Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Verification and Design of Zero-Sum Potential Games ⋆

Changxi Li, **Fenghua He**, **Ning Hao**

Control and Simulation Center, Harbin Institute of Technology, Harbin 150001, P. R. China, Email: changxi1989@163.com(Changxi Li), hefenghua@gmail.com(Fenghua He), 18B904055@stu.hit.edu.cn(Ning Hao).

Abstract: Zero-sum game is a class of game where one player's gain is equivalent to another's loss, which can be used in competitive situation. But pure Nash equilibrium maybe not exist in general zero-sum games. Potential games have nice properties, such as existence of pure Nash equilibrium. To combine advantages of zero-sum games and potential games, zero-sum potential game is proposed in this paper. Verification for a finite non-cooperative game being a zero-sum potential game is considered. Conversely, how to design a zero-sum potential game is also studied when the potential function is given. We show that verification and design of zero-sum potential game can be realized by solving linear equations. Furthermore, we find that if any two players play the zero-sum potential game in a network, then the networked game is also a zero-sum potential game.

Keywords: Zero-sum game, potential game, networked evolutionary game, semi-tensor product of matrices, linear equation, verification and design.

1. INTRODUCTION

Zero-sum games and potential games maybe two well-known class of games in game theory (Washburn (1995), Monderer (1996)). They are widely studied by researchers from different fields. Examples of zero-sum games include gambling games, card games, sports games (Morrow (1994), Giuseppe (2018)), and generative adversarial network (GAN) (Goodfellow (2014)). Applications of potential game include: (i) consensus/synchronization of multi-agent systems (Marden (2009)); (ii) distributed optimization (Li (2013); Yang (2010)); (iii) control in wireless networks (Candogan (2010)), just to name a few.

Zero-sum game is a class of game where one player's gain is equivalent to another's loss, which can be used in competitive situation. But there maybe no pure Nash equilibrium in general zero-sum games. Potential games have nice properties, such as the finite improvement property and the existence of pure Nash equilibrium (Rosenthal (1973)). To combine advantages of zero-sum games and potential games, zero-sum potential game is proposed in this paper. Zero-sum potential game can be used in competitive situation and existence of pure Nash equilibrium is also guaranteed. Two-player zero-sum potential game was studied by Branzei (2003), which showed that two-person zero-sum potential games can be transformed into supermodular games. Hwang (2016) proposed a similar concept, named zero-sum equivalent potential game.

Recently, game-theoretical control has drawn considerable attention due to its widespread applications (Marden (2009), Li (2013)). Addressing control related problems via game theory needs two steps: (i) viewing interacting agents as intelligent rational decision-makers of a game by defining a set of available strategies and incentives for each agent, (ii) specifying a learning rule for the game so that agents can converge to a desirable situation. Compared with traditional methods, the advantage of game-theoretical control is that a modularized design architecture is provided, which is shown in Fig. 1. Gopalakrishnan (2011) described the modularized design architecture as an hourglass architecture.

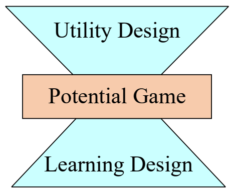

Fig. 1: Hourglass architecture of game-theoretical control

According to the hourglass architecture of game-theoretical control, the first step is to design utility for each agent to make sure that the designed game falls under some special category games, such as potential games. Several researchers have studied potential game design in distributed control, such as Marden (2009), Li (2013), and Liu (2019). Different with existing works, we consider design method for zero-sum potential games, which can be used in competitive case. In fact, verification and design of zero-sum potential game are interdependent. Because only when one know how to check a given game is a zero-

⋆ This work is supported partly by the National Natural Science Foundation of China (NSFC) under Grants 61473099, 61773371, 61733018 and 61333001. Corresponding author: Changxi Li. Tel.: +86 0451-86402947; fax.: +86 0451-86414580.

sum potential game, can one design a zero-sum potential game. Therefore, verification and design of finite zero-sum potential games are the research emphases of this paper.

The contributions of this manuscript are threefold: (i) A necessary and sufficient condition for a finite game being zero-sum potential game is obtained, which is equivalent to existence of solutions for a linear equation. (ii) An approach for designing finite zero-sum potential game is presented when the potential function is given. (iii) We prove that if the edge-related game (fundamental network game) is a zero-sum potential game, then the networked evolutionary game is also a zero-sum potential game.

The rest of this paper is organized as follows: Section 2 is some preliminaries, including game theory, semi-tensor product (STP) of matrices, and problem description. Section 3 considers verification of finite zero-sum potential games. Section 4 investigates design of finite zero-sum potential games. Section 5 studies zero-sum potential networked evolutionary games. A brief conclusion is given in Section 6.

Notations: $\mathbb{R}^n$ is the Euclidean space of all n-dimensional vectors. $\mathcal{M}_{m\times n}$ is the set of $m \times n$ real matrices. $\mathbf{1}_n$ is a n-dimensional vector with all elements being 1. I_m is the $m\times m$-dimensional identity matrix. δ_n^i is the i-th column of the identity matrix I_n. $\mathcal{D}_k := \{1, 2, \cdots, k\}, k \geq 2$. Col$(M)$ is the set of columns of matrix M. The transposition of matrix $M \in \mathcal{M}_{m\times n}$ is denoted by $M^T \in \mathcal{M}_{n\times m}$. Span$(M)$ is the subspace spanned by all columns of matrix M.

2. PRELIMINARIES

2.1 Potential Games

The research object in this paper is finite non-cooperative games. A finite non-cooperative game is a triple

$$G = \{N, \{S_i\}_{i\in N}, \{c_i\}_{i\in N}\},$$

where $N = \{1, 2, \cdots, n\}$ is the set of players, $S_i = \{1, 2, \cdots, k_i\}$ is the set of strategies of player $i \in N$, and $c_i : S \to \mathbb{R}$ is the utility function of player i, with $S := \times_{i=1}^n S_i$ being the strategy profile of the game.

The concept of potential game was firstly proposed by Rosenthal (1973), whose definition is as follows:

Definition 1. A finite non-cooperative game G is a potential game if there exists a function $P : S \to \mathbb{R}$, such that for every player $i \in N$ and $\forall s_{-i} \in S_{-i}, \forall x_i, y_i \in S_i$

$$c_i(x_i, s_{-i}) - c_i(y_i, s_{-i}) = P(x_i, s_{-i}) - P(y_i, s_{-i}),$$

where P is called the potential function of G, and $S_{-i} := \times_{j\neq i} S_j$ is the set of partial strategy profiles other than player i.

The following Lemma is obvious according to Definition 1.

Lemma 2. A finite game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is potential if and only if there exist functions $d_i : S_{-i} \to \mathbb{R}, i \in N$ such that for every $x \in S$

$$P(s) = c_i(s) - d_i(s_{-i}),\ \forall i \in N, \tag{1}$$

where $P(s)$ is the potential function, and $s_{-i} \in S_{-i}$.

Zero-sum game is a class of game which describes situation in which one player's gain is equivalent to another's loss. A finite non-cooperative game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is a zero sum game if and only if

$$\sum_{i=1}^n c_i(s) = 0,\ \forall s \in S.$$

Combining potential game and zero sum game, we obtain a new class of games, called zero-sum potential game.

Definition 3. A finite non-cooperative game G is called zero-sum potential if and only if G is a potential game and a zero-sum game simultaneously.

2.2 Semi-tensor Product of Matrices

The technical tool used in this paper is semi-tensor product of matrices. Here is a brief introduction on STP of matrices. Please refer to Cheng (2012) for more details.

Definition 4. Suppose $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and l be the least common multiple of n and p. The STP of A and B is defined by

$$A \ltimes B := \left(A \otimes I_{l/n}\right)\left(B \otimes I_{l/p}\right) \in \mathcal{M}_{ml/n\times ql/p},$$

where $\otimes$ is the Kronecker product.

Assume $i \in \mathcal{D}_k$. By identifying $i \sim \delta_k^i$ we call δ_k^i the vector form of integer i. A function $f : \times_{i=1}^n \mathcal{D}_{k_i} \to \mathbb{R}$ is called a mix-valued pseudo-logical function.

Definition 5. Let $f : \times_{i=1}^n \mathcal{D}_{k_i} \to \mathbb{R}$ be a mix-valued pseudo-logical function. Then there exists a unique row vector $M_f \in \mathbb{R}^k$, such that

$$f(x_1, \cdots, x_n) = M_f \ltimes_{i=1}^n x_i.$$

M_f is called the structure vector of f, and $k = \times_{i=1}^n k_i$.

Using the vector expression of strategies, the utility function $c_i(x)$ of a finite game can be expressed as

$$c_i(x) = V_i \ltimes_{j=1}^n x_j,$$

where $x_i \in S_i$, and $V_i \in \mathbb{R}^k$ is called the structure vector of c_i, $k = \times_{i=1}^n k_i$. Denote by $V_G = [V_1, \cdots, V_n]$, which is called the utility vector of game G.

2.3 Problem Description

This paper aims at providing a systematic approach for verification and design a zero-sum potential game. Verification tries to answer the question of how to identify a game is a zero-sum potential game. Furthermore, if the game is a zero-sum potential game, then the potential function should be provided, which can be described as follows.

$$c_i(x), i = 1, 2, \cdots, n. \Rightarrow P(x).$$

Design of zero-sum potential games is an inverse procedure of verification. Design of zero-sum potential games studies how to design a zero-sum potential game according to a given potential function, which can be described as follows.

$$P(x) \Rightarrow c_i(x), i = 1, 2, \cdots, n.$$

3. VERIFICATION OF ZERO-SUM POTENTIAL GAMES

Denote $\mathcal{G}_{[n;k_1,\cdots,k_n]}$ by set of finite games with $|N| = n$, $|S_i| = k_i$, $i = 1, \cdots, n$. In this section we consider how to verify a given finite non-cooperative game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$

is a zero-sum potential game. According to the definition of zero-sum potential game, we know that a finite non-cooperative game G is a zero-sum potential game if and only if

$$\sum_{i=1}^{n} c_i(s) = 0, \ \forall s \in S. \tag{2}$$

$$c_i(s) = P(s) + d_i(s_{-i}), \ \forall i \in N, \forall s \in S. \tag{3}$$

Let

$$E_i := I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}}, \ i = 1, \cdots, n,$$

where

$$k^{[p,q]} := \begin{cases} \times_{j=p}^{q} k_j, & q \geq p \\ 1, & q < p. \end{cases}$$

The following theorem reveals the discriminate criterion for zero-sum potential game.

Theorem 6. A given finite non-cooperative game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is a zero-sum potential game, if and only if, the following linear equation

$$Z\xi = nV_G^T, \tag{4}$$

has a solution ξ, where V_G is the utility vector of G, and

$$Z = \begin{bmatrix} (n-1)E_1 & -E_2 & \cdots & -E_n \\ -E_1 & (n-1)E_2 & \cdots & -E_n \\ \vdots & \vdots & \ddots & \vdots \\ -E_1 & -E_2 & \cdots & (n-1)E_n \end{bmatrix}. \tag{5}$$

Moreover, if the solution ξ exists, then the potential function vector V_P is

$$V_P = V_1 - \xi_1 E_1^T$$

where ξ_1 is the sub-vector of the first $\times_{j=2}^{n} k_j$ elements of ξ.

Proof. Using the vector expression of strategies, equation (2) and (3) have the following form

$$\sum_{i=1}^{n} V_i \ltimes_{j=1}^{n} s_j = 0, \ \forall s \in S.$$

$$\begin{aligned} V_P \ltimes_{j=1}^{n} s_j &= V_i \ltimes_{j=1}^{n} s_j - V_i^d \ltimes_{j \neq i}^{n} s_j, \\ &= V_i \ltimes_{j=1}^{n} s_j - V_i^d E_i^T \ltimes_{j \neq i}^{n} s_j, \ \forall s \in S, \end{aligned}$$

where V_P and V_i^d are the structure vectors of $P(x)$ and $d_i(s_{-i})$, respectively.

Since $s \in S$ are arbitrary, we have

$$\sum_{i=1}^{n} V_i = 0. \tag{6}$$

$$V_P = V_i - V_i^d E_i^T. \tag{7}$$

Let V_G be the utility vector of G. Combing (6) and (7), we have

$$\sum_{i=1}^{n} \left((V_P)^T + E_i (V_i^d)^T \right) = 0. \tag{8}$$

Equation (8) implies that

$$n(V_P)^T = -\sum_{i=1}^{n} E_i (V_i^d)^T. \tag{9}$$

Substituting (9) into (7), we have

$$Z\xi = n(V_G)^T, \tag{10}$$

where $\xi = [V_1^d, \ldots, V_n^d]^T$.

Equation (10) implies that a given finite non-cooperative game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is a zero-sum potential game if and only if it has a solution, which completes the proof.

□

Based on the proof, we know that zero-sum potential games is a subspace of finite games $\mathcal{G}_{[n;k_1,\cdots,k_n]}$. Denote by $\mathcal{Z}_{[n;k_1,\cdots,k_n]}$ the subspace of zero-sum potential games

By calculation, we know

$$Z \cdot \mathbf{1}_\tau = 0,$$

where $\tau = \sum_{i=1}^{n} \frac{k}{k_i}$. It is easy to verify that

$$\text{rank}(Z) = \sum_{i=1}^{n} \frac{k}{k_i} - 1.$$

which means that deleting any one column of Z the remaining columns form a basis of $\mathcal{Z}_{[n;k_1,\cdots,k_n]}$. Delete the last column of Z and denote the remaining part of Z by Z_0. Then we have

Theorem 7. The subspace of zero-sum potential games $\mathcal{Z}_{[n;k_1,\cdots,k_n]}$ is

$$\mathcal{Z}_{[n;k_1,\cdots,k_n]} = \text{Span}(Z_0),$$

which has $\text{Col}(Z_0)$ as its basis. And the dimension of $\mathcal{Z}_{[n;k_1,\cdots,k_n]}$ is

$$\dim(\mathcal{Z}_{[n;k_1,\cdots,k_n]}) = \sum_{i=1}^{n} \frac{k}{k_i} - 1.$$

Remark 8. For general zero-sum game, the existence of pure Nash equilibrium cannot be guaranteed. But each zero-sum potential game has at least one pure Nash equilibrium.

Example 9. Consider a two-player three-strategies game $G \in \mathcal{G}_{[2;3,3]}$. The payoff matrix of G is

$$\begin{bmatrix} (2,-2) & (14,-14) & (-6,6) \\ (10,-10) & (22,-22) & (2,-2) \\ (-14,14) & (-2,2) & (-22,22) \end{bmatrix}.$$

Using the vector expression of strategies, we obtain the utility vector of G as follows

$$\begin{aligned} V_G = [&2, 14, -6, 10, 22, 2, -14, -2, -22, \\ &-2, -14, 6, -10, -22, -2, 14, 2, 22]. \end{aligned}$$

According to equation (5) in Theorem 6, we can construct matrix Z as follows

$$Z = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ \vdots & & & & & \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

By theorem 6, we have the solution of equation (4),

$$\xi = [6, 14, -10, 4, -8, 12].$$

Therefore, the potential function vector of G is

$$\begin{aligned} V_P &= V_1 - \xi_1 E_1^T \\ &= [-5, 1, -9, -9, -3, -13, 3, 9, -1]. \end{aligned}$$

4. DESIGN OF ZERO-SUM POTENTIAL GAMES

In this section we consider when a potential function $P(s)$ is given, how to design the utility function for each player to make it a zero-sum potential game with potential function $P(s)$. The following theorem answers this question.

Theorem 10. Consider a utility-adjustable game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ with objective function $P(s)$

$$P(s) = V_P \ltimes_{j=1}^n s_j.$$

Then utility function can be designed as a zero-sum potential game if and only if the following $n+1$ equations

$$\begin{cases} T\zeta^1 = 0, \\ T_i \zeta_i = (V_P)^T, i \in N, \end{cases} \tag{11}$$

have a solution, where $T = \mathbf{1}_n \otimes I_k$, $T_i = [I_k, E_i]$, $\zeta_i = [(\zeta_i^1)^T, (\zeta_i^2)^T]^T$, $\zeta_i^1 \in \mathbb{R}^k$, $\zeta_i^2 \in \mathbb{R}^{-k}$, $\zeta^1 = [(\zeta_1^1)^T, \cdots, (\zeta_n^1)^T]$, $k = \times_{j=1}^n k_j$, $k_{-i} = \times_{j\neq i} k_j$.

Moreover if the solution ζ_i, $\forall i \in N$ exists, the local information based utility function of player i is

$$c_i(s) = (\zeta_i^1)^T \ltimes_{j=1}^n s_j, \forall i \in N. \tag{12}$$

Proof. (Necessary:) According to equation (6) and equation (7), we have

$$\begin{cases} \sum_{i=1}^n V_i = 0, \\ V_P = V_i - V_i^d E_i^T, i \in N, \end{cases} \tag{13}$$

which are equivalent to the following form

$$\begin{cases} T V_G^T = 0, \\ T_i [V_i^T, (V_i^d)^T]^T = (V_P)^T, i \in N, \end{cases} \tag{14}$$

where $T = \mathbf{1}_n \otimes I_k$, $T_i = [I_k, E_i]$. Equation (10) implies equation (11).

(Sufficiency:) If the equation (11) has a solution, then conditions in (13) are satisfied. Furthermore, the utility vector of player i is

$$V_i = (\zeta_i^1)^T, \ \forall i \in N.$$

□

Example 11. Consider the subspace of two-player zero-sum potential games $\mathcal{Z}_{[2;k_1,k_2]}$. Matrix E has the following form

$$\begin{aligned} Z &= \begin{bmatrix} E_1 & -E_2 \\ -E_1 & E_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1}_{k_1} \otimes I_{k_2} & -I_{k_1} \otimes \mathbf{1}_{k_2} \\ -\mathbf{1}_{k_1} \otimes I_{k_2} & I_{k_1} \otimes \mathbf{1}_{k_2} \end{bmatrix} \end{aligned}$$

Observing the form of matrix Z, it is easy to find that the utility function of player 1 for any two-player zero-sum potential game satisfies

$$c_1(s_1, s_2) = f(s_1) - g(s_2), \tag{15}$$

for some function $f : S_1 \to \mathbb{R}$, and $g : S_2 \to \mathbb{R}$.

Conversely, if the utility functions of a two-player game G have the following form

$$\begin{cases} c_1(s_1, s_2) = f(s_1) - g(s_2), \\ c_2(s_1, s_2) = g(s_2) - f(s_1). \end{cases} \tag{16}$$

for some function $f : S_1 \to \mathbb{R}$ and $g : S_2 \to \mathbb{R}$. Then $G \in \mathcal{E}_{[2;k_1,k_2]}$ is a zero-sum potential game with potential function

$$P(s_1, s_2) = f(s_1) + g(s_2).$$

The proof is very easy. For any $s_1, \hat{s}_1 \in S_1$ and any $s_2, \hat{s}_2 \in S_2$,

$$\begin{aligned} & c_1(s_1, s_2) - c_1(\hat{s}_1, s_2) \\ &= [f(s_1) - g(s_2)] - [f(\hat{s}_1) - g(s_2)] \\ &= [f(s_1) + g(s_2)] - [f(\hat{s}_1) + g(s_2)] \\ &= P(s_1, s_2) - P(\hat{s}_1, s_2), \end{aligned}$$

$$\begin{aligned} & c_2(s_1, s_2) - c_2(s_1, \hat{s}_2) \\ &= [g(s_2) - f(s_1)] - [g(\hat{s}_2) - f(s_1)] \\ &= [g(s_2) + f(s_1)] - [g(\hat{s}_2) + f(s_1)] \\ &= P(s_1, s_2) - P(s_1, \hat{s}_2). \end{aligned}$$

Conditions (16) are called separation property of two-player game in Branzei (2003).

Now we consider the design of two-player zero-sum potential games when the potential function $P(s_1, s_2)$ is given. Using separation property of two-player game, the utility functions of player 1 can be designed as follows

$$\begin{aligned} c_1(s_1, s_2) &= f(s_1) - g(s_2) \\ &= f(s_1) - [P(s_1, s_2) - f(s_1)] \\ &= 2f(s_1) - P(s_1, s_2), \end{aligned}$$

where $f : S_1 \to \mathbb{R}$ is an arbitrary function. The utility functions of player 2 can be designed as

$$c_2(s_1, s_2) = -c_1(s_1, s_2).$$

5. ZERO-SUM POTENTIAL NETWORKED EVOLUTIONARY GAMES

Assume a non-cooperative finite game is repeated infinitely. Then each player can update his strategy by using the game historical knowledge, which is called strategy updating rule (SUR). The repeated game along with strategy updating rule is called evolutionary game.

If the evolutionary game is played on network, then it is networked evolutionary game (NEG). Networked evolutionary game exists extensively in the real world. In the networked evolutionary game, there is a network graph which describes the game relationship. Player set is the node set of the network graph. If player i and player j play a game G, then (i, j) is an edge of the network graph. The game G is called fundamental network game (FNG) of the NEG. The definition of NEG is as follows.

Definition 12. (Cheng (2015)) A networked evolutionary game, denoted by $\mathcal{G} = ((N, E), G, \Pi)$, consists of three parts:

(i) a network graph (N, E), where N is the node set (player set) of NEG, E is the edge set;
(ii) a fundamental network game G. Players i and j play game G if $(i, j) \in E$.
(iii) a local information based strategy updating rule $\Pi = \{\Pi_i, i \in N\}$, where Π_i is the SUR for player i.

Denote by $N_i = \{j \in N | (i,j) \in E\}$ the neighbours of player i. The utility of player i in the NEG is

$$c_i(s) = \sum_{j \in N_i} c_{i,j}(s_i, s_j),$$

where $c_{i,j}(s_i, s_j)$ is the payoff of player i when he plays the FNG with player j using strategy s_i to compete with strategy s_j.

We are interested in the properties of NEG when its FNG is a zero-sum potential game.

Theorem 13. Consider a networked evolutionary game $\mathcal{G} = ((N,E), G, \Pi)$. If the fundamental network game G is zero-sum potential, then the networked evolutionary game $\mathcal{G}$ is also a zero-sum potential game. Moreover, if the potential function of the game between player i and j is $P_{(i,j)}$, then the overall network potential is:

$$P_{\mathcal{G}} = \sum_{(i,j) \in E} P_{(i,j)}. \tag{17}$$

Proof. Consider the networked evolutionary game $\mathcal{G} = ((N,E), G, \Pi)$. First, we prove that if the fundamental network game G is zero-sum game, then the NEG is also a zero-sum game.

$$\begin{aligned} &\sum_{i=1}^{n} c_i(s_1, \cdots, s_n) \\ &= \sum_{i=1}^{n} \sum_{j \in N_i} c_{i,j}(s_i, s_j) \\ &= \sum_{(i,j) \in E} [c_{i,j}(s_i, s_j) + c_{j,i}(s_i, s_j)] \\ &= 0. \end{aligned} \tag{18}$$

The last equation comes from the fact that each fundamental network game is a zero-sum game.

Cheng (2014) proved that if the fundamental network game is potential game, then the NEG is also a potential game. Moreover, if the potential function of the game between player i and j is $P_{(i,j)}$, then the overall network potential is:

$$P_{\mathcal{G}} = \sum_{(i,j) \in E} P_{(i,j)}. \tag{19}$$

Combining (18) and (19), we can obtain Theorem 13.

□

To explain the results of Theorem 13, we present an example.

Example 14. Consider a networked evolutionary game $\mathcal{G} = ((N,E), G, \Pi)$, where (i) the network graph is shown in Fig. 2; (ii) G is a two-player two-strategy symmetric game. whose payoff bi-matrix is shown in Table 1; (iii) the SUR Π in this problem is ignored.

$s_1 \backslash s_2$	1	2
1	0, 0	3, −3
2	−3, 3	0, 0

Table 1: Payoff bi-matrix

We use two ways to prove that $\mathcal{G}$ is a zero-sum potential game.

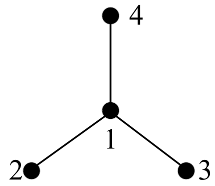

Fig. 2: Network graph of Example 14

(1) Global Verification (Consider the NEG as an integrated game and use Theorem 6 to verify.)
First, it is easy to calculate the payoff structure vector of the overall game, which is shown as follows

$$\begin{aligned} V_G = [&0,3,3,6,3,6,6,9,-9,-6,-6,-3,-6,-3,-3, \\ &0,0,0,0,0,-3,-3,-3,-3,3,3,3,3,0,0,0,0, \\ &0,0,-3,-3,0,0,-3,-3,3,3,0,0,3,3,0,0,0, \\ &-3,0,-3,0,-3,0,-3,3,0,3,0,3,0,3,0] \in \mathbb{R}^{64}. \end{aligned}$$

According to Theorem 6, equation (4) has a solution ξ

$$\begin{aligned} \xi = [&-3,3,3,9,3,9,9,15,-3,0,0,3,9,12,12,15,-3, \\ &0,0,3,9,12,12,15,-3,0,0,3,9,12,12,15] \in \mathbb{R}^{32}. \end{aligned}$$

Then the structure vector V_P of potential function $P(s)$ is

$$\begin{aligned} V_P &= V_1 - \xi_1 E_1^T \\ &= [3,0,0,-3,0,-3,-3,-6,-6,-9, \\ &\quad -9,-12,-9,-12,-12,-15]. \end{aligned}$$

where ξ_1 is the sub-vector of the first 8 elements of ξ. Therefore, $\mathcal{G}$ is a zero-sum potential game.

(2) Edge by Edge Verification (Verification by Theorem 13)
According to the payoff bi-matrix in Table 1, we know that the FNG G is a symmetric zero-sum potential game (Duersch (2012)). The potential function of G is

$$V_P^G = [4, 1, 1-2].$$

Therefore, the NEG $\mathcal{G}$ is a zero-sum potential game.
Using (17), we can obtain the global potential function $\tilde{P}(s)$ of $\mathcal{G}$, whose potential function vector is

$$V_{\tilde{P}} = [12,9,9,6,9,6,6,3,3,0,0,-3,0,-3,-3,-6].$$

Comparing with the global verification, we have

$$\tilde{P}(s) = P(s) + 9, \ \forall s \in S.$$

Therefore, the global verification is consistent with edge by edge verification.

6. CONCLUSION

The concept of zero-sum potential games is proposed to combine the advantages of zero-sum games and potential games. Verification and design of zero-sum potential games are studied, which can be realized by solving corresponding linear equations. We proved that if the fundamental network game is a zero-sum potential game, then the networked evolutionary game is also a zero-sum potential game.

Open and interesting questions for further investigations include: (i) design learning rules for zero-sum potential games which can converge to its Nash equilibrium; (ii) applications of zero-sum potential games on engineering related problems.

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