Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Evolution of opinion dynamics with eccentric agents ⋆

Qi Zhang * **Lin Wang** * **Xiaofan Wang** *,**

* *Department of Automation, Shanghai Jiao Tong University, and Key Laboratory of System Control and Information Processing, Ministry of Education of China, Shanghai 200240, China. (e-mail: douuuu,wanglin,xfwang@sjtu.edu.cn).*
** *Shanghai Key Laboratory of Power Station Automation Technology, School of Mechatronic Engineering and Automation, Shanghai University, Shanghai, China.*

Abstract:
More recently, there has been a surge of studies that seek mechanisms of the opinion evolution. While many studies have been dedicated to this field, much less attention has been paid to the joint influence of diverse agents on the opinion evolution. In this paper, we proposed an opinion dynamic model based on the Deffuant Weisbuch(DW) model with the existence of eccentric agents. The eccentric agent will change its opinion if the eccentric agent is selected and the opinion difference between two selected agents is beyond the bounded confidence. Previous studies have demonstrated that consensus usually cannot be achieved in the DW model. However, our study suggests that the existence of a single eccentric agent is able to promote consensus in numerical simulations, regardless of any bounded confidence and initial opinion distribution. We further proved that the DW model with the single eccentric agent achieves quasi-consensus. The equilibrium of the system was also proposed. Lastly, we analyzed the final opinion distribution and convergence time with varying bounded confidence and convergence parameters.

Keywords: Social network, Opinion dynamics, Bounded confidence, the Deffuant Weisbuch model, Quasi-Consensus

1. INTRODUCTION

Recently, the dynamics of social networks and mechanisms of the opinion evolution have been widely studied in various fields, such as statistical physics(Castellano et al. (2009)), social psychology(Dandekar et al. (2013)), and control theory(Tian and Wang (2018)). The reasons why individuals change their opinions are complicated. Therefore, a rich, extensive body of research establishes models based on social psychology and empirical studies to illustrate the mechanism of opinion formation.

The DeGroot model (DeGroot (1974)) is an early study of the opinion dynamic model, which investigates how the system achieves consensus based on the repeated average algorithm. Many extended models have been proposed, and many impressive results have been drawn. (Friedkin and Johnsen (1999)) considers the influence of initial bias on the process of opinion formation. On this basis, (Friedkin et al. (2016)) further considers the impact of statements under logical constraints on the evolution of opinions and explains the changes in Americans' opinions towards the Iraq war. (Jia et al. (2015)) studies the influence of the evolution of social power on the opinion formation process over issue sequence. (Dandekar et al. (2013)) proposes a biased opinion formation model on homophily networks, which results in polarization if agents are sufficiently biased.

In real lifetime, individuals are more willing to communicate with individuals with similar opinions because of misunderstanding, social conflicts, and some other reasons. Based on this, the bounded confidence(BC) model is proposed to describe this common phenomenon. In this type of model, every agent(denoted as i) has bounded confidence d_i, and its opinions can only be influenced by agents whose opinion values are in i's confidence area. That is to say, the communication between agents is state-dependent in the bounded confidence model. The Hegselmann Krause(HK) model considers the bounded confidence model based on the DeGroot model(Hegselmann (2002)). As long as the agents' bounded confidence is determined, the final opinion of each agent is also known. While in many cases, individuals cannot obtain information from multiple neighbors at the same time. Hence, the DW model (Deffuant et al. (2000)), on which our proposed analysis is based, investigates the effects of random pairwise communications between agents. In this model, if the opinion distance between the two interacting agents is smaller than the bounded confidence, then their opinions

⋆ This work was supported by the National Natural Science Foundation of China under GrantNos 61773255, 61873167, the Natural Science Foundation of Shanghai (No. 17ZR1445200), and the Science Fund for Creative Research Groups of the National Natural Science Foundation of China (No. 61521063).

will become closer; otherwise, nothing happens. Although it has explicit mathematical formulation, it does not have explicit mathematical opinion evolution result due to the randomness selection. The evolution of the opinion has two outcomes: consensus and fragmentation. If the confidence value is sufficiently large, the agents' communication range is wide, and the global consensus will be achieved. Instead, the small confidence value will lead to a coexistence of several incommunicable clusters.

The existing literature on the DW model is extensive, ranging from the theoretical analysis of the original model to the complex simulations combining real phenomena. Different from the above models, the DW model is hard to analyze theoretically, since many useful mathematical tools cannot be used, such as Matrix Theory and Markov chains. (Fortunato (2004)) shows that there exists a threshold d_c. When the bounded confidence is beyond the threshold, the system will achieve consensus. (Fortunato (2004),Lanchier (2012)) further proves that the critical value for consensus in the DW model on $\mathbb{Z}$ is $\frac{1}{2}$. (Nguyen et al. (2019)) investigates the equilibrium set of the DW model when the convergence parameter sets to $\frac{1}{2}$ and applies this gossip algorithm to data clustering. (Lin et al. (2013)) studies the convergence properties of the DW model with the convergence parameter, which is a decreasing function of the distance between the individuals' opinions, using the probability method. (Li et al. (2018)) proposes a model in which agents' opinion depends on all its neighbors' in activity-driven networks. If the difference between an agent and its neighbors' average opinion is greater than bounded confidence, the agent will change its opinion. (Vicario et al. (2017)) extends the DW model with confirmation bias to study online social debates, explaining the coexistence of two stable final opinions, which is often observed in reality by utilizing the mean-field theory. (Kozma and Barrat (2008)) investigates the coevolution of agents' opinions and adaptive networks based on the DW model.

While many studies have been dedicated to the mechanism of the opinion evolution, much less attention has been paid to the joint influence of different type(identities) of agents on the opinion evolution. Therefore, in this paper, we extend the DW model with eccentric agents(the DWEA model). Unlike the limited communication range of normal agents in the default DW model(proposed in Deffuant et al. (2000)), eccentric agents are more willing to communicate with agents who have different opinions. Precisely, when the eccentric agent and one of the other agents are selected to meet, the eccentric agents will change its opinion if and only if the opinion distance between these two selected agents is greater than the bounded confidence. Previous studies have demonstrated that consensus usually is hard to achieve in the default DW model(Weisbuch et al. (2002)). Intuitively, eccentric agents may make the evolution of group opinion more chaotic. However, our study suggests that the existence of the eccentric agent is able to promote consensus in numerical simulations, regardless of any bounded confidence and initial opinion distribution. The rules for updating opinions of eccentric agents are to seek common ground while shelving differences and play a key role in the process of achieving group consensus. We further prove that the DW model with a single eccentric agent achieves quasi-consensus. The equilibrium set of the DWEA model is also proposed. Lastly, the final opinion distribution and convergence time with varying bounded confidence and convergence parameters are analyzed through numerical simulations.

The remaining part of the paper proceeds as follows. Preliminaries and a brief description of the DW model is presented in Section 2. Then, Section 3 presents the DW model with eccentric agents, and the theoretical results of the DWEA model is also proposed. In Section 4, the final opinion distribution and convergence time with different parameters are analyzed. We conclude with Section 5.

2. PRELIMINARIES AND NOTATIONS

In the following section, we will illustrate some preliminaries, notations, and a brief description of the DW model.

2.1 Preliminaries and notations

Let's consider a group of agents, and their social network is represented by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. $\mathcal{V}$ stands for the set of agents in this network and it consists of agents with two identities: eccentric agents and normal agents, represented by $\mathcal{V}_e$ and $\mathcal{V}_n$ respectively. $\mathcal{E}$ is the set of edges. In this paper, assume that the graph is fully connected with none self-loop, i.e. $\mathcal{E} = \{(i,j) : i,j \in \mathcal{V}, i \neq j\}$. $I_i(t)$ represents the set of agents whose opinion is in the agent i's communication range at time t. The agents' opinions can be attributed to support or opposite to a statement. Thus, in the mathematic model, the agents' opinion $x(t)$ is mapped into the interval $[0,1]$ indicating the degree of support for the statement. The symbol $\mathbf{1}_n \in \mathbb{R}^n$ denotes a vector in which all the elements are 1. The set $\mathbb{Z}^+$ stands for all positive integers. The cardinality of a set A is denoted by $|A|$.

2.2 The introduction of the DW model

The DW model is firstly proposed in (Deffuant et al. (2000)). Here is a brief description of this model. Consider a group of N normal agents with continuous opinion $x_i \in [0,1]$. At each time step $t \in \mathbb{Z}^+$, a pair of agents is selected randomly. They change their opinions if and only if the opinion difference between them is smaller than the bounded confidence d. The set $I_i(t)$ describes a set of normal agents whose opinions are in agent i's confidence area at time t, defined as follow:

$$I_i(t) = \{j \in \mathcal{V} : |x_i(t) - x_j(t)| < d\}, i \in \mathcal{V} \tag{1}$$

Suppose that a pair of agents (i,j) is selected at time t. If $|x_i(t) - x_j(t)| < d$, i.e. $j \in I_i(t)$ and $i \in I_j(t)$, the opinion will be changed as follow:

$$\begin{cases} x_i(t+1) = x_i(t) + \mu(x_j(t) - x_i(t)), \\ x_j(t+1) = x_j(t) + \mu(x_i(t) - x_j(t)). \end{cases} \tag{2}$$

where $\mu \in (0, 0.5]$ is the convergence parameter. Based on the fact that individuals always trust themselves more than others, the convergence parameter is set to be $\mu \in (0, 0.5]$. If $|x_i(t) - x_j(t)| \geq d$,

$$x_i(t+1) = x_i(t), x_j(t+1) = x_j(t).$$

And the other unselected agents' opinion remains unchanged, i.e.

$$x_l(t+1) = x_l(t), l \neq i, j.$$

Lack of understanding, social pressure, and some other reasons may lead agents to be unwilling to communicate with the agents when the distance between their opinion is too large. The bounded confidence d is usually used to describe the openness of the agents. The bounded confidence of the agents is one of the crucial factors which influence the final opinion distribution. We simulate a group of $N = 100$ agents' opinion evolution under the two different bounded confidence $d = 0.2, 0.5$ and $\mu = 0.4$ which is shown in Fig.1(a), (b), respectively. From the simulation, we find that opinions gather together into three clusters when $d = 0.2$, which is plotted in three colors. In Fig.1(b), the system achieves global consensus eventually when $d = 0.5$. In order to show the evolution of opinions intuitively, we use red(yellow) line representing the agents whose initial opinion $x(0) \geq 0(x(0) < 0.5)$. Compared with Fig.1(a), the wider confidence area makes the agents' interaction more thoroughly, leading to a global consensus.

3. THE DW MODEL WITH ECCENTRIC AGENTS

In this section, the DW model with eccentric agents (the DWEA model) will be presented. Based on this, we will show some theoretical results of the DWEA model.

3.1 Introduction of the DWEA model

Before introducing the DWEA model, let's give the definition of the eccentric agents first.

Definition 1. An agent $i \in \mathcal{V}$ is called eccentric if it only communicates with the agents when the opinion difference between them is beyond the bounded confidence d. For a specific description, the eccentric agent i's opinion is only influenced by the agents in the set

$$I_i(t) = \{l \in \mathcal{V} : |x_i(t) - x_l(t)| \geq d\}, i \in \mathcal{V}_e.$$

Consider a group of agents $\mathcal{V} = \mathcal{V}_n \cup \mathcal{V}_e$. $\mathcal{V}_n = \{1, 2, \cdots, N\}$ denotes the normal agents who communicate with the agents when opinion distance between them is within the bounded confidence d. It follows that

$$I_i(t) = \{l \in \mathcal{V} : |x_i(t) - x_l(t)| < d\}, i \in \mathcal{V}_n, \forall t \in \mathbb{Z}^+. \quad (3)$$

In this paper, assume that $\mathcal{V}_e = \{0\}$ represents the eccentric agent. So, we have

$$I_0(t) = \{l \in \mathcal{V} : |x_l(t) - x_0(t)| \geq d\}, \forall t \in \mathbb{Z}^+. \quad (4)$$

At each time step t, two agents are selected randomly, denoted by (i, j). The selected agents change its opinion if and only if the other selected agent belongs to the set $I(t)$. That is to say

$$x_i(t+1) = \begin{cases} x_i(t), & j \notin I_i(t) \\ x_i(t) + \mu(x_j(t) - x_i(t)), & j \in I_i(t) \end{cases} \quad (5)$$

Similarly, the agent j updates as follow:

$$x_j(t+1) = \begin{cases} x_j(t), & i \notin I_j(t) \\ x_j(t) + \mu(x_i(t) - x_j(t)), & i \in I_j(t) \end{cases} \quad (6)$$

And the other unselected agents' opinion value remains unchanged, i.e.

$$x_l(t+1) = x_l(t), l \neq i, j, l \in \mathcal{V}$$

Hence, there are four cases available for the opinion update at each time step, which is shown in Fig 2.

Remark 1. In this paper, we consider a fully connected network, which implies that any pair of agents have the access to meet. At each time step, the probability of choosing each agents is uniform. So, the pair of agents (i, j) are chosen at time t with probability $P_{ij} = \frac{2}{N(N+1)}$. It needs to be emphasized that the agents' selection is independent of agents' opinions.

Before turning to the precise theoretical analysis of the DWEA model, we present the different behaviors observed from the situation, based on whether the eccentric agent exists or not in Fig.1. For the DW model, which is shown in Fig.1(a), since the bounded confidence is not big enough to make the agents communicate thoroughly, the local convergence takes place, which leads to three opinion clusters. In Fig.1(c), the eccentric agent does not make any big difference compared with the default DW model in the beginning. The group opinion separates into three quasi-clusters, as plotted in blue, yellow and red. Then, the eccentric agent plays a communicator between clusters with very different opinions, and it seeks common ground while shelving differences, breaking the limit of homogeneity of received information. The group opinions achieve consensus under a single eccentric agent, and this opinion formation process costs much time. The eccentric agent makes the opinion evolution more complicated in a certain way.

3.2 Monotonous properties

In this part, we will illustrate the monotonous properties of the model. Before this, we need to prove that the interaction between the agents makes opinions closer.

Lemma 1. Suppose that a pair of agents, denoted by (i, j), are selected at time t. Without loss of generality, assume that $x_i(t) \leq x_j(t)$. Then

$$x_i(t) \leq x_i(t+1) \leq x_j(t+1) \leq x_j(t). \quad (7)$$

Proof. After the interaction between the selected agents, there are two kinds of opinion statement available, based on whether the agent is eccentric agent or not. Recall that the convergence parameter $\mu \in (0, 0.5]$, according to the assumption $x_i(t) \leq x_j(t)$, we have

Case 1. If the agent i's opinion changes, then

$$x_i(t+1) = x_i(t) + \mu(x_j(t) - x_i(t)) > x_i(t).$$

Case 2. If the agent i's opinion remains unchanged, then

$$x_i(t+1) = x_i(t).$$

To a conclusion, $x_i(t+1) \leq x_i(t) + \mu(x_j(t) - x_i(t))$. Similarly, we have $x_j(t+1) \geq x_j(t) + \mu(x_i(t) - x_j(t))$. So, we have

$$x_i(t+1) - x_j(t+1) \leq x_i(t) - x_j(t) \leq 0.$$

Based on this, we get

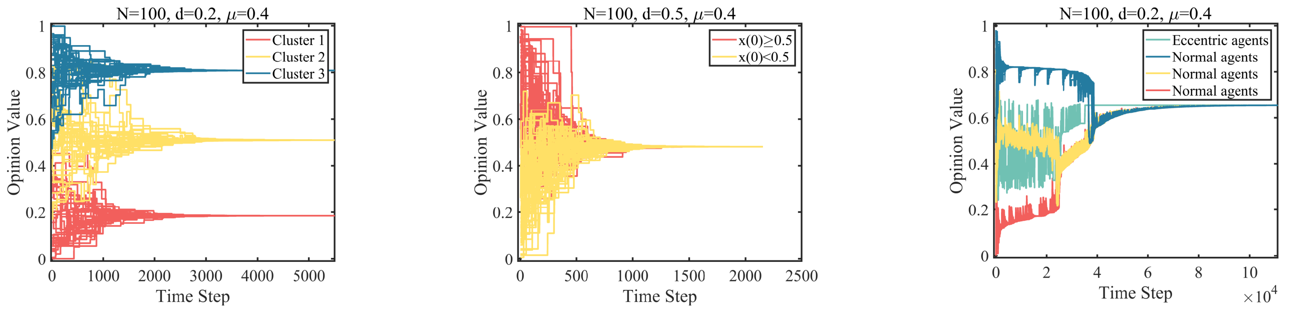
$$x_i(t) \leq x_i(t+1) \leq x_j(t+1) \leq x_j(t).$$

This completes the proof.

Define

$$f(t) = \max_{i \in \mathcal{V}} x_i(t)$$

$$g(t) = \min_{i \in \mathcal{V}} x_j(t)$$



(a) Simulation of the DW model (b) Simulation of the DW model (c) Simulation of the DWEA model

Fig. 1. The trajectories of opinion evolution under the DW model and DWEA model. (a)Agents' opinions separate into three clusters. (b),(c) The group opinions achieve consensus.

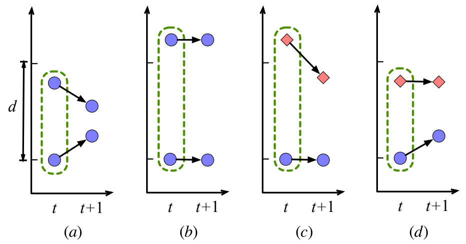


Fig. 2. The opinion update rules. Blue circles represent the normal agents and red rhombuses stands for the eccentric agents. Two agents circled with a dotted line indicate that they are selected to meet. Arrows represents the change of the opinion.

Lemma 2. Consider the functions $f(t)$ and $g(t)$. For all $t \geq 0$,

1. $f(t)$ is monotonous non-increasing, i.e. $f(t+1) \leq f(t)$,
2. $g(t)$ is monotonous non-decreasing, i.e. $g(t+1) \geq g(t)$.

Proof. By contradiction.
Assume that there exists a time step t_1, s.t.

$$f(t_1+1) > f(t_1). \tag{8}$$

Suppose that a pair of agents are randomly selected at time t_1, denoted by (i,j). Without loss of generality, assume that $x_i(t) \leq x_j(t)$, from the Lemma 1, we have

$$x_i(t_1) \leq x_i(t_1+1) \leq x_j(t_1+1) \leq x_j(t_1).$$

Recall that

$$f(t_1) = \max_{l \in \mathcal{V}} x_l(t_1) \geq x_j(t_1). \tag{9}$$

Based on the assumption $f(t_1+1) - f(t_1) > 0$, i.e. the largest opinion value in the group has changed at time t_1+1. At time t_1, only a pair of agents meet and may change their opinion, while the other agents remain their previous opinion. According to the Lemma (9), it implies that

$$f(t_1+1) = x_j(t_1+1) \leq x_j(t_1) \leq f(t_1),$$

which contradicts assumption (8). This follows that

$$f(t+1) \leq f(t), \forall t \in \mathbb{Z}^+.$$

By the similar argument, $g(t+1) \geq g(t), \forall t \in \mathbb{R}^+$ can also be proved. This completes the proof.

3.3 Quasi-Consensus of the system

In this part, we consider the quasi-consensus of the DWEA model with a single eccentric agent. The definition of the quasi-consensus is presented first.

Definition 2. The dynamic system is said to achieve quasi-consensus with respect to a bound $\gamma > 0$, if

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| \leq \gamma, \forall i, j \in \mathcal{V}.$$

Theorem 1. Consider the DWEA model with a single eccentric agent, for any given initial value $X(0) \in [0,1]^n$, fixed bounded confidence d, and convergence parameter $\mu \in (0, 0.5]$, the system achieves quasi-consensus.

Proof. From the definition of the quasi-consensus, we need to prove that for any pair of agents (i,j),

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| \leq \gamma.$$

It's clear that

$$|x_i(t) - x_j(t)| \leq f(t) - g(t).$$

So. the theroem will be proved if there exists $\gamma \geq 0$, such that

$$\lim_{t\to\infty} f(t) - g(t) \leq \gamma.$$

First of all, we need to prove convergence of the functions $f(t)$ and $g(t)$.

According to the Lemma 2, we know that $f(t)$ is monotonically non-increasing with lower bound 0, and $g(t)$ is monotonically non-decreasing with upper bound 1. By the monotone convergence theorem, $f(t)$ and $g(t)$ achieve convergence, i.e.

$$\lim_{t\to\infty} f(t) = f^*, \tag{10}$$

$$\lim_{t\to\infty} g(t) = g^*. \tag{11}$$

Based on the algorithm of limit, we know that

$$\lim_{t\to\infty} (f(t) - g(t)) = \lim_{t\to\infty} f(t) - \lim_{t\to\infty} g(t) = f^* - g^*.$$

Therefor, let $\gamma = f^* - g^*$, it follows that

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| \leq \lim_{t\to\infty} (f(t) - g(t)) = \gamma.$$

which completes the proof.

3.4 The equilibrium point of the system

Definition 3. (Nguyen et al. (2019)) A point $X^* \in \mathbb{R}^n$ is an equilibrium point of the system if for any pair of agents $(i, j) \in \mathcal{E}$ are selected, it remains unchanged after the interaction.

Theorem 2. $X^* = x^* \mathbf{1}_{N+1}$ is an equilibrium point of the DWEA model.

Proof. Suppose that $X(t) = x^* \mathbf{1}_{N+1}$ and the agent i and j are selected at time t. We will discuss agent i's opinion value $x_i(t+1)$, and $x_j(t+1)$ can be obtained in a similar way.

Case 1. The agent i is an eccentric agent, i.e. $i \in \mathcal{V}_e$. Because of the fact $x_i(t) = x_j(t)$, $j \neq I_i(t)$, then the agent i's opinion remains unchanged at time $t+1$,i.e. $x_i(t+1) = x_i(t) = x^*$.

Case 2. The agent i is a normal agent, i.e. $i \in \mathcal{V}_n$. Due to $j \in I_i(t)$, we have $x_i(t+1) = x_i(t) + \mu(x_j(t) - x_i(t)) = x^*$.

In conclusion, $x_i(t+1) = x^*$. Similarly, we have $x_j(t+1) = x^*$. As for the agents who are not selected at time t, according to the opinion updating rule, we know that $x_l(t+1) = x_l(t), l \neq i, j$.

To sum up, $X(t+1) = X(t)$. So, $X^* = x_* \mathbf{1}_n$ is an equilibrium point. This completes the proof.

Theorem 3. The euqilibrium points $X^* \in \mathbb{R}^n$ of the DWEA model with an eccentric agent have the form:

$$x_i^* = x_j^*, \forall i, j \in \mathcal{V}. \tag{12}$$

Proof. Let $X^* \in \mathbb{R}^n$ denote the equilibrium point of the system. We will prove this lemma by contradiction. Assume that there exists $i, j \in \mathcal{V}$ such that $x_i^* \neq x_j^*$. This proof will be claimed precisely based on the following four cases:

Case 1. $i, j \in \mathcal{V}_n$ and $0 < |x_i^* - x_j^*| < d$. Let agents i and j are selected, then $x_i^* + \mu(x_j^* - x_i^*) \neq x_i^*$, which contradicts the definition of equilibrium.

Case 2. $i, j \in \mathcal{V}_n$ and $|x_i^* - x_j^*| \geq d$. Denote $J_i^- = \{l \in \mathcal{V}\backslash\{i\} : 0 < |x_l^* - x_i^*| \leq d\}$, $J_i^+ = \{l \in \mathcal{V}\backslash\{i\} : |x_l^* - x_i^*| \geq d\}$ and J_j^-, J_j^+ similarly. Then, $(J_i^- \cup (J_i^+ \cap \mathcal{V}_e)) \bigcup (J_j^- \cup (J_j^+ \cap \mathcal{V}_e)) \neq \varnothing$. Without loss of generality, suppose $(J_i^- \cup (J_i^+ \cap \mathcal{V}_e)) \neq \varnothing$. Then, let the agent i and one of agents $l \in J_i^- \cup (J_i^+ \cap \mathcal{V}_e)$ are selected. If $l \in J_i^-$, $x_i^* + \mu(x_l^* - x_i^*) \neq x_i^*$. If $l \in J_i^+ \cap \mathcal{V}_e$, then $i \in I_l$ such that $x_l^* + \mu(x_i^* - x_l^*) \neq x_l^*$. The above two situations both contradict to the definition of equilibrium.

Case 3. $i \in \mathcal{V}_e$, $j \in \mathcal{V}_n$ and $0 < |x_i^* - x_j^*| < d$. Let agents i and j are selected, then $i \in I_j$ such that $x_j^* + \mu(x_i^* - x_j^*) \neq x_j^*$, which contradicts the definition of equilibrium.

Case 4. $i \in \mathcal{V}_e$, $j \in \mathcal{V}_n$ and $|x_i^* - x_j^*| \geq d$. Let agents i and j are selected, then $j \in I_i$ such that $x_i^* + \mu(x_j^* - x_i^*) \neq x_i^*$, which contradicts the definition of equilibrium.

To a conclusion, the assumption is error and the equilibrium points of the DWEA model have the form (12), which completes the proof.

4. NUMERICAL RESULT

In this section, we will analyze the final opinion distribution compared with the DW model, and how the parameters affect the convergence time.

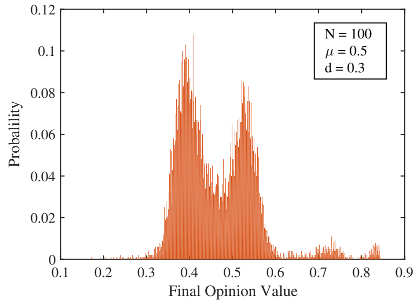


Fig. 3. The probability distribution of the final opinion value under the DW model in the presence of a single eccentric agent.

4.1 Final opinion distribution

In the DW model, suppose that the agent i and j are selected at time t, according to the opinion update rule (2), we know that for $t \geq 0$,

$$\sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} x_i(0).$$

Therefore, if the DW model reaches consensus, the final opinion will be the average of initial opinions, i.e.

$$\lim_{t \to \infty} x_i(t) = \frac{1}{|\mathcal{V}|} \sum_{l \in \mathcal{V}} x_l(0), i \in \mathcal{V}.$$

However, a different phenomenon is observed in the DWEA model. Suppose that the agent i and j are chosen at time t. From the opinion update rule (5), we know that

Caes 1. If the agent $i, j \in \mathcal{V}_n$, then $x_i(t+1) + x_j(t+1) = x_i(t) + x_j(t)$.

Case 2. If the eccentric agent is selected at time t, without loss of generality, assume that $i \in \mathcal{V}_e$. $x_i(t+1) + x_j(t+1) = x_i(t) + x_j(t)$ if and only if $x_i(t) = x_j(t)$, and $x_i(t+1) + x_j(t+1) \neq x_i(t) + x_j(t)$ otherwise.

To sum up, the average of group opinions is time-varying.

According to the simulation results, the system will achieve consensus with the presence of the single eccentric agent. Based on the Remark 1, the selection of the agents is uniform. So, the difference in the selection of the agents at each time step may result in different final opinion values. We simulate the opinion evolution under the DWEA model to investigate the final opinion value and the simulation result is presented in Fig.3. We consider a group of $N = 100$ agents, and let $\mu = 0.5$, $d = 0.3$ and the initial opinion is evenly distributed on the interval $[0, 1]$. Then, we count the final opinion under the fixed random selected initial opinion and the assumed parameter through 1000 times simulations.

4.2 Convergence time

Based on the simulation in Fig.1 at each step, the opinion evolution depends on multiple factors, including the initial opinion, convergence parameter μ, bounded confidence d, and the agents' selection. Therefore, we analyze the effect of convergence parameter μ and bounded confidence d on convergence time based on the same initial opinion and agents' selection rule, which is presented in Remark 1.

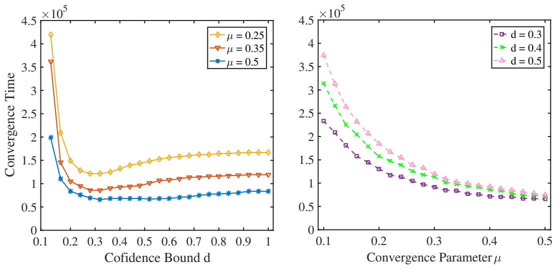


Fig. 4. The convergence time of the DWEA model under different parameters.

We consider a group of $N = 100$ agents, and the initial opinion is evenly distributed on the interval $[0, 1]$. The simulation results are presented in Fig.4 which is the average of over 200 realizations. In this simulation, we say the system achieves consensus if and only if $f(t) - g(t) \leq 0.0001$. In Fig.4, we find that the convergence time becomes smaller first and turns to larger afterwards as the the bounded confidence increases. Fig.4 also displays the same result that the system converges faster when $d = 0.3$ compared with the cases where $d = 0.4, 0.5$ under the same convergence parameter. Actually, according to the eccentric agent opinion update rule (4) and (5), if there exists $\tau > 0$, such that

$$f(\tau) - x_0(\tau) < d, x_0(\tau) - g(\tau) < d$$

then, we have

$$x_0(t) = x_0(t + 1) = \hat{x}, \forall t > \tau.$$

It means that if the bounded confidence is sufficient large, the eccentric agent may not change its opinion during the opinion evolution. And for $t > \tau$, it's clear that $\mathcal{V}_e \subset I_i(t), i \in \mathcal{V}_n$. It follows that, the normal agents will be close to the eccentric agent as time goes by as it's shown in Fig.1(c), which costs lots of time.

Based on the fact that individuals always trust themselves more than others, the convergence parameter is chosen to be $\mu \in (0, 0.5]$. Besides, (Weisbuch et al. (2002)) shows that the value of μ only influences the convergence time to equilibrium based on a large number of simulations. We find a similar result in the DWEA model that the bigger the convergence parameter μ, the faster the convergence time which is presented in Fig.4(b).

5. CONCLUSION

In this paper, we studied the DWEA model analytically and theoretically under a single eccentric agent who only communicates with the agents when the difference between them is beyond the bounded confidence d. In the previous studies, a larger bounded confidence could lead to a global consensus in the default DW model. Small bounded confidence results in the coexistence of several clusters in the final opinion distribution, and the distance between each cluster is beyond the bounded confidence, which makes agents from different clusters cannot communicate with each other. However our study suggests that the existence of the eccentric agent is able to promote to achieve consensus in numerical simulations, regardless of any bounded confidence and initial opinion distribution. Moreover, we provided a theoretical analysis of the system achieving the quasi-consensus, in which the opinion distance between any pair of agents is below a fixed parameter. Then, the equilibrium point of the system is proposed. Finally, we analyzed the final opinion distribution compared with the DW model. We study the convergence time with difference bounded confidence and convergence parameters utilizing numerical simulations.

REFERENCES

Castellano, C., Fortunato, S., and Loreto, V. (2009). Statistical physics of social dynamics. *Reviews of modern physics*, 81(2), 591.

Dandekar, P., Goel, A., and Lee, D.T. (2013). Biased assimilation, homophily, and the dynamics of polarization. *Proceedings of the National Academy of Sciences*, 110(15), 5791–5796.

Deffuant, G., Neau, D., Amblard, F., and Weisbuch, G. (2000). Mixing beliefs among interacting agents. *Advances in Complex Systems*, 03(01n04), 87–98.

DeGroot, M.H. (1974). Reaching a consensus. *Journal of the American Statistical Association*, 69(345), 118–121.

Fortunato, S. (2004). Universality of the threshold for complete consensus for the opinion dynamics of deffuant et al. *International Journal of Modern Physics C*, 15(09), 1301–1307.

Friedkin, N.E., Proskurnikov, A.V., Tempo, R., and Parsegov, S.E. (2016). Network science on belief system dynamics under logic constraints. *Science*, 354(6310), 321–326.

Friedkin, N.E. and Johnsen, E.C. (1999). Social influence networks and opinion change. *Adv Group Process*, 16(1).

Hegselmann, R. (2002). Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies & Social Simulation*, 5(3), 2.

Jia, P., MirTabatabaei, A., Friedkin, N.E., and Bullo, F. (2015). Opinion dynamics and the evolution of social power in influence networks. *SIAM review*, 57(3), 367–397.

Kozma, B. and Barrat, A. (2008). Consensus formation on adaptive networks. *Physical Review E*, 77(1), 016102.

Lanchier, N. (2012). The critical value of the deffuant model equals one half. *Latin American Journal of Probability & Mathematical Statistics*, 9(2), 383–402.

Li, D., Han, D., Ma, J., Sun, M., Tian, L., Khouw, T., and Stanley, H.E. (2018). Opinion dynamics in activity-driven networks. *EPL (Europhysics Letters)*, 120(2), 28002.

Lin, L., Scaglione, A., Swami, A., and Zhao, Q. (2013). Consensus, polarization and clustering of opinions in social networks. *IEEE Journal on Selected Areas in Communications*, 31(6), 1072–1083.

Nguyen, T.H.L., Wada, T., Masubuchi, I., Asai, T., and Fujisaki, Y. (2019). Bounded confidence gossip algorithms for opinion formation and data clustering. *IEEE Transactions on Automatic Control*, 64(3), 1150–1155.

Tian, Y. and Wang, L. (2018). Opinion dynamics in social networks with stubborn agents: An issue-based perspective. *Automatica*, 96, 213–223.

Vicario, M.D., Scala, A., Caldarelli, G., Stanley, H.E., and Quattrociocchi, W. (2017). Modeling confirmation bias and polarization. *Scientific Reports*, 7, 40391.

Weisbuch, G., Deffuant, G., Amblard, F., and Nadal, J.P. (2002). Meet, discuss, and segregate! *Complexity*, 7(3), 55–63.