb{R}^{n \times l}$ is unknown. The pair (A, B) is controllable and the pair (L, A) is observable.

Introduce the control law in the form

$$u = K_1 y + K_2 \varepsilon, \qquad (16)$$

where $K_1 \in \mathbb{R}^{m \times v}$ and $K_2 \in \mathbb{R}^{m \times v}$ are chosen by the designer. In particular, K_1 and K_2 can be chosen such that the matrices $A + BK_1L$ and LBK_2 are Hurwitz. Taking into account (3) and (16), rewrite (4) and (15) in the forms

$$\begin{aligned} \dot{x} &= (A + BK_1L + T_1L)x + BK_2\varepsilon \\ &\quad + Df - T_1\Phi(\varepsilon, t), \\ \dot{\varepsilon} &= \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\big[LBK_2\varepsilon + (LA + LBK_1L \\ &\quad + T_2L)x + LDf \\ &\quad - \frac{\partial\Phi(\varepsilon,t)}{\partial t} - T_2\Phi(\varepsilon, t)\big]. \end{aligned} \qquad (17)$$

Here $T_1 \in \mathbb{R}^{n \times v}$ and $T_2 \in \mathbb{R}^{v \times v}$. Introduce the following notation

$$
\begin{aligned}
x_e &= col\{x,\ \varepsilon\}, \quad f_e = col\left\{f,\ \frac{\partial\Phi(\varepsilon,t)}{\partial t},\ \Phi(\varepsilon,t)\right\},\\
A_{21}(\varepsilon,t) &= \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}(LA + LBK_1L + T_2L),\\
A_{22}(\varepsilon,t) &= \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} LBK_2,\\
A_e(\varepsilon,t) &= \begin{bmatrix} A + BK_1L + T_1L & BK_2\\ A_{21} & A_{22}\end{bmatrix},\\
D_{21}(\varepsilon,t) &= \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} LD,\\
D_{22}(\varepsilon,t) &= -\left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1},\\
D_{23}(\varepsilon,t) &= -\left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} T_2,\\
D_e(\varepsilon,t) &= \begin{bmatrix} D & 0 & -T_1\\ D_{21}(\varepsilon,t) & D_{22}(\varepsilon,t) & D_{23}(\varepsilon,t)\end{bmatrix}.
\end{aligned} \tag{18}
$$

Considering (18), rewrite (17) as follows

$$\dot{x}_e = A_e(\varepsilon,t)x_e + D_e(\varepsilon,t)f_e. \tag{19}$$

Theorem 3. Let conditions (a)-(d) hold for transformation (3), $\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon} > 0$ for all ε and t. Given $\alpha > 0$, K_1, K_2, T_1 and T_2 there exist the coefficient $\beta > 0$ and the matrix $P = P^{\mathrm{T}} > 0$ such that the following matrix inequality holds

$$\begin{bmatrix}\Psi_{11}(\varepsilon,t) & PD_e(\varepsilon,t)\\ * & -\beta I\end{bmatrix} \leq 0. \tag{20}$$

Here " $*$ " defines the symmetric block of the symmetric matrix, $E = [I\ 0]$, $\Psi_{11}(\varepsilon,t) = A_e(\varepsilon,t)^{\mathrm{T}}P + PA_e(\varepsilon,t) + \alpha P$. Then control law (16) ensures goal (2).

Proof 3. For the ISS analysis of (19) consider Lyapunov function in the form $V = x_e^{\mathrm{T}}Px_e$. Considering (19) and substituting the expression for V in the inequality

$$\dot{V} + \alpha V - \beta f_e^{\mathrm{T}} f_e \leq 0, \tag{21}$$

we get

$$
\begin{aligned}
&x_e^{\mathrm{T}}[A_e(\varepsilon,t)^{\mathrm{T}}P + PA_e(\varepsilon,t) + \alpha P]x_e - \beta f_e^{\mathrm{T}} f_e\\
&+2x_e^{\mathrm{T}}PD_e(\varepsilon,t)f_e \leq 0.
\end{aligned} \tag{22}
$$

Introduce the new vector $z = col\{x_e, f_e\}$ and rewrite inequality (22) as

$$z^{\mathrm{T}}\begin{bmatrix}\Psi_{11}(\varepsilon,t) & PD_e(\varepsilon,t)\\ * & -\beta I\end{bmatrix} z \leq 0. \tag{23}$$

Inequality (23) is satisfied if inequality (20) holds. Therefore, the function $x_e(t)$ is bounded from (21). Thus, the signals $x(t)$ and $\varepsilon(t)$ are bounded. Then control law (16) is bounded. Tacking into account Theorem 1, goal (3) is satisfied. Theorem 3 is proved.

Example 6. Let in (15)

$$
A = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0.1 & -2 & -3\end{bmatrix},\quad B = \begin{bmatrix}1 & 2\\ 1 & 1\\ 1 & 2\end{bmatrix},\quad D = \begin{bmatrix}1\\1\\1\end{bmatrix},
$$
$$
L = \begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\end{bmatrix},\quad G = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0.1 & 0.1 & 0.1\end{bmatrix},
$$

the disturbance $f(t)$ is given by (12).

Choose $K_1 = 0.01\begin{bmatrix}0 & 0\\ -1 & -1\end{bmatrix}$, $K_2 = \begin{bmatrix}1.5 & -1.75\\ -1 & 1\end{bmatrix}$ in control law (16). Additionally, choose $T_1 = \begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 1\end{bmatrix}$ $T_2 = \begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$.

Let $\Phi(\varepsilon,t) = diag\{\Phi_1(\varepsilon_1,t), \Phi_2(\varepsilon_2,t)\}$, where Φ_i is given in example 3: $\Phi_i(\varepsilon_i,t) = \frac{\overline{g}_i(t)e^{\varepsilon_i} + \underline{g}_i(t)}{e^{\varepsilon_i}+1}$, $i = 1,2$. Therefore, $\Phi(\varepsilon,t) > 0$ for all ε and t. Then $\frac{\partial\Phi_i(\varepsilon_i,t)}{\partial\varepsilon_i} = \frac{e^{\varepsilon_i}(\overline{g}_i(t) - \underline{g}_i(t))}{(e^{\varepsilon_i}+1)^2} > 0$ since $\overline{g}_i(t) > \underline{g}_i(t)$. Additionally, $\left(\frac{\partial\Phi_i(\varepsilon_i,t)}{\partial\varepsilon_i}\right)^{-1} \to +\infty$ at $\varepsilon \to +\infty$ and the smallest value of $\left(\frac{\partial\Phi_i(\varepsilon_i,t)}{\partial\varepsilon_i}\right)^{-1}\Big|_{\varepsilon=0} = \frac{4}{\overline{g}_i(t) - \underline{g}_i(t)} > 0$.

According to Fridman (2010), if LMI is feasible on the vertices of a polytope, then inside the polytope LMI also is feasible. In our case for every fixed $\frac{\partial\Phi_i(\varepsilon_i,t)}{\partial\varepsilon_i}$ the matrix inequality (20) is linear. However, the polytop is unbounded, since $\left(\frac{\partial\Phi_i(\varepsilon_i,t)}{\partial\varepsilon_i}\right)^{-1} \to +\infty$ at $\varepsilon \to +\infty$. The simulations with increasing $\left(\frac{\partial\Phi_i(\varepsilon_i,t)}{\partial\varepsilon_i}\right)^{-1}$ show that the eigenvalues of the matrix P converge to some positive values. At the vertices $\frac{4}{\overline{g}_i(t) - \underline{g}_i(t)}$ the matrix inequality (20) holds too.

Choose the parameters of the function $\Phi(\varepsilon,t)$ in the form

$$
\begin{aligned}
\overline{g}_1(t) &= (g_0 - g_1)e^{-kt} + g_1,\\
\overline{g}_2(t) &= (g_0 - g_2)\cos(kt) + g_4,\\
\underline{g}_1(t) &= (g_0 - g_2)e^{-kt} + g_3,\\
\underline{g}_2(t) &= \cos(kt) + g_5,
\end{aligned} \tag{24}
$$

where $g_0 = \sqrt{y^{\mathrm{T}}(0)y(0)} + 0.01$, $g_1 = 0.1$, $g_2 = 2$, $g_3 = -0.2$, $g_4 = g_0 - 0.1$, $g_5 = 0.8$ and $k = 0.5$. Fig. 4, 5 show the transients in $y_1(t)$, $y_2(t)$ and $u(t) = col\{u_1(t), u_2(t)\}$ for $x(0) = col\{\frac{5}{3}, \frac{2}{3}, -1\}$.

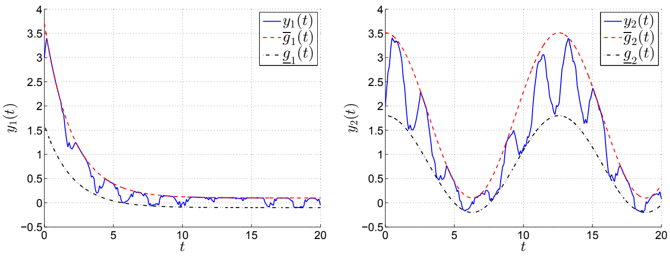

Fig. 4. The transients in $y_1(t)$ and $y_2(t)$ for $\Phi(\varepsilon,t)$ with (24) and $x(0) = col\{\frac{5}{3}, \frac{2}{3}, -1\}$.

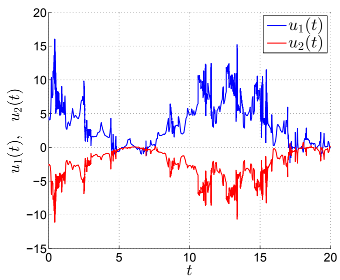

Fig. 5. The transients in $u(t) = col\{u_1(t), u_2(t)\}$ for $\Phi(\varepsilon,t)$ with (24) and $x(0) = col\{\frac{5}{3}, \frac{2}{3}, -1\}$.

Note that the control law $u = K_1 y + K_2 \varepsilon$ does not depend on the parameters of plant (5). The simulations show the proposed control low is robust under unknown parameters of (5). Thus, the closed-loop system remains stable for $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}$ and $G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g_{\varphi 1} & g_{\varphi 2} & g_{\varphi 3} \end{bmatrix}$, where $a_1 \in [-5; 0.1]$, $a_2 \in [-5; -2]$, $a_3 \in [-5; -3]$, $b \in [0.5; 10]$ $g_{\varphi 1} \in [-3; 3]$, $g_{\varphi 2} \in [-3; 3]$ and $g_{\varphi 3} \in [-3; 3]$.

According to (3) and (a), the initial value $y(0)$ must belong to the sets $\underline{g}_i(0) < y_i(0) < \overline{g}_i(0)$, $i = 1, 2$. If the initial conditions have significant uncertainty, then the functions $\underline{g}_i(t)$ and $\overline{g}_i(t)$ can be specified with a margin at the initial time. For example, the functions $\underline{g}_i$ and $\overline{g}_i$ can be presented in the form

$$\begin{aligned} \overline{g}_1(t) &= (g_0 - g_1)e^{-kt} + g_1 + g_6 e^{-k_0 t}, \\ \overline{g}_2(t) &= (g_0 - g_2)\cos(kt) + g_4 + g_6 e^{-k_0 t}, \\ \underline{g}_1(t) &= (g_0 - g_2)e^{-kt} + g_3 - g_6 e^{-k_0 t}, \\ \underline{g}_2(t) &= \cos(kt) + g_5 - g_6 e^{-k_0 t}, \end{aligned} \tag{25}$$

where $g_6 = 3$ and $k_0 = 2$. Fig. 6 illustrates the plots of the output signals $y_1(t)$ and $y_2(t)$ with $x(0) = col\{\frac{10}{3}, -\frac{5}{3}, -1\}$.

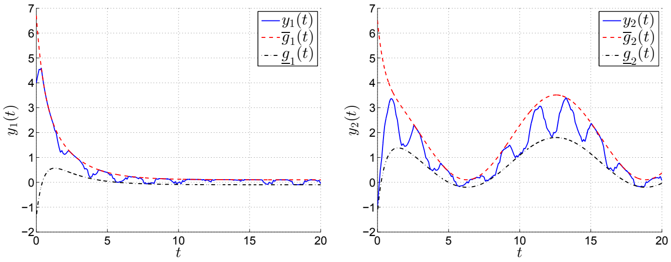

Fig. 6. The transients in $y_1(t)$ and $y_2(t)$ for $\Phi(\varepsilon, t)$ with (25) and $x(0) = col\{\frac{10}{3}, -\frac{5}{3}, -1\}$.

The simulations show that the transients in y can be close to the boundaries of $\overline{g}(t)$ and $\underline{g}(t)$. From $\varepsilon = \ln \frac{\underline{g} - y}{y - \overline{g}}$ it follows that the value of $|\varepsilon(t)|$ can take large values. Therefore, the computational load of the controller is increased. As a result, Matlab work is increased and sometimes Matlab gives an error in the calculations. To prevent this problem, it is recommended to select the parameters of the loop of ε more than the parameters of the loop of y. Thereby, the transient time in $\varepsilon(t)$ is reduced in comparison with the transient time for $y(t)$. Moreover, it increases robustness w.r.t. uncertainty of plant parameters and the large value of the disturbance f. Let us demonstrate this fact. Rewrite the control law as $u = K_1 y + \gamma K_2 \varepsilon$, $\gamma > 0$. Increasing γ, the transients in y keep away from the boundaries $\overline{g}(t)$ and $\underline{g}(t)$ (see Fig. 7).

6. CONCLUSION

The method for control of dynamical systems based on a special change of coordinates is proposed. According to this method, the initial control problem with the given restriction on an output variable leads to the problem of the input-to-state stability analysis of a new extended system without restrictions. As a result, a plant output signal belongs to a given set at any time in the closed-loop system. The examples of change of coordinates that can be used for design algorithms are presented. Based on the proposed method, the new control laws for linear plants are designed.

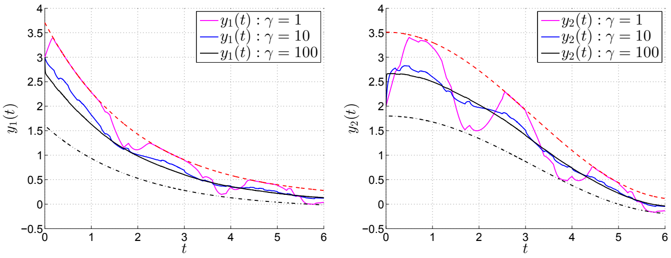

Fig. 7. The transients in $y_1(t)$ and $y_2(t)$ for $\gamma = 1$, $\gamma = 10$ and $\gamma = 100$ and $x(0) = col\{1\ 1\ 0\}$.

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