Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Adaptive Boundary Observer Design for coupled ODEs-Hyperbolic PDEs systems ⋆

Mohammad Ghousein * **Emmanuel Witrant** *

* *Univ. Grenoble Alpes, GIPSA-Lab, 11 Rue des Mathématiques 38400 Saint-Martin-d'Hères, France(e-mail: mohammad.ghousein@gipsa-lab.fr, emmanuel.witrant@gipsa-lab.fr).*

Abstract: We consider the state estimation of n_ξ hyperbolic PDEs coupled with n_X ordinary differential equations at the boundary. The hyperbolic system is linear and propagates in the positive x-axis direction. The ODE system is linear time varying (LTV) and includes a set of n_θ unknown constant parameters, which are to be estimated simultaneously with the PDE and the ODE states using boundary sensing. We design a Luenberger state observer, and our method is mainly based on the decoupling of the PDE estimation error states from that of the ODEs via swapping design. We then derive the observer gains through the Lyapunov analysis of the decoupled system. Furthermore, we give sufficient conditions of the exponential convergence of the adaptive observer through differential Lyapunov inequalities (DLIs) and we illustrate the theoretical results by numerical simulations.

Keywords: Hyperbolic partial differential equations, Adaptive boundary Observers, Boundary Control.

1. INTRODUCTION

Many physical processes are modeled using linear hyperbolic partial differential equations coupled with linear ordinary differential equations. The infinite state which is modeled by the PDE represents the transport in space, and its value at the boundary is usually constrained to some exterior dynamics represented by the ODEs. The mentioned coupling topology mostly appears in networks, where the edges are modeled using transport PDEs and the nodes are modeled using ODEs. Examples of such systems can be found in road traffic in Goatin (2006), gas flow in pipelines in Gugat and Dick (2011), flow in open channels in Coron et al. (2007), exhaust gas regulation (EGR) in car engines in Castillo et al. (2014), etc. Practically speaking, boundary control and observation of these kinds of systems is more realistic than the distributed ones, since actuators and sensors are placed naturally at the extremities of the domain. In addition, in several real applications, we may not have complete knowledge of the system's parameters on both the PDE and the ODE sides. This adds more complexity to the control and the observer designs in view of the limited amount of available measurements. In short, the idea of developing adaptive boundary controls and observers for coupled ODEs-hyperbolic PDEs systems is a necessity if we consider the significant number of physical applications.

Boundary observers for ODEs-coupled hyperbolic PDEs are not widely investigated in the literature. The authors in Castillo et al. (2013) designed a Luenberger observer for systems of linear and quasilinear hyperbolic systems with dynamic boundary conditions which are asymptotically stable. This approach was later extended by the authors in Ferrante and Cristofaro (2019) to linear hyperbolic systems coupled with possibly unstable LTI systems. By keeping the same observer architecture in Castillo et al. (2013) but using a non-diagonal quadratic Lyapunov function, the authors in Ferrante and Cristofaro (2019) have derived sufficient conditions for the exponential stability of the observer through bilinear matrix inequalities (BMIs). On the other hand, backstepping boundary observer designs are also investigated for coupled ODEs-hyperbolic PDEs systems. The authors in Krstic and Smyshlyaev (2008) synthesized an observer for LTI systems with arbitrary constant delay in the sensor measurement. The delay is interpreted as a first order transport equation and backstepping observer design is used on the resulting coupled LTI-PDE system. This work was later extended by the authors in Hasan et al. (2016) to a 2×2 hyperbolic system coupled with a linear LTI system at the boundary.

All the results mentioned so far assume a perfect knowledge of the system. In many practical cases, some model parameters are unknown, which motivates the need for adaptive estimators. The objective of an adaptive boundary observer is to simultaneously construct the distributed PDE states, the ODE states and the unknown parameters from only boundary sensing. In fact, few results exist in the literature on the adaptive design for coupled ODEs-hyperbolic PDEs system. The authors in Anfinsen and Aamo (2017) synthesize an adaptive observer for a 2×2 hyperbolic system coupled with an uncertain LTI system. The design was done in several steps. The first step is to estimate the unknown parameters by extracting some delayed measurements from the system. The second step is to build a Lunberger state observer for the ODE states and the third step is to use swapping filters to generate estimates of the PDE states. In this framework, we con-

⋆ This work was sponsored within the ITEA3 European project, 15016 EMPHYSIS (Embedded systems with physical models in the production code software).

sider the observer design of a system of linear positive speed transport equations coupled with linear time varying ODEs at the boundary. The system involves a set of unknown constant disturbances to be estimated. Such class of systems can be extended to model the air-path in exhaust gas systems equipped with dual-loop (EGR) for diesel car engines (see e.g. Castillo et al. (2014)). We address the estimation problem using a different methodology than the one presented in Anfinsen and Aamo (2017). We propose an adaptive observer architecture that is built directly on the plant model, so that all states are estimated simultaneously in one step and with no necessity to require asymptotic stability of the ODE states. Inspired by the swapping design techniques (see Kreisselmeier (1977) for ODEs and Smyshlyaev and Krstic (2010) for PDEs), we decouple the state estimation error of the infinite PDE states from the finite dimensional states of the ODEs and the parameters. Then we give sufficient conditions through DLIs to ensure the exponential convergence of the error system using Lyapunov analysis.
The paper is organized as follows: the problem description is given in Section 2, providing the class of systems under study and the estimation problem to be solved. The adaptive observer architecture with the estimation convergence analysis is presented in Section 3. Section 4 is dedicated to the simulation results for a showcase example and some concluding remarks are given in Section 5.

Notation
The symbols S_+^n and D_+^n represent the set of real $n \times n$ symmetric positive definite matrices and the set of real $n \times n$ diagonal positive definite matrices, respectively. For a symmetric matrix A, positive and negative definiteness are denoted, respectively, by $A \succ 0$ and $A \prec 0$. In partitioned symmetric matrices, the $\bullet$ stands for symmetric blocks. For a vector $z \in \mathbb{R}^n$, $|z|$ is the euclidean norm. Given a matrix $A \in \mathbb{R}^{n \times m}$, $||A||_\infty =$ max $|a_{ij}|$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Let $V \subseteq \mathbb{R}^n$ and $f : [0,1] \longmapsto V$, we denote by $||f||_{\mathbb{L}^2([0,1])^n} = \sqrt{\int_0^1 |f(x)|^2 dx}$ the $\mathbb{L}^2$ norm of f. If $f \in \mathbb{L}^2([0,1])^n$, then $||f||_{\mathbb{L}^2([0,1])^n} < +\infty$.

2. PROBLEM DESCRIPTION

We consider the following class of cascade ODEs-hyperbolic PDEs systems evolving in Ω=$[0,1]\times[0,+\infty)$:

$$\partial_t \xi(x,t) + \Lambda^+ \partial_x \xi(x,t) = F\xi(x,t) \tag{1}$$

$$\xi(0,t) = C(t)X(t) + D(t)u(t) + \psi_1(t)\theta \tag{2}$$

$$\dot{X}(t) = A(t)X(t) + B(t)u(t) + \psi_2(t)\theta \tag{3}$$

where ∂_t and ∂_x denote the partial derivatives with respect to time and space respectively. $\xi(x,t) : \Omega \longmapsto \mathbb{R}^{n_\xi}$ is the PDE state vector. $X(t)$:$[0,+\infty) \longmapsto \mathbb{R}^{n_X}$ is the ODE state vector. $\theta \in \mathbb{R}^{n_\theta}$ is the vector of the unknown parameters. $u(t)$:$[0,+\infty) \longmapsto \mathbb{R}^{n_u}$ is a known input vector. $\Lambda^+ \in D_+^{n_\xi}$ is the matrix of the constant transport speeds:

$$\Lambda^+ = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n_\xi} \end{bmatrix} \quad \text{with} \quad 0 < \lambda_1 < .. < \lambda_{n_\xi} \tag{4}$$

$F \in \mathbb{R}^{n_\xi \times n_\xi}$. We assume that all the time-dependent matrices: $A(t) \in \mathbb{R}^{n_X \times n_X}$, $B(t) \in \mathbb{R}^{n_X \times n_u}$, $C(t) \in \mathbb{R}^{n_\xi \times n_X}$, $D(t) \in \mathbb{R}^{n_\xi \times n_u}$, $\psi_1(t) \in \mathbb{R}^{n_\xi \times n_\theta}$ and $\psi_2(t) \in \mathbb{R}^{n_X \times n_\theta}$ are bounded and piece-wise continuous in time. Our goal is to estimate $\xi(x,t)$, $X(t)$ and θ assuming that the following measurements are available:

$$y(t) = M\xi(1,t) \tag{5}$$

where $M \in \mathbb{R}^{n_y \times n_\xi}$ is the output matrix.

3. ADAPTIVE OBSERVER DESIGN

We introduce the following adaptive observer design:

$$\begin{aligned} \partial_t \hat{\xi}(x,t) + \Lambda^+ \partial_x \hat{\xi}(x,t) &= F\hat{\xi}(x,t) \\ &+ p(x,t)(y(t) - M\hat{\xi}(1,t)) + K_1(x,t) \end{aligned} \tag{6}$$

$$\hat{\xi}(0,t) = C(t)\hat{X}(t) + D(t)u(t) + \psi_1(t)\hat{\theta}(t) \tag{7}$$

$$\begin{aligned} \dot{\hat{X}}(t) &= A(t)\hat{X}(t) + B(t)u(t) + \psi_2(t)\hat{\theta}(t) \\ &+ L(t)(y(t) - M\hat{\xi}(1,t)) \end{aligned} \tag{8}$$

where $p(x,t)$:$\Omega \longmapsto \mathbb{R}^{n_\xi \times n_y}$ and $L(t)$:$[0,+\infty) \longmapsto \mathbb{R}^{n_X \times n_y}$ are the observer gains. $K_1(x,t)$:$\Omega \longmapsto \mathbb{R}^{n_\xi}$ is an additional feedback gain to be defined later. We denote the estimates by hat, and we define the estimation error variables $\tilde{\xi}(x,t)$=$\xi(x,t)$-$\hat{\xi}(x,t)$, $\tilde{X}(t)$=$X(t)$-$\hat{X}(t)$ and $\tilde{\theta}(t)$=θ- $\hat{\theta}(t)$. By subtracting (6)-(8) from (1)-(3), we have the following error dynamics:

$$\begin{aligned} \partial_t \tilde{\xi}(x,t) + \Lambda^+ \partial_x \tilde{\xi}(x,t) &= F\tilde{\xi}(x,t) \\ &- p(x,t)M\tilde{\xi}(1,t)) - K_1(x,t) \end{aligned} \tag{9}$$

$$\tilde{\xi}(0,t) = C(t)\tilde{X}(t) + \psi_1(t)\tilde{\theta}(t) \tag{10}$$

$$\dot{\tilde{X}}(t) = A(t)\tilde{X}(t) + \psi_2(t)\tilde{\theta}(t) - L(t)M\tilde{\xi}(1,t) \tag{11}$$

The observer designed in (6)-(8) is of Luenberger-type, which is copy of the original system with output injections $y(t)$, and an additional feedback gain $K_1(x,t)$. Our objective is then to find the observer gains $p(x,t)$ and $L(t)$, and a proper parameter estimation law that can guarantee the exponential convergence of the estimation error in (9)-(11).

3.1 Decoupling using swapping design

We parameterize the PDE state estimation error $\tilde{\xi}(x,t)$ in (9)-(11) using K-filters (see Kreisselmeier (1977) for ODEs and Smyshlyaev and Krstic (2010) for PDEs) as follows:

$$\tilde{\phi}(x,t) = \tilde{\xi}(x,t) - T(x,t)\tilde{X}(t) - R(x,t)\tilde{\theta}(t) \tag{12}$$

The swapping filters: $T(x,t)$: $\Omega \longmapsto \mathbb{R}^{n_\xi \times n_X}$ and $R(x,t)$: $\Omega \longmapsto \mathbb{R}^{n_\xi \times n_\theta}$ are to be defined later. Differentiating (12) with respect to time and space and substituting with (9)-(11) wet get:

$$\begin{aligned} \partial_t \tilde{\phi}(x,t) + \Lambda^+ \partial_x \tilde{\phi}(x,t) &= F\tilde{\phi}(x,t) - \big(K_1(x,t) \\ &+ R(x,t)\dot{\tilde{\theta}}(t)\big) + \big(T(x,t)L(t) - p(x,t)\big)M\tilde{\xi}(1,t) \\ &\Big(- \partial_t T(x,t) - \Lambda^+ \partial_x T(x,t) + FT(x,t) \\ &- T(x,t)A(t)\Big)\tilde{X}(t) + \Big(- \partial_t R(x,t) \\ &- \Lambda^+ \partial_x R(x,t) + FR(x,t) - T(x,t)\psi_2(t)\Big)\tilde{\theta}(t) \end{aligned} \tag{13}$$

Equation (13) suggests to choose: $K_1(x,t){=}{-}R(x,t)\dot{\hat{\theta}}(t){=}R(x,t)\dot{\tilde{\theta}}(t)$, $p(x,t){=}T(x,t)L(t)$ and the following dynamics for the swapping filters

$$\partial_t T(x,t) + \Lambda^+ \partial_x T(x,t) = FT(x,t) - T(x,t)A(t) \quad (14)$$

$$\partial_t R(x,t) + \Lambda^+ \partial_x R(x,t) = FR(x,t) - T(x,t)\psi_2(t) \quad (15)$$

we also impose the following boundary conditions on the filters

$$T(0,t) = C(t), \quad R(0,t) = \psi_1(t) \quad (16)$$

Doing so, and using (13) and (14)-(16) the dynamics of $\tilde{\phi}(x,t)$ become

$$\partial_t \tilde{\phi}(x,t) + \Lambda^+ \partial_x \tilde{\phi}(x,t) = F\tilde{\phi}(x,t) \quad (17)$$

$$\tilde{\phi}(0,t) = 0 \quad (18)$$

In view of equation (12) and the derived dynamics (17)-(18), the infinite state estimation error $\tilde{\xi}(x,t)$ splits into three parts: 1) an observation error $\tilde{\phi}(x,t)$ that is totally decoupled from the ODE state estimation errors, 2) $T(x,t)\tilde{X}(t)$, which is proportional to the estimation error on the ODE states $\tilde{X}(t)$, and 3) the induced error due to the parameters mismatch $R(x,t)\tilde{\theta}(t)$ which is also proportional to the parameter estimation errors $\tilde{\theta}(t)$. To prove the exponential convergence of ($\tilde{\xi}(x,t)$, $\tilde{X}(t)$, $\tilde{\theta}(t)$), it is sufficient to prove the exponential convergence of ($\tilde{\phi}(x,t)$, $\tilde{X}(t)$, $\tilde{\theta}(t)$) and the boundedness of the filters $T(x,t)$ and $R(x,t)$. This is what we establish in the following series of Lemmas.

Lemma 1. Consider the system (17)-(18) with initial condition $\tilde{\phi}_0(x) \in (\mathbb{L}^2([0,1]))^{n_\xi}$. Then for all $\gamma_\phi > 0$, there exists $C_\phi > 0$ such that:

$$||\tilde{\phi}(.,t)||_{(\mathbb{L}^2([0,1]))^{n_\xi}} \leq C_\phi e^{-\gamma_\phi t}||\tilde{\phi}_0(x)||_{\mathbb{L}^2([0,1])^{n_\xi}} \quad (19)$$

Furthermore, the equilibrium $\tilde{\phi} \equiv 0$ is reached in finite time $t_f = \frac{1}{\lambda_1}$.

Proof 1. Consider the following Lyapunov function

$$V_1(t) = \int_0^1 \big(\tilde{\phi}^T(x,t)P_1\tilde{\phi}(x,t)\big)e^{-\mu x}dx \quad (20)$$

where $P_1 \in D_+^{n_\xi}$ and $\mu > 0$. Deriving (20) in time, substituting with (17), integrating by parts and then substituting with (18) yields to:

$$\dot{V}_1(t) = -\tilde{\phi}^T(1,t)\Lambda^+ P_1 e^{-\mu}\tilde{\phi}(1,t) + \int_0^1 \tilde{\phi}^T(x,t)\big[-\mu\Lambda^+P_1 + F^TP_1 + P_1F\big]e^{-\mu x}\tilde{\phi}(x,t)dx \quad (21)$$

The matrix $\Lambda^+P_1e^{-\mu}$ is always positive definite for any $P_1 \in D_+^{n_\xi}$. In addition, for all $\gamma_\phi > 0$ we can always choose μ large enough to have $-\mu\Lambda^+P_1 + F^TP_1 + P_1F \leq -\gamma_\phi P_1$. Thus, $\dot{V}_1(t) \leq -\gamma_\phi V_1(t)$ which shows the exponential convergence of $\tilde{\phi}$ in the $\mathbb{L}^2$-norm. Given that $\Lambda^+ \in D_+^{n_\xi}$, we can change the status of t and x and rewrite (17) as:

$$\partial_x\tilde{\phi}(x,t) + (\Lambda^+)^{-1}\partial_t\tilde{\phi}(x,t) = (\Lambda^+)^{-1}F\tilde{\phi}(x,t) \quad (22)$$

and then (18) becomes a zero initial condition for (22). Then the uniqueness of solutions of (22)-(18) and the order of the transport speeds given in equation (4) imply that $\tilde{\phi}(x,t)$ vanishes after $t \geq \frac{1}{\lambda_1}$ (see Lemma 3.1 in Hu et al. (2016) for further details) and this concludes the proof.

Lemma 2. Consider the filter systems $T(x,t)$ and $R(x,t)$ defined in (14)-(15) with boundary conditions (16). Then for all initial conditions $T_0(x) \in (\mathbb{L}^2([0,1]))^{n_\xi \times n_X}$ and $R_0(x) \in (\mathbb{L}^2([0,1]))^{n_\xi \times n_\theta}$, the PDE filters $T(x,t)$ and $R(x,t)$ are bounded in the $\mathbb{L}^2$ sense.

Proof 2. We start by $T(x,t)$. We write (14)-(16) using the index notation: for all $1 \leq i \leq n_\xi$, $1 \leq j \leq n_X$, we have

$$\partial_t T_{ij}(x,t) + \lambda_i\partial_x T_{ij}(x,t) = \sum_{k=1}^{n_\xi} F_{ik}T_{kj}(x,t) - \sum_{k=1}^{n_X} T_{ik}(x,t)a_{kj}(t) \quad (23)$$

$$T_{ij}(0,t) = c_{ij}(t) \quad (24)$$

Now, we consider the following Lyapunov function

$$V_2(t) = \sum_{i=1}^{n_\xi}\sum_{j=1}^{n_X} V_{ij}(t) = \frac{1}{2}\sum_{i=1}^{n_\xi}\sum_{j=1}^{n_X}\int_0^1 e^{-\mu x}T_{ij}^2(x,t)dx \quad (25)$$

with $\mu > 0$. Deriving (25) with respect to time, replacing by (23), integrating by parts and substituting by (24), then applying Young's inequality one gets

$$\dot{V}_2(t) \leq \sum_{i=1}^{n_\xi}\sum_{j=1}^{n_X}\Bigg[\frac{1}{2}\lambda_i c_{ij}^2(t) + \Bigg(-\mu\lambda_i + \sum_{k=1}^{n_\xi}\big(|F_{ik}| + |F_{ki}|\big) + \sum_{k=1}^{n_X}\big(|a_{kj}(t)| + |a_{jk}(t)|\big)\Bigg)V_{ij}(t)\Bigg] \quad (26)$$

Denoting by $F_{max}{=}\|F\|_\infty$ and $A_{max}{=}\underset{t\geq 0}{max}\ \|A(t)\|_\infty$, we can further write (26) as

$$\dot{V}_2(t) \leq \sum_{i=1}^{n_\xi}\sum_{j=1}^{n_X}\Big[\frac{1}{2}\lambda_i c_{ij}^2(t) + \big(-\mu\lambda_1 + 2n_\xi F_{max} + 2n_X A_{max}\big)V_{ij}(t)\Big] \quad (27)$$

We can choose μ large enough to have $-\mu\lambda_1 + 2n_\xi F_{max} + 2n_X A_{max} \leq -\gamma_2$ for every $\gamma_2 > 0$. Doing so, (27) becomes

$$\dot{V}_2(t) \leq -\gamma_2 V_2(t) + \frac{\lambda_{n_\xi}}{2}\|C(t)\|_2^2 \quad (28)$$

with $\|C(t)\|_2^2 = \sum_{i=1}^{n_\xi}\sum_{j=1}^{n_X} c_{ij}^2(t)$. Inequality (28) shows that $V_2(t)$ is bounded as a direct consequence of the boundedness of the system matrices $A(t)$ and $C(t)$. Since $V_2(t)$ is the weighted $\mathbb{L}^2$ norm of $T(x,t)$, then by (28) we can deduce that $T(x,t)$ is bounded in the $\mathbb{L}^2$ sense. Following exactly the same procedure, and using the $\mathbb{L}^2$ boundedness of $T(x,t)$ proved above together with the boundedness of $\psi_1(t)$ and $\psi_2(t)$, one can also derive an $\mathbb{L}^2$ bound on $R(x,t)$ and this completes the proof.

3.2 ODE error dynamics and the parameter adaptation law

The ODE dynamics are investigated as follows. We evaluate (12) at x=1, multiply by M on both sides, and then substitute in (11) to have

$$\dot{\tilde{X}}(t) = A_d(t)\tilde{X}(t) + \big(\psi_2(t) - L(t)MR(1,t)\big)\tilde{\theta}(t) - L(t)M\tilde{\phi}(1,t) \quad (29)$$

with $A_d(t)$=$A(t)$-$L(t)MT(1,t)$. We introduce the piecewise continuous shift operator $s(t) = \begin{cases} 1 & \text{if} \quad t \geq t_f \\ 0 & \text{else} \end{cases}$ in the observer gain $L(t)$ computation, i.e. we write $L(t) = s(t)l(t)$. The main reason is to remove the effect of the initial conditions of the filters $T(x,t)$ and $R(x,t)$ on the overall adaptive design. Doing so, (29) becomes

$$\begin{aligned}\dot{\tilde{X}}(t) &= \big(A(t) - s(t)l(t)MT(1,t)\big)\tilde{X}(t) \\ &+ \big(\psi_2(t) - s(t)l(t)MR(1,t)\big)\tilde{\theta}(t) - s(t)l(t)M\tilde{\phi}(1,t)\end{aligned} \tag{30}$$

Equation (12) at x=1 also suggests the following normalized adaptation law:

$$\dot{\hat{\theta}}(t) = -\dot{\tilde{\theta}}(t) = \frac{s(t)P_\theta(t)\Phi^T(t)}{1 + \|\Phi^T(t)\Phi(t)\|^2} M\tilde{\xi}(1,t) \tag{31}$$

$$\dot{P}_\theta(t) = s(t)\left[\beta P_\theta(t) - \frac{P_\theta(t)\Phi^T(t)\Phi(t)P_\theta(t)}{1 + \|\Phi^T(t)\Phi(t)\|^2}\right] \tag{32}$$

where the regressor $\Phi(t)$ is given by $\Phi(t)$=$MR(1,t)$. $P_\theta(t)$: $[0,+\infty) \longmapsto \mathbb{R}^{n_\theta \times n_\theta}$ and $\beta > 0$ is the forgetting factor. The initial conditions $\hat{\theta}(0) = \hat{\theta}_0$ and $P_\theta(0)$=$P_{\theta,0}$=$P_{\theta,0}^T$ are chosen arbitrary. The adaptive law (31)-(32) is called continuous time recursive least square estimator with a forgetting factor (see Ioannou and Sun (1996) for various linear regression estimation techniques). Using (31) and (12), we now compute the dynamics of $\tilde{\theta}(t)$ as follows

$$\begin{aligned}\dot{\tilde{\theta}}(t) = &-\frac{s(t)P_\theta(t)}{1 + \|\Phi^T(t)\Phi(t)\|^2}\Big[\Phi^T(t)\Phi(t)\tilde{\theta}(t) \\ &+ \Phi^T(t)MT(1,t)\tilde{X}(t) + \Phi^T(t)M\tilde{\phi}(1,t)\Big].\end{aligned} \tag{33}$$

Remark 1. The formulation of (30) and (33) as a function of $s(t)$ implies that the ODE error stabilization and the parameter adaptation start functioning when the maximum delay time due to transport in space (t_f=$\frac{1}{\lambda_1}$) is passed.

We are now at a point where we can state the stability result of the ($\tilde{\phi}(x,t)$, $\tilde{X}(t)$, $\tilde{\theta}(t)$) system.

Lemma 3. Consider the system (17)-(18) and (30)-(33) with initial conditions ($\tilde{\phi}_0(x) \in (\mathbb{L}^2([0,1]))^{n_\xi}$, $\tilde{X}_0 \in \mathbb{R}^{n_X}$, $\tilde{\theta}_0 \in \mathbb{R}^{n_\theta}$). If $\Phi(t)$ is bounded and persistently exciting (PE), i.e. for all $t \geq t_f$ there exist positive constants T_0, c_0 and c_1 so that:

$$c_0 I \leq \int_t^{t+T_0} \Phi^T(\tau)\Phi(\tau)d\tau \leq c_1 I \tag{34}$$

In addition, if there exist an observer gain $L(t) \in \mathbb{R}^{n_X \times n_y}$ and a bounded matrix $P_X(t) \in S_+^{n_X \times n_X}$ such that, for all $t \geq t_f$:

$$Z(t) \leq -Q(t) \tag{35}$$

where $Z(t)$ is given in (36) and $Q(t)$ is a predefined bounded positive definite matrix. Then for all $t \geq t_f$, the system (17)-(18) and (30)-(33) is exponentially stable in the $|\tilde{X}|^2 + |\tilde{\theta}|^2 + \|\tilde{\phi}(.,t)\|^2_{(\mathbb{L}^2([0,1]))^{n_\xi}}$ norm.

Proof 3. We combine the ODE error dynamics and the parameter error dynamics in one vector $\tilde{X}_c(t) = \begin{pmatrix} \tilde{X}(t) \\ \tilde{\theta}(t) \end{pmatrix}$ written in the following state-space representation:

$$\dot{\tilde{X}}_c(t) = A_c(t)\tilde{X}_c(t) + B_c(t)\tilde{\phi}(1,t) \tag{37}$$

where:

$$A_c(t) = \begin{bmatrix} A(t) - s(t)l(t)MT(1,t) & \psi_2(t) - s(t)l(t)\Phi(t) \\ -\dfrac{s(t)P_\theta(t)\Phi^T(t)MT(1,t)}{1 + \|\Phi^T(t)\Phi(t)\|^2} & -\dfrac{s(t)P_\theta(t)\Phi^T(t)\Phi(t)}{1 + \|\Phi^T(t)\Phi(t)\|^2} \end{bmatrix},$$

$$B_c(t) = \begin{bmatrix} -s(t)l(t)M \\ -\dfrac{s(t)P_\theta(t)\Phi^T(t)M}{1 + \|\Phi^T(t)\Phi(t)\|^2} \end{bmatrix}$$

Moreover, using (32) we compute the dynamics of $P_\theta^{-1}(t)$ (the inverse of $P_\theta(t)$):

$$\frac{d}{dt}P_\theta^{-1}(t) = s(t)\left[-\beta P_\theta^{-1}(t) + \frac{\Phi^T(t)\Phi(t)}{1 + \|\Phi^T(t)\Phi(t)\|^2}\right] \tag{38}$$

It can be shown (see Ioannou and Sun (1996)) that if (34) is satisfied, then $P_\theta(t)$ and $P_\theta^{-1}(t)$ are both bounded and positive definite for all $t \geq 0$. Now, let us consider the following Lyapunov function:

$$V_3(t) = \tilde{X}_c^T(t)P_c(t)\tilde{X}_c(t) + V_1(t) \tag{39}$$

where $P_c(t) = \begin{pmatrix} P_X(t) & 0 \\ 0 & P_\theta^{-1}(t) \end{pmatrix}$. Deriving (39) with respect to time, we have

$$\begin{aligned}\dot{V}_3(t) &= \dot{\tilde{X}}_c^T(t)P_c(t)\tilde{X}_c(t) + \tilde{X}_c^T(t)\dot{P}_c(t)\tilde{X}_c(t) \\ &+ \tilde{X}_c^T(t)P_c(t)\dot{\tilde{X}}_c(t) + \dot{V}_1(t)\end{aligned} \tag{40}$$

Replacing (37) in (40) leads to:

$$\begin{aligned}\dot{V}_3(t) &= \tilde{X}_c^T(t)\Big(\dot{P}_c(t) + A_c^T P_c(t) + P_c(t)A_c(t)\Big)\tilde{X}_c(t) \\ &+ \tilde{\phi}^T(1,t)B_c^T(t)P_c(t)\tilde{X}_c(t) + \tilde{X}_c^T(t)P_c(t)B_c(t)\tilde{\phi}(1,t) \\ &+ \dot{V}_1(t)\end{aligned} \tag{41}$$

If $t < t_f$, equation (41) becomes

$$\begin{aligned}\dot{V}_3(t) &= \tilde{X}^T(t)\Big(\dot{P}_X(t) + A^T(t)P_X(t) + P_X(t)A(t)\Big)\tilde{X}(t) \\ &+ \tilde{\theta}^T(t)\psi_2^T(t)P_X(t)\tilde{X}(t) + \tilde{X}^T(t)P_X(t)\psi_2(t)\tilde{\theta}(t) + \dot{V}_1(t)\end{aligned} \tag{42}$$

By Lemma 1, $V_1(t)$ is exponentially stable for all times, then if $A(t)$ is uniformly exponentially stable (UES) we can guarantee that there exists $P_X(t)$ such that $\dot{P}_X(t) + A^T(t)P_X(t) + P_X(t)A(t) \prec 0$ (see Theorem 7.4 in Rugh (1996))) and as a result, the Lyapunov function $V_3(t)$ is upper bounded by the magnitude of $\psi_2(t)$ for all t<t_f, otherwise i.e. if $A(t)$ is not UES, we choose $P_X(t)$=P_X∈$S_+^{n_X \times n_X}$ arbitrary, and $V_3(t)$ can be growing for all $t < t_f$.
However, the interesting part of the analysis is when the maximum delay time due to transport in space is passed i.e. when $t \geq t_f$. Using (38) it is easy to verify that $Z(t)$=$\dot{P}_c(t) + A_c^T P_c(t) + P_c(t)A_c(t)$ for t≥t_f. Furthermore, using Lemma 1, $\tilde{\phi}(x,t)$ is $\mathbb{L}_2$-stable and $\tilde{\phi}(1,t) \equiv 0$ is reached in finite time $t_f = \frac{1}{\lambda_1}$. Hence, if (35) is satisfied, using (41) one gets

$$\dot{V}_3(t) \leq -\tilde{X}_c^T(t)Q(t)\tilde{X}_c(t) - \gamma_\phi V_1(t) \tag{43}$$

for all $t \geq t_f$. Thus, by the boundedness of $Q(t)$ there exists a positive constant $\gamma_{tot} > 0$ such that $\dot{V}_3(t) \leq -\gamma_{tot}V_3(t)$, which shows the exponential stability of (17)-(18) and (30)-(33) in the $|\tilde{X}|^2 + |\tilde{\theta}|^2 + \|\tilde{\phi}(.,t)\|^2_{(\mathbb{L}^2([0,1]))^{n_\xi}}$ norm for $t \geq t_f$ and completes the proof.

$$Z(t) = \begin{bmatrix} \dot{P}_X(t) + A_d^T(t)P_X(t) + P_X(t)A_d(t) & P_X(t)\big(\psi_2(t) - l(t)MR(1,t)\big) - \dfrac{T^T(1,t)M^T\Phi(t)}{1+\|\Phi^T(t)\Phi(t)\|^2} \\ \bullet & -\beta P_\theta^{-1}(t) - \dfrac{\Phi^T(t)\Phi(t)}{1+\|\Phi^T(t)\Phi(t)\|^2} \end{bmatrix} \quad (36)$$

Remark 2. The existence of the Lyapunov function (39) on the interval $[t_f, +\infty)$ for our observer architecture depends on two intrinsic properties of the system. One is the detectability given by the existence of $P_X(t)$. The other is the persistency of parameter excitation given by the existence of $P_\theta^{-1}(t)$. To illustrate the point, let us reconsider inequality (35). A necessary condition for (35) to have solutions is that the diagonal elements in $Z(t)$ must be negative definite. If we start by $Z_{11}(t)$, we must have

$$\dot{P}_X(t) + A_d^T(t)P_X(t) + P_X(t)A_d(t) \prec 0 \quad (44)$$

which is the differential Lyapunov equation in $A_d(t)$. It is well known that (44) has a unique solution $P_X(t)$ if $A_d(t)$ is UES. Any time-varying state matrix which is 1) continuously differentiable, 2) bounded, 3) slowly varying and 4) the real part of its Eigen-values is negative for all times is UES (see e.g. Theorem 8.7 in Rugh (1996). For instance if we assume that the first three conditions of Theorem 8.7 in Rugh (1996)) are satisfied for $A_d(t)$ in the interval of time $[t_f, +\infty)$, we still require that the real part of its eigen-values to be negative. Let us recall that for $t \geq t_f$, $A_d(t)$=$A(t)$-$l(t)MT(1,t)$. We can always choose $l(t)$ such that $A_d(t)$ is Hurwitz if the pair $(A(t), MT(1,t))$ is detectable. If we look into the $T(x,t)$ filter (14)-(16), we can observe that $T(1,t)$ is a delayed version of $C(t)$ with a change in magnitude due to the coupling (F and $A(t)$). Hence, finding $P_X(t)$ is directly related to the detectability of the system ($A(t)$, M, $C(t)$) through the pair ($A(t)$, $MT(1,t)$). On the other hand, $Z_{22}(t)$ is always negative-definite, since $P_\theta^{-1}(t)$ is positive definite and bounded based on the (PE) assumption (34). It is important to mention that the condition (34) is directly related to the values of $\psi_1(t)$ and $\psi_2(t)$ through $R(1,t)$. For instance, if $\psi_1 \equiv \psi_2 \equiv 0$ then by (15)-(16), after t_f, $R(1,t) \equiv 0$ which gives $\Phi \equiv 0$ then (34) cannot be satisfied. This completely coincides with the logic that we cannot estimate θ if ψ_1 and ψ_2 are zero (see equations (2) and (3)).

We can now state the stability result of the original error system ($\tilde{\xi}(x,t)$, $\tilde{X}(t)$, $\tilde{\theta}(t)$).

Theorem 1. Consider the error system (6)-(8) with initial conditions ($\tilde{\xi}_0(x) \in (\mathbb{L}^2([0,1]))^{n_\xi}$, $\tilde{X}_0 \in \mathbb{R}^{n_X}$, $\tilde{\theta}_0 \in \mathbb{R}^{n_\theta}$). Under Lemma 1, Lemma 2 and if the conditions of Lemma 3 are satisfied, then the error system ($\tilde{\xi}(x,t)$,$\tilde{X}(t)$,$\tilde{\theta}(t)$) is exponentially stable in the $|\tilde{X}|^2 + |\tilde{\theta}|^2 + \|\tilde{\xi}(.,t)\|^2_{(\mathbb{L}^2([0,1]))^{n_\xi}}$ norm for all $t \geq t_f$.

Proof 4. Consider the following Lyapunov function

$$V(t) = \tilde{X}_c^T(t)P_c(t)\tilde{X}_c(t) + \int_0^1 \big(\tilde{\xi}^T(x,t)P\tilde{\xi}(x,t)\big)e^{-\mu x}dx \quad (45)$$

In view of (12), the result falls directly from Lemma 3 with the L^2 boundedness of the filters $T(x,t)$ and $R(x,t)$ proved in Lemma 2.

4. SIMULATION RESULTS

We implement the adaptive observer in MATLAB for the scalar case $n_\xi = n_X = 1$ and $n_\theta = 1$. The system is given by:

$$\partial_t \xi(x,t) + 2\partial_x \xi(x,t) = 0.02\xi(x,t) \quad (46)$$

$$\xi(0,t) = X(t) + \frac{\sqrt{3}}{2}\theta \quad (47)$$

$$\dot{X}(t) = sin(t)X(t) + cos(t)u(t) + \frac{1}{2}\theta \quad (48)$$

The control input is constant $u(t) = 2$ and the parameter to be estimated is $\theta = 1$. The system initial conditions are

$$\xi_0(x) = 10x \quad \text{and} \quad X_0 = 5. \quad (49)$$

The states $\xi(x,t)$, $X(t)$ and the parameter $\theta = 1$ are to be estimated using the available measurement $y(t) = \xi(1,t)$. The system (46)-(48) corresponds to a transport equation with first order time-varying boundary conditions. It is clear that the plant is open loop - unstable looking into the ODE dynamics $A(t) = sin(t)$. To implement the adaptive observer (6)-(8) with the adaptation law (31)-(32), we need to find two gains $L(t)$ and β. The forgetting factor β is fixed to 0.1. The dynamic observer gains $L(t)$ is calculated at each time step to ensure that $Z(t)$ is negative definite for $t \geq t_f$. This is done in the following order: 1) use a pole placement method to calculate $L(t)$ that guarantees the existence of $P_X(t)$ that satisfies (44), 2) verify that (35) is satisfied for a predefined value of $Q(t)$. Condition (35) was satisfied for all $t \geq t_f$ for a constant value of $P_X(t)$=P_X=0.5 and for $Q(t)$=$0.0125P_c(t)$ where $L(t)$ is calculated by locating the poles of $A_d(t)$ at -1 for all $t \geq t_f$. The values corresponding to $L(t)$ are plotted on Fig.1. The placement starts after t_f=0.5 s and $L(t)$ exhibits an oscillatory behavior due to the dynamics of $A(t)$. We start the adaptive observer from the following initial conditions

$$\hat{\xi}_0(x) = 9(x+1) \quad \text{and} \quad \hat{X}_0 = 10$$

A finite difference (FD) scheme in space and time is implemented to approximate both the infinite dimensional states ($\xi(x,t)$, $\hat{\xi}(x,t)$) and the finite dimensional states ($X(t)$, $\hat{X}(t)$ and $\hat{\theta}(t)$). The exponential convergence of the estimation error on both the ODE and the PDE sides is shown on Figures 2 and 3. After t_f=0.5 s, the estimation errors converge to zero after exhibiting some oscillatory transients. Furthermore, the Lyapunov function $V(t)$ shown on Fig. 4 increases on the interval of time $[0, 0.5\,s]$ due to the unstable dynamics of $A(t)$ and the presence of no observer gain $L(t)$=0, but afterwards it starts its exponential decay towards zero when measurement corrections are introduced for $t \geq t_f$. Finally, we show the estimation of the parameter θ starting from an initial condition $\hat{\theta}_0 = 6$ on Fig.5. The adaptation starts after t_f=0.5 s and $\hat{\theta}(t)$ converges to θ in approximately 50 s.

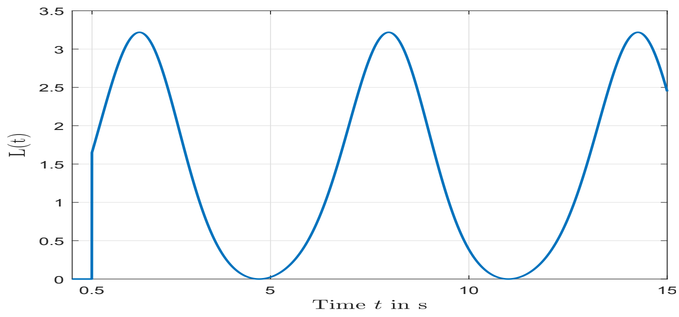

Figure 1. Calculated observer gain $L(t)$

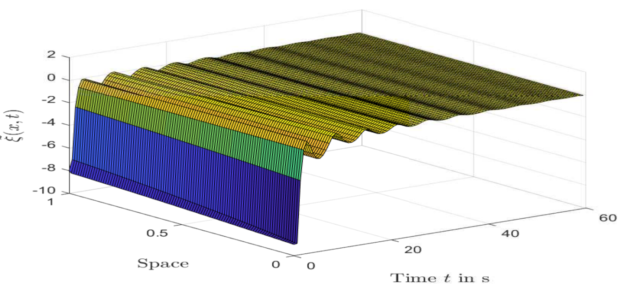

Figure 2. Estimation error of the PDE state $\tilde{\xi}(x,t)$

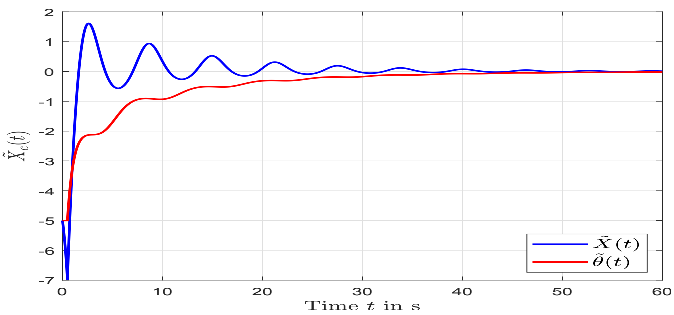

Figure 3. Estimation error of the ODE states $\tilde{X}(t)$ and $\tilde{\theta}(t)$

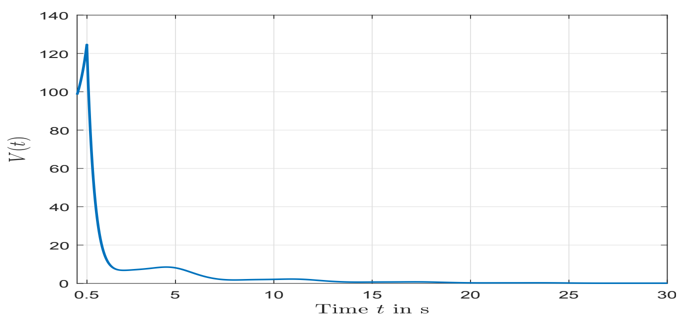

Figure 4. Lyapunov function $V(t)$

5. CONCLUSION

We have proposed an adaptive observer for a system of linear transport PDEs coupled with time-varying ODEs at the boundary. The system involves constant parameters that are to be estimated together with the PDE-ODE states using boundary sensing only. We have used swapping design to decouple the estimation error of the infinite states (PDEs) from the finite states (ODEs). We thus proved boundedness of regressors filters and obtained sufficient conditions for the exponential stability of the estimation error using DLIs.

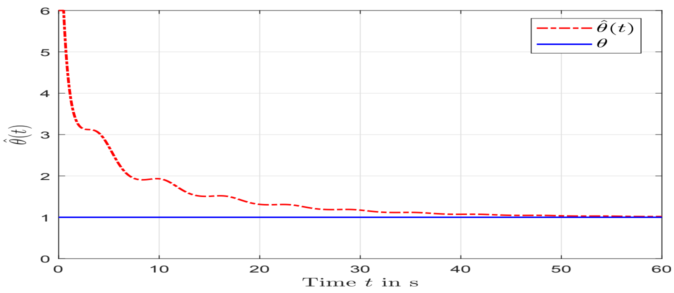

Figure 5. Estimation of the parameter state $\hat{\theta}(t)$ and $\tilde{X}_2(t)$

REFERENCES

Anfinsen, H. and Aamo, O.M. (2017). State estimation in hyperbolic pdes coupled with an uncertain lti system. In *2017 American Control Conference (ACC)*, 3821–3827. IEEE.

Castillo, F., Witrant, E., Prieur, C., and Dugard, L. (2013). Boundary observers for linear and quasi-linear hyperbolic systems with application to flow control. *Automatica*, 49(11), 3180–3188.

Castillo, F., Witrant, E., Prieur, C., Talon, V., and Dugard, L. (2014). Fresh air fraction control in engines using dynamic boundary stabilization of lpv hyperbolic systems. *IEEE Transactions on Control Systems Technology*, 23(3), 963–974.

Coron, J.M., d'Andrea Novel, B., and Bastin, G. (2007). A strict lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic control*, 52(1), 2–11.

Ferrante, F. and Cristofaro, A. (2019). Boundary observer design for coupled ODEs–hyperbolic PDEs systems. In *2019 18th European Control Conference (ECC)*, 2418–2423.

Goatin, P. (2006). The aw–rascle vehicular traffic flow model with phase transitions. *Mathematical and computer modelling*, 44(3-4), 287–303.

Gugat, M. and Dick, M. (2011). Time-delayed boundary feedback stabilization of the isothermal Euler equations with friction. *Math. Control Relat. Fields*, 1(4), 469–491.

Hasan, A., Aamo, O.M., and Krstic, M. (2016). Boundary observer design for hyperbolic PDE–ODE cascade systems. *Automatica*, 68, 75–86.

Hu, L., Di Meglio, F., Vazquez, R., and Krstic, M. (2016). Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. *IEEE Transactions on Automatic Control*, 61(11), 3301–3314.

Ioannou, P. and Sun, J. (1996). Robust adaptive control, Vol. 1, PTR Prentice-Hall Upper Saddle River.

Kreisselmeier, G. (1977). Adaptive observers with exponential rate of convergence. *IEEE Transactions on Automatic Control*, 22(1), 2–8.

Krstic, M. and Smyshlyaev, A. (2008). Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57(9), 750–758.

Rugh, W. (1996). *Linear system theory*, volume 2. prentice hall Upper Saddle River, NJ.

Smyshlyaev, A. and Krstic, M. (2010). *Adaptive control of parabolic PDEs*. Princeton University Press.