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A new form of rate independence for hysteresis systems ⋆

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Abstract: In mathematical texts hysteresis is defined as a rate independent phenomenon, and many mathematical models of hysteresis exhibit this rate independence property. However, experiments suggest that this property is only an approximation of the hysteresis behaviour when the excitation is slow enough. Using a generalized form of the hysteretic Duhem model, we show that, although this model is not rate independent, it still satisfies a new form of rate independence. We also explore the relationship between this new form and the usual property of rate independence.

Keywords: Rate dependent, Rate independent, Hysteresis

1. INTRODUCTION

Let U, X, Y be arbitrary sets. Let $I \subset \mathbb{R}$ be an interval, $\mathscr{U}$ be the set of functions $u : I \to U$, and $\mathscr{X}$ the set of functions $x : I \to X$. Let Φ be an operator such that $\Phi : \mathscr{U} \times X \to \mathscr{X}$, and $\mathscr{T} : X \to Y$ be a function. In this paper we consider operators $\mathcal{H}$ such that $\mathcal{H}(u, x_0) = \mathscr{T} \circ [\Phi(u, x_0)]$ where $(u, x_0) \in \mathcal{U} \times X$.

The elements of $\mathscr{U}$ are called inputs, and the argument $t \in I$ of the input $u \in \mathscr{U}$ is called time. This means that the value of the input u at time t is $u(t)$.

The function $\mathcal{H}(u, x_0) : I \to Y$ is called the output that corresponds to the input $u \in \mathscr{U}$ and the initial state $x_0 \in X$. The set of all functions $y : I \to Y$ is denoted $\mathscr{Y}$ so that $\mathcal{H} : \mathscr{U} \times X \to \mathscr{Y}$.

The function $\Phi(u, x_0) : I \to X$ is called the state that corresponds to the input $u \in \mathscr{U}$ and the initial state $x_0 \in X$. If $\mathscr{Y} = \mathscr{X}$ and $\mathscr{T}$ is the identity function, then the state is also the output.

In mathematical textbooks, hysteresis is defined by the so-called *rate independence* property which states that $\mathcal{H}(u \circ \phi, x_0) = \mathcal{H}(u, x_0) \circ \phi$ for any homeomorphism $\phi : I \to I$, any $u \in \mathscr{U}$, and any $x_0 \in X$; see for instance (Visintin, 1994, p. 60).

Define $y = \mathcal{H}(u, x_0)$ and consider the graph ouput-versus-input $G_{u,x_0} = \big\{\big(u(t), y(t)\big), t \in I\big\}$. If instead of u the input is $u \circ \phi$ where $\phi : I \to I$ is a homeomorphism, we get $G_{u\circ\phi,x_0} = \big\{\big(u \circ \phi(t), y \circ \phi(t)\big), t \in I\big\} = G_{u,x_0}$. This means that, for a rate-independent operator $\mathcal{H}$, the graph ouput-versus-input is invariant under the action of a change in time scale.

Consider the relation $\mathscr{E}_1$ defined on the set $\mathscr{U}$ as follows: for $u_1, u_2 \in \mathscr{U}$ we say that $u_1 \mathscr{E}_1 u_2$ if there exists a homeomorphism $\phi : I \to I$ such that $u_2 = u_1 \circ \phi$. Then $\mathscr{E}_1$ is an equivalence relation. Rate independence can be interpreted as follows: *equivalent inputs with respect to the equivalence relation $\mathscr{E}_1$ lead to the same graph ouput-versus-input.*

On the other hand, experimental evidence suggests that rate independence is but an approximation of real hysteretic systems when the exciting input varies slowly, see for instance (Gan and Zhang, 2015, Fig. 7). This fact led to the use of mathematical models of hysteresis that are not rate independent as in Aljanaideh et al. (2016); Gan and Zhang (2015); Aljanaideh et al. (2016); Zhang and Ma (2016) among others, and to the emergence of mathematical frameworks for the study of hysteresis systems when the property of rate independence does not hold, see Ikhouane (2013, 2020); Oh and Bernstein (2013).

In this paper we consider the generalized version of the Duhem model proposed in Oh and Bernstein (2013) and reviewed in Ikhouane (2018). We show that although the generalized Duhem model is not rate independent in general, this model does satisfy a new form of rate independence.

The main result of the paper states as follows: *equivalent inputs with respect to a new equivalence relation $\mathscr{E}_2$ lead to the same hysteresis loop.* The main ingredient to prove this result is the mathematical framework proposed in Ikhouane (2013).

This paper is organized as follows. Section 2 presents in a concise manner historical developments detailed in (Ikhouane, 2018, Section 2). The mathematical notation used in this paper is presented in Section 3. The equivalence relation $\mathscr{E}_2$ is defined in Section 4. Based on this equivalence relation, the new form of rate independence is introduced in Section 6 using the tools of Section 5.

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The obtained results are interpreted in Section 7 and the conclusions are given in Section 8.

2. HISTORICAL DEVELOPMENTS

The term *hysteresis* appears for the first time in 1881 in a paper by J. A. Ewing that describes the relationship between the torsion of a magnetized wire and its polarization, see Ewing (1881). The following paragraph is a quotation from Ewing's paper: "The curves for the back and forth twists are irreversible, and include a wide area between them. The change of polarization lags behind the change of torsion. To this action ... the author now gives the name Hysterēsis (... to be behind)".

We can see that the term 'hysteresis' is proposed by Ewing to describe the loop that is observed in the plot output versus input. This description is qualitative in the sense that no mathematical definition is proposed by Ewing for the observed phenomenon.

In 1971 Bouc proposes a mathematical characterization of hysteresis in the following terms (Bouc, 1971, p. 17): "Consider the graph with hysteresis of Fig. 1 where $\mathscr{F}$ is not a function of x. To the value $x = x_0$ correspond four values of $\mathscr{F}$.

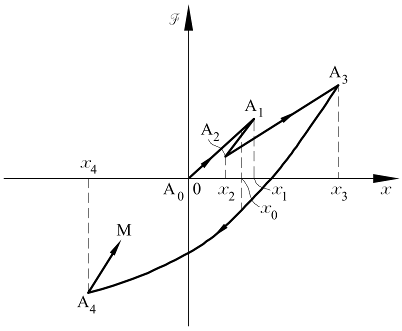

Fig. 1. *Graph "Force–Displacement" with hysteresis.*

... we denote $\mathscr{F}(t) = \mathscr{A}\big(x(\cdot), t\big)$ ... Our aim is to explicit functional $\mathscr{A}\big(x(\cdot), t\big)$.

To this end, we make the following assumption: the graph of Fig. 1 remains the same for all increasing function $x(\cdot)$ between 0 and x_1, decreasing between the values x_1 and x_2, etc.

... We can also say: If $\phi : \mathbb{R} \to \mathbb{R}$ is a class C^1 function whose derivative is strictly positive ... and if we consider the function $y(t) = x\big(\phi(t)\big)$ ... then ... we have

$$\mathscr{A}\big(x(\cdot)\big)(t) = \mathscr{A}\big(y(\cdot)\big)\big(\phi^{-1}(t)\big)."$$

The last equation is precisely the definition of rate independence. We can see that, for Bouc, rate independence is the consequence of an assumption.

In 1994, Visintin writes: "Definition. Hysteresis = Rate Independent Memory Effect." (Visintin, 1994, p. 13). It is thus not surprising that the author notes that "although most typical examples of hysteresis phenomena exhibit hysteresis loops, the occurrence of loops should not be regarded as an essential feature of hysteresis." (Visintin, 2005, p. 6). This opinion is also shared by Mayergoyz (Mayergoyz, 2003, pp. xvi–xvii).

We make the following comments.

To study the phenomenon of hysteresis mathematically, a mathematical definition of that phenomenon is needed. Ewing describes qualitatively hysteresis loops but does not provide a mathematical formulation of hysteresis. Bouc, on the other hand, makes an assumption on the graph output-versus-input of the hysteresis process, then deduces rate independence as a consequence of that assumption.

Is the assumption made by Bouc valid for the phenomenon observed by Ewing? To shed light upon this question we ask ourselves: 'what do experiments say?'.

3. MATHEMATICAL NOTATION

The Lebesgue measure on $\mathbb{R}$ is denoted μ. We say that a subset of $\mathbb{R}$ is measurable when it is Lebesgue measurable. Let $I \subset \mathbb{R}_+$ be an interval, and consider a function $\phi : I \to \mathbb{R}^l$ where $l > 0$ is an integer. We say that ϕ is measurable when ϕ is (M_μ, B)–measurable where B is the class of Borel sets of $\mathbb{R}^l$ and M_μ is the class of measurable sets of $\mathbb{R}_+$, see Yen and Van Der Vaart (1966). For a measurable function $\phi : I \to \mathbb{R}^l$, $\|\phi\|$ denotes the essential supremum of the function $|\phi|$ on I where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^l$.

$\mathcal{S}(\mathbb{R}_+, \mathbb{R}^l)$ denotes the space of absolutely continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}^l$ such that $\|\phi\| < \infty$ and $\|\dot{\phi}\| < \infty$ where $\dot{\phi}$ is the derivative of ϕ.

$L^\infty(\mathbb{R}_+, \mathbb{R}^l)$ denotes the Banach space of measurable and essentially bounded functions $\phi : \mathbb{R}_+ \to \mathbb{R}^l$ endowed with the norm $\|\cdot\|$.

$C^0(\mathbb{R}_+, \mathbb{R}^l)$ denotes the space of continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}^l$ endowed with the norm $\|\cdot\|$.

$\forall \gamma \in]0, \infty[$, the linear time-scale-change $s_\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by the relation $s_\gamma(t) = t/\gamma, \forall t \in \mathbb{R}_+$.

$\lim\limits_{x \uparrow a}$ sets for $\lim\limits_{\substack{x \to a \\ x < a}}$ whilst $\lim\limits_{x \downarrow a}$ sets for $\lim\limits_{\substack{x \to a \\ x > a}}$.

Let U be a set and let $T \in]0, \infty[$. The function $\phi : \mathbb{R}_+ \to U$ is said to be T–periodic if $\phi(t) = \phi(t + T), \forall t \in \mathbb{R}_+$.

4. THE NORMALIZED INPUT AND THE EQUIVALENCE RELATION $\mathscr{E}_2$

The normalization of the input is one of the tools proposed in Ikhouane (2013, 2018) to study the hysteresis properties of the generalized Duhem model. In Section 4.1 we present the properties of the normalized input that are used in the text. An illustrative example is presented in Section 4.2. In Section 4.3 we construct the new equivalence relation $\mathscr{E}_2$ using the concept of normalized input, and study the relationship between $\mathscr{E}_2$ and $\mathscr{E}_1$.

4.1 background results

For $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$, let $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ be the total variation of u on $[0, t]$, that is $\rho_u(t) = \int_0^t |\dot{u}(\tau)| \,\mathrm{d}\tau \in \mathbb{R}_+$, $\forall t \in \mathbb{R}_+$. The function ρ_u is well defined, nondecreasing and absolutely continuous. Observe that ρ_u may not be invertible (this happens when the input u is constant on some interval or intervals). Denote $\rho_{u,\max} = \lim_{t\to\infty} \rho_u(t)$ and let

- $I_u = [0, \rho_{u,\max}]$ if $\rho_{u,\max} = \rho_u(t)$ for some $t \in \mathbb{R}_+$ (in this case the interval I_u is finite),
- $I_u = [0, \rho_{u,\max}[$ if $\rho_{u,\max} > \rho_u(t)$ for all $t \in \mathbb{R}_+$ (in this case the interval I_u may be finite or infinite).

Lemma 1. Ikhouane (2013) Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be non constant so that the interval I_u is not reduced to a single point. Then there exists a unique function $\psi_u \in \mathcal{S}(I_u, \mathbb{R})$ that satisfies $\psi_u \circ \rho_u = u$. Moreover, the function ψ_u satisfies $\|\dot{\psi}\|_u = 1$ and

$$\mu\left(\left\{\varrho \in I_u \mid \dot{\psi}_u(\varrho) \text{ is not defined or } |\dot{\psi}_u(\varrho)| \neq 1\right\}\right) = 0.$$

The function ψ_u is constructed as follows. Let $\varrho \in I_u$, then there exists $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$ (note that t_ϱ is not necessarily unique as ρ_u is not necessarily invertible). Then, $u(t_\varrho)$ is independent of the particular choice of t_ϱ, and $\psi_u(\varrho)$ is defined by the relation $\psi_u(\varrho) = u(t_\varrho)$ Ikhouane (2013).

Lemma 1 shows that the input u has been "normalized" so that the resulting function ψ_u is such that $\dot{\psi}_u$ has norm 1 with respect to the new time variable ϱ. For this reason, we call function ψ_u the *normalized input*.

For every $\gamma \in]0, \infty[$ recall the linear time-scale-change s_γ.

Lemma 2. Ikhouane (2013) $\forall \gamma \in]0, \infty[, I_{u\circ s_\gamma} = I_u$ and $\psi_{u\circ s_\gamma} = \psi_u$.

Lemma 3. Ikhouane (2013) Let $T \in]0, \infty[$. If $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ is non constant and T–periodic, then $I_u = \mathbb{R}_+$ and $\psi_u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ is $\rho_u(T)$–periodic.

4.2 An illustrative example

In this section we present an example of calculation of a normalized input.

Let $u_{\min,1}, u_{\min,2}, u_{\max,1}, u_{\max,2}, t_1, t_2, t_3, t_4 \in \mathbb{R}$ be such that $u_{\min,1} < u_{\min,2} < u_{\max,1} < u_{\max,2}$ and $0 < t_1 < t_2 < t_3 < t_4$. Let the input $u \in \mathcal{S}([0, t_4], \mathbb{R})$ be strictly increasing on the intervals $[0, t_1]$ and $[t_2, t_3]$, strictly decreasing on the intervals $[t_1, t_2]$ and $[t_3, t_4]$, with $u(0) = u_{\min,1}$, $u(t_1) = u_{\max,1}$, $u(t_2) = u_{\min,2}$, $u(t_3) = u_{\max,2}$, $u(t_4) = u_{\min,1}$, and $u(t) = u_{\min,1}$.To find its corresponding normalized function ψ_u we proceed as follows. Note that ρ_u is strictly increasing so that it is invertible, and ρ_u^{-1} is also strictly increasing. From Lemma 1 it comes that $\psi_u = u\circ\rho_u^{-1}$ so that ψ_u is strictly increasing on the interval $[0, \varrho_1]$, where $\varrho_1 = \rho_u(t_1)$. Thus $\dot{\psi}_u(\varrho) \geq 0$ when $\varrho \in (0, \varrho_1)$ and $\dot{\psi}_u(\varrho)$ exists. On the other hand, by Lemma 1 the set on which $\dot{\psi}_u$ is not defined or is different from ± 1 has measure zero. Thus $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in (0, \varrho_1)$. Using the fact that ψ_u is absolutely continuous we obtain from the Fundamental Theorem of Calculus that

$$\psi_u(\varrho) - \psi_u(0) = \int_0^\varrho \dot{\psi}_u(\tau)\, d\tau = \int_0^\varrho d\tau = \varrho, \forall \varrho \in [0, \varrho_1].$$

Taking into account that $\psi_u\big(\rho_u(0)\big) = u(0)$ it comes that $\psi_u(0) = u_{\min,1}$ so that

$$\psi_u(\varrho) = \varrho + u_{\min,1}, \text{ for all } \varrho \in [0, \varrho_1].$$

Proceeding in an analogous way for the rest of the intervals we reach the expression of the normalized input as:

$$\psi_u(\varrho) = \begin{cases} \varrho + u_{\min,1} & \text{for } \varrho \in [0, \varrho_1], \\ -\varrho + \varrho_1 + u_{\max,1} & \text{for } \rho \in [\varrho_1, \varrho_2], \\ \varrho - \varrho_2 + u_{\min,2} & \text{for } \varrho \in [\varrho_2, \varrho_3], \\ -\varrho + \varrho_3 + u_{\max,2} & \text{for } \varrho \in [\varrho_3, \varrho_4], \end{cases} \quad (1)$$

where $\varrho_1 = u_{\max,1} - u_{\min,1} > 0$, $\varrho_2 = \varrho_1 + u_{\max,1} - u_{\min,2} > \varrho_1$, $\varrho_3 = \varrho_2 + u_{\max,2} - u_{\min,2} > \varrho_2$, and $\varrho_4 = \varrho_3 + u_{\max,2} - u_{\min,1} > \varrho_3$. Note that the function ψ_u is defined on the interval $I_u = [0, \varrho_4]$.

4.3 The equivalence relation $\mathscr{E}_2$

To get the equivalence relation $\mathscr{E}_2$ and analyze its relationship with the equivalence relation $\mathscr{E}_1$ we need first to construct an appropriate set of time-change scales and an appropriate set of periodic inputs.

The set $\mathbb{I}_T$ Let $T \in]0, \infty[$ and $\phi_T \in \mathcal{S}(\mathbb{R}_+, \mathbb{R}_+)$ be a function that satisfies (a)–(c).

(a) ϕ_T is strictly increasing on $\mathbb{R}_+$.
(b) $\phi_T(0) = 0$, $\lim_{t\to\infty} \phi_T(t) = \infty$, and $\phi_T(t + kT) = \phi_T(t) + kT, \forall t \in [0, T], \forall k \in \mathbb{N}$.
(c) $\phi_T^{-1} \in \mathcal{S}(\mathbb{R}_+, \mathbb{R}_+)$.

The set of all such functions ϕ_T is denoted $\mathbb{I}_T$.

Proposition 4. $\phi_T^{-1} \in \mathbb{I}_T$.

Proof. Straightforward. □

Proposition 5. Suppose that $\phi_{1T}, \phi_{2T} \in \mathbb{I}_T$. Then we have $\phi_{1T} \circ \phi_{2T} \in \mathbb{I}_T$.

Proof. This is a consequence of (Ikhouane, 2020, Proposition A1). □

The set $\mathbb{W}_T$ Let $p \in \mathbb{N} \setminus \{0, 1\}$ and $T \in]0, \infty[$. Let $t_0, t_1, \ldots, t_p \in [0, T]$ with $t_0 = 0$, $t_p = T$, and $t_k < t_{k+1}$ for all $k = 0, 1, \ldots, p-1$.

Definition 6. The $p+1$–tuple $(t_0, t_1, \ldots, t_p)$ is called a partition of the interval $[0, T]$.

Let $\mathbb{W}_{t_0,t_1,\ldots,t_p}$ be the set of all T–periodic functions $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ such that (i) and (ii) hold.

(i) u is either strictly increasing or strictly decreasing on each interval $[t_k, t_{k+1}]$ for all $k = 0, 1, \ldots, p-1$.
(ii) On each interval $[t_k, t_{k+1}]$ the inverse of the restriction of u to that interval belongs to the space $\mathcal{S}\Big([u(t_k), u(t_{k+1})], [t_k, t_{k+1}]\Big)$.

Note that owing to the periodicity of u we have $u(T) = u(0)$.

Define the set $\mathbb{W}_T = \bigcup \mathbb{W}_{t_0,t_1,\ldots,t_p}$ as the union of all possible sets $\mathbb{W}_{t_0,t_1,\ldots,t_p}$ obtained for all possible partitions of the interval $[0, T]$.

Proposition 7. Let $u \in \mathbb{W}_{t_0,t_1,\ldots,t_p}$. Then $I_u = \mathbb{R}_+$; the normalized input ψ_u is $\rho_u(T)$–periodic and is independent of the partition in the sense that ψ_u depends only on $u(t_0), u(t_1), \ldots, u(t_{p-1})$. Moreover, $\rho_u \in \mathbb{I}_T$.

Proof. The first part of the proposition comes from Lemma 3 whilst the second part follows as in Section 4.2. □

For $p = 4$ an example of an input $u \in \mathbb{W}_{t_0,t_1,\ldots,t_p}$ is provided in Figure ??. Since u is t_4–periodic its graph is plotted on one period, that is on the interval $[0, t_4]$. The corresponding normalized input ψ_u is ϱ_4–periodic where $\varrho_4 = \rho_u(t_4)$ and is plotted on one period in Figure ??.

On $\mathscr{E}_1$ and $\mathscr{E}_2$ We define the relation $\mathscr{E}_1$ on the set $\mathbb{W}_T$ as follows.

Definition 8. For $u_1, u_2 \in \mathbb{W}_T$ we say that $u_1 \mathscr{E}_1 u_2$ when there exists $\phi_T \in \mathbb{I}_T$ such that $u_1 = u_2 \circ \phi_T$.

Proposition 9. $\mathscr{E}_1$ is an equivalence relation on $\mathbb{W}_T$.

Proof. Observe first that $u_2 \circ \phi_T \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ owing to (Ikhouane, 2020, Proposition A1) and that $u_2 \circ \phi_T$ is T–periodic. Suppose that $u_2 \in \mathbb{W}_{t_0,t_1,\ldots,t_p}$, then $u_2 \circ \phi_T$ is strictly increasing (resp. decreasing) on the interval $[t_k, t_{k+1}]$ when u_2 is strictly increasing (resp. decreasing) on the same interval. This means that $u_1 \in \mathbb{W}_{t_0,t_1,\ldots,t_p}$ so that the equivalence relation $\mathscr{E}_1$ makes sense.

$u \mathscr{E}_1 u$ by taking ϕ_T as the identity function.

$u_1 \mathscr{E}_1 u_2 \Leftrightarrow u_2 \mathscr{E}_1 u_1$ using Proposition 4.

If $u_1 \mathscr{E}_1 u_2$ and $u_2 \mathscr{E}_1 u_3$ then by Proposition 5 we have $u_1 \mathscr{E}_1 u_3$. □

We define the relation $\mathscr{E}_2$ on the set $\mathbb{W}_T$ as follows.

Definition 10. For $u_1, u_2 \in \mathbb{W}_T$ we say that $u_1 \mathscr{E}_2 u_2$ when $\psi_{u_1} = \psi_{u_2}$.

Proposition 11. $\mathscr{E}_2$ is an equivalence relation on $\mathbb{W}_T$.

Proof. Straightforward. □

Proposition 12. For any $u_1, u_2 \in \mathbb{W}_T$ the following holds: $u_1 \mathscr{E}_2 u_2 \Leftrightarrow u_1 \mathscr{E}_1 u_2$.

Proof. $\Rightarrow$
$u_1 \mathscr{E}_2 u_2 \Leftrightarrow \psi_{u_1} = \psi_{u_2}$. Then

$$\underbrace{\psi_{u_1} \circ \rho_{u_1}}_{= u_1} = \underbrace{\psi_{u_2} \circ \rho_{u_2}}_{= u_2} \circ \underbrace{\rho_{u_2}^{-1} \circ \rho_{u_1}}_{\in \mathbb{I}_T},$$

so that $u_1 \mathscr{E}_1 u_2$.
$\Leftarrow$
$u_1 \mathscr{E}_1 u_2 \Leftrightarrow u_1 = u_2 \circ \phi_T$. Considering the change of variable $\nu = \phi_T(\tau)$ we get

$$\rho_{u_1}(t) = \int_0^t \left|\dot{u}_1(\tau)\right| \mathrm{d}\tau = \int_0^{\phi_T(t)} \left|\dot{u}_2(\nu)\right| \mathrm{d}\nu = \rho_{u_2}\big(\phi_T(t)\big),$$

so that $\rho_{u_1} = \rho_{u_2} \circ \phi_T$. On the other hand,

$$\psi_{u_1} = u_1 \circ \rho_{u_1}^{-1} = (u_2 \circ \phi_T) \circ (\rho_{u_2} \circ \phi_T)^{-1} = u_2 \circ \rho_{u_2}^{-1} = \psi_{u_2}.$$

Thus $u_1 \mathscr{E}_2 u_2$. □

5. BACKGROUND RESULTS

In this section we introduce the concepts of normalized state and output, along with the concepts of consistency and strong consistency which are needed to define formally the hysteresis loop of the generalized Duhem model. These concepts have been introduced in Ikhouane (2013) and further developped in Ikhouane (2018).

5.1 The generalized Duhem model

The *generalized* Duhem model with input u, state x and output y consists of a differential equation that describes the state x as follows, see Oh and Bernstein (2013):

$$\dot{x}(t) = f\big(x(t), u(t)\big) g\big(\dot{u}(t)\big), \text{ for almost all } t \in \mathbb{R}_+, \quad (2)$$

$$x(0) = x_0, \quad (3)$$

and an algebraic equation that describes the output y as

$$y(t) = h\big(x(t), u(t)\big), \forall t \in \mathbb{R}_+. \quad (4)$$

In Equations (2)–(4) the input $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$; the function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times n'}$ is continuous; n and n' are strictly positive integers; the function $g : \mathbb{R} \to \mathbb{R}^{n'}$ is continuous and satisfies $g(0) = 0$; the function $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous; and the initial state $x_0 \in \mathbb{R}^n$.

Hypothesis 13. for any homeomorphism $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and any continuous functions x,u we have $h(x \circ \phi, u \circ \phi) = h(x, u) \circ \phi$.

An example of a function h that satisfies Assumption 13 is $h(\alpha, \beta) = A\alpha + B\beta$ where A and B are matrices of appropriate dimensions.

Hypothesis 14. For every $(u, x_0) \in \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n$ there exists a unique solution $x \in \mathcal{S}(\mathbb{R}_+, \mathbb{R}^n)$ that satisfies Equations (2)–(3).

From Assumption 14 we get $y \in C^0(\mathbb{R}_+, \mathbb{R}) \cap L^\infty(\mathbb{R}_+, \mathbb{R})$.

Define the operator

$$\mathcal{H}_o : \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n \to C^0(\mathbb{R}_+, \mathbb{R}) \cap L^\infty(\mathbb{R}_+, \mathbb{R})$$

by the relation $\mathcal{H}_o(u, x_0) = y$; and the operator

$$\mathcal{H}_s : \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n \to \mathcal{S}(\mathbb{R}_+, \mathbb{R}^n)$$

by the relation $\mathcal{H}_s(u, x_0) = x$.

5.2 The normalized state and output

Lemma 15. Ikhouane (2013) Let $(u, x_0) \in \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n$. Then, there exists unique functions $x_u \in \mathcal{S}(I_u, \mathbb{R}^n)$ and $\varphi_u \in L^\infty(I_u, \mathbb{R}) \cap C^0(I_u, \mathbb{R})$ that satisfy $x_u \circ \rho_u = \mathcal{H}_s(u, x_0)$ and $\varphi_u \circ \rho_u = \mathcal{H}_o(u, x_0)$.

The functions x_u and φ_u are called the *normalized state* and *normalized output* respectively.

5.3 Definition of consistency

The concept of consistency is introduced in Ikhouane (2013) as follows.

Definition 16. Ikhouane (2013) Let $(u, x_0) \in \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n$. The operator $\mathcal{H}_s$ (resp. $\mathcal{H}_o$) is said to be *consistent with respect to* (u, x_0) if there exists a function $x_u^\star \in L^\infty(I_u, \mathbb{R}^n) \cap C^0(I_u, \mathbb{R}^n)$ (resp. $\varphi_u^\star \in L^\infty(I_u, \mathbb{R}) \cap C^0(I_u, \mathbb{R})$) such that $\lim_{\gamma \to \infty} \|x_{u \circ s_\gamma} - x_u^\star\| = 0$. (resp. $\lim_{\gamma \to \infty} \|\varphi_{u \circ s_\gamma} - \varphi_u^\star\| = 0$).

Note that if $\mathcal{H}_s$ is consistent with respect to (u, x_0) then $\mathcal{H}_o$ is consistent with respect to (u, x_0).

5.4 Definition of strong consistency

Observe that, in Definition 16 of consistency, the input u may be periodic or not. However, to characterize the hysteresis loop of the operator $\mathcal{H}_o$, the input u needs to be periodic. For this reason Ikhouane (2013) introduces the concept of strong consistency (this is Definition 17) in relation with periodic inputs.

Definition 17. Ikhouane (2013) Let $x_0 \in \mathbb{R}^n$. Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be such that the input u is non constant and T–periodic where $T \in]0, \infty[$. Assume furthermore that the operator $\mathcal{H}_o$ is consistent with respect to (u, x_0). For any nonnegative integer k, define the function $\varphi^\star_{u,k} \in L^\infty\big([0, \rho_u(T)], \mathbb{R}\big)$ by $\varphi^\star_{u,k}(\varrho) = \varphi^\star_u\big(\rho_u(T)k + \varrho\big), \forall \varrho \in [0, \rho_u(T)]$. The operator $\mathcal{H}_o$ is said to be *strongly consistent with respect to* (u, x_0) if there exists $\varphi^\circ_u \in L^\infty\big([0, \rho_u(T)], \mathbb{R}\big)$ such that $\lim_{k\to\infty} \|\varphi^\star_{u,k} - \varphi^\circ_u\| = 0$.

In Definition 17 if we substitute $\mathcal{H}_o$ by $\mathcal{H}_s$ and φ_u by x_u we get the definition of strong consistency for the operator $\mathcal{H}_s$.

Also, note that if $\mathcal{H}_s$ is strongly consistent with respect to (u, x_0) then $\mathcal{H}_o$ is strongly consistent with respect to (u, x_0).

Definition 18. Ikhouane (2013, 2018) Let $x_0 \in \mathbb{R}^n$ and let $T > 0$. Let $u \in \mathcal{S}(\mathbb{R}_+, \mathbb{R})$ be non constant and T–periodic. Assume that the operator $\mathcal{H}_o$ (resp. $\mathcal{H}_s$) is strongly consistent with respect to (u, x_0). We call *hysteresis loop of the operator $\mathcal{H}_o$ (resp. $\mathcal{H}_s$) with respect to* (u, x_0) the set

$$\mathcal{G}_u = \left\{\big(\psi_u(\varrho), \varphi^\circ_u(\varrho)\big), \varrho \in [0, \rho_u(T)]\right\}. \tag{5}$$

(resp. $\mathcal{G}_u = \left\{\big(\psi_u(\varrho), x^\circ_u(\varrho)\big), \varrho \in [0, \rho_u(T)]\right\}$.)

5.5 Examples

Detailed numerical simulations and examples that illustrate the concepts introduced in Section 5 are provided in Ikhouane (2018). In particular these examples show that, in general, the generalized Duhem model (2)–(4) is not rate independent.

6. A NEW FORM OF RATE INDEPENDENCE

In this section we show that the generalized Duhem model satisfies a new form of rate independence.

Proposition 19. Suppose that (i)–(iii) hold.

(i) The quantities $\ell_1 = \lim_{\nu\downarrow 0} \frac{g(\nu)}{\nu}$ and $\ell_2 = \lim_{\nu\uparrow 0} \frac{g(\nu)}{\nu}$ exist, are finite, and at least one of them is nonzero.
(ii) There exists a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|x\| \le Q(|x_0|, \|u\|, \|\dot{u}\|)$.
(iii) The operator $\mathcal{H}_s$ is consistent and strongly consistent with respect to all $(T, u, x_0) \in]0, \infty[\times \mathbb{W}_T \times \mathbb{R}^n$.

Then, for any $u_1, u_2 \in \mathbb{W}_T$ such that $u_1 \mathscr{E}_2 u_2$ we have $\mathcal{G}_{u_1} = \mathcal{G}_{u_2}$.

Proof. By (Naser and Ikhouane, 2013, Lemma 12) we have for all $\varrho \in \mathbb{R}_+$ that

$$x^\star_{u_1}(\varrho) = x_0 + \int_0^\varrho f\big(x^\star_{u_1}(\nu), \psi_{u_1}(\nu)\big) g^\star\big(\dot{\psi}_{u_1}(\nu)\big)\, \mathrm{d}\nu, \tag{6}$$

$$x^\star_{u_2}(\varrho) = x_0 + \int_0^\varrho f\big(x^\star_{u_2}(\nu), \psi_{u_2}(\nu)\big) g^\star\big(\dot{\psi}_{u_2}(\nu)\big)\, \mathrm{d}\nu, \tag{7}$$

where $g^\star : \mathbb{R} \to \mathbb{R}$ is defined by

$$g^\star(\nu) = \begin{cases} \ell_1 \nu, & \nu \ge 0 \\ \ell_2 \nu, & \nu \le 0. \end{cases} \tag{8}$$

Since $\psi_{u_1} = \psi_{u_2}$ it comes from Equations (6)–(7) that $x^\star_{u_1} = x^\star_{u_2}$. This fact leads to $x^\circ_{u_1} = x^\circ_{u_2}$ and $\varphi^\circ_{u_1} = \varphi^\circ_{u_2}$, which proves the proposition. □

We have seen that the usual rate independence property can be interpreted as follows: equivalent inputs -in the sense of the equivalence relation $\mathscr{E}_1$- have the same graph output-versus-input.

On the other hand, Proposition 12 says that the equivalence relations $\mathscr{E}_1$ and $\mathscr{E}_2$ are the same on the set $\mathbb{W}_T$. This fact along with Proposition 19 implies that, although the generalized Duhem model (2)–(4) is not rate independent in the usual sense, it satisfies a new form of rate independence: *equivalent inputs -in the sense of the equivalence relation $\mathscr{E}_2$- have the same hysteresis loop.*

7. INTERPRETATION OF THE RESULTS

Let the operators $\mathcal{H}^\star_s : \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n \to L^\infty(\mathbb{R}_+, \mathbb{R}^n) \cap C^0(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathcal{H}^\star_o : \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n \to L^\infty(\mathbb{R}_+, \mathbb{R}) \cap C^0(\mathbb{R}_+, \mathbb{R})$ be defined by the relations $\mathcal{H}^\star_s(u, x_0) = x^\star_u \circ \rho_u$ and $\mathcal{H}^\star_o(u, x_0) = \varphi^\star_u \circ \rho_u$ respectively.

Proposition 20. Let $\gamma > 0$ and $T > 0$. Let $(u, \phi_T, x_0) \in \mathbb{W}_T \times \mathbb{I}_T \times \mathbb{R}^n$. Define $\phi_{T;\gamma} = \phi_T \circ s_\gamma$. Suppose that the conditions (i)–(iii) of Proposition 19 hold. Then

$$\mathcal{H}^\star_s(u \circ \phi_{T;\gamma}, x_0) = \mathcal{H}^\star_s(u, x_0) \circ \phi_{T;\gamma}$$

and

$$\mathcal{H}^\star_o(u \circ \phi_{T;\gamma}, x_0) = \mathcal{H}^\star_o(u, x_0) \circ \phi_{T;\gamma}.$$

Proof. Define $y^\star_u = \mathcal{H}^\star_s(u, x_0)$ and $z^\star_u = \mathcal{H}^\star_s(u \circ \phi_{T;\gamma}, x_0)$. Then, by (Naser and Ikhouane, 2013, Lemma 12) we have

$$\dot{z}^\star_u(\tau) = f\big(z^\star_u(\tau), u \circ \phi_{T;\gamma}(\tau)\big) g^\star\big(\dot{\overbrace{u \circ \phi_{T;\gamma}}}(\tau)\big), \tag{9}$$

$$\dot{y}^\star_u(\tau) = f\big(y^\star_u(\tau), u(\tau)\big) g^\star\big(\dot{u}(\tau)\big). \tag{10}$$

Define $\nu = \phi_{T;\gamma}(\tau)$ and $w^\star_u = z^\star_u \circ \phi^{-1}_{T;\gamma}$. Then (9) gives

$$\dot{w}^\star_u(\nu) = f\big(w^\star_u(\nu), u(\nu)\big) g^\star\big(\dot{u}(\nu)\big), \tag{11}$$

which is the same as (10). Since $w^\star_u(0) = y^\star_u(0) = x_0$ it comes from the uniqueness of solutions that $w^\star_u = y^\star_u$, that is $z^\star_u \circ \phi^{-1}_{T;\gamma} = y^\star_u$ which proves the proposition. □

Proposition 20 says that the operators $\mathcal{H}^\star_s$ and $\mathcal{H}^\star_o$ are rate independent.

Define the operator $\mathcal{H}^\dagger_s : \mathcal{S}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}^n \to L^\infty(\mathbb{R}_+, \mathbb{R}^n) \cap C^0(\mathbb{R}_+, \mathbb{R}^n)$ by the relation $\mathcal{H}^\dagger_s = \mathcal{H}_s - \mathcal{H}^\star_s$.

Proposition 21. For all $u \in \mathbb{W}_T$ we have

$$\lim_{\gamma\to\infty} \|\mathcal{H}^\dagger_s(u \circ \phi_{T;\gamma}, x_0) \circ \phi^{-1}_{T;\gamma}\| = 0.$$

Proof. Define $v = u \circ \phi_T$, then $v \in \mathbb{W}_T$. We have

$$\mathcal{H}^\dagger_s(v \circ s_\gamma, x_0) = \mathcal{H}_s(v \circ s_\gamma, x_0) - \mathcal{H}^\star_s(v \circ s_\gamma, x_0).$$

Proposition 20 gives

$$\mathcal{H}^\star_s(v \circ s_\gamma, x_0) = \mathcal{H}^\star_s(v, x_0) \circ s_\gamma.$$

Also $\mathcal{H}_s(v \circ s_\gamma, x_0) = x_{v\circ s_\gamma} \circ \rho_{v\circ s_\gamma} = x_{v\circ s_\gamma} \circ \rho_v \circ s_\gamma$ and $\mathcal{H}^\star_s(v, x_0) = x^\star_v \circ \rho_v$.

By (iii) of Proposition 19 it comes from Definition 16 that

$$\lim_{\gamma\to\infty} \|x_{v\circ s_\gamma} - x^\star_v\| = 0$$

so that

$$\lim_{\gamma\to\infty} \| \underbrace{x_{v\circ s_\gamma} \circ \rho_v}_{\mathcal{H}_s(v\circ s_\gamma, x_0)\circ s_\gamma^{-1}} - \underbrace{x^\star_v \circ \rho_v}_{\mathcal{H}^\star_s(v, x_0)} \| = 0.$$

which gives

$$\lim_{\gamma\to\infty} \|\mathcal{H}_s(u \circ \phi_{T;s_\gamma}, x_0) \circ \underbrace{s_\gamma^{-1} \circ \phi_T^{-1}}_{\phi_{T;s_\gamma}^{-1}} - \mathcal{H}_s^\star(u, x_0)\| = 0,$$

where we have used the fact that $\mathcal{H}_s^\star(u \circ \phi_T, x_0) \circ \phi_T^{-1} = \mathcal{H}_s^\star(u, x_0)$.

Finally, noting that $\mathcal{H}_s^\star(u, x_0) = \mathcal{H}_s^\star(u \circ \phi_{T;\gamma}, x_0) \circ \phi_{T;\gamma}^{-1}$ we get the proposition. □

Propositions 20 and 21 say that the operator $\mathcal{H}_s$ has been decomposed into the sum of two operators:

(i) The operator $\mathcal{H}_s^\star$ which is rate independent with respect to the changes in time scale $\phi_{T;\gamma}$. Note that taking ϕ_T as the identity function makes $\mathcal{H}_s^\star$ rate independent with respect to the linear changes in time scale s_γ.
(ii) The operator $\mathcal{H}_s^\dagger$ such that $\mathcal{H}_s^\dagger(u \circ \phi_{T;\gamma}, x_0) \circ \phi_{T;\gamma}^{-1}$ vanishes when $\gamma \to \infty$ (loosely speaking, the output vanishes when the frequency of the input goes to zero).

The same can be said about the operator $\mathcal{H}_o$.

Let us go back to the experiments of (Gan and Zhang, 2015, Fig. 7). The plot in each subfigure is the steady state graph output-versus-input obtained by eliminating the transient and keeping the steady-state response. Far from the transient this steady-state part is close to the graph output-versus-input, and is also close to the hysteresis loop for inputs with low frequencies. The experiments show that the loops obtained at low frequencies change little with frequency. This observation is compatible with our decomposition $\mathcal{H}_o = \mathcal{H}_o^\star + \mathcal{H}_o^\dagger$ since a change in the frequency is a linear change in time scale, $\mathcal{H}_o^\star$ is rate independent with respect to linear changes in time scale, and $\mathcal{H}_o^\dagger$ is small at low frequencies.

As the frequency of the input increases, the experiments of (Gan and Zhang, 2015, Fig. 7) show that the loops become clearly dependent on frequency. Note that these loops are not the hysteresis loop $\mathcal{G}_u$ since the latter is obtained precisely for $\gamma \to \infty$, that is at low frequencies.

These observations show that the new form of rate independence that we consider in this paper is more compatible with the experimental observations of (Gan and Zhang, 2015, Fig. 7) than rate independence understood as the invariance of the graph output-versus-input under the equivalence relation $\mathscr{E}_1$.

8. CONCLUSIONS

Rate independence is the usual property used to define hysteresis operators. However, this property is but an approximation of the behavior of real hysteresis processes. Using the generalized Duhem model as a case study, we showed that, although this model is not rate independent in general, it does satisfy a new form of rate independence. We interpreted the obtained results in relation with experimental observations.

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