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Finite-dimensional observer-based controller for linear 1-D heat equation: an LMI approach ⋆

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Abstract: The objective of the present paper is finite-dimensional observer-based control of 1-D linear heat equation with constructive and easily implementable design conditions. We propose a modal decomposition approach in the cases of bounded observation and control operators (i.e, non-local sensing and actuation). The dimension of the controller is equal to the number of modes which decay slower than a given decay rate $\delta > 0$. The observer may have a larger dimension N. The observer and controller gains are found separately of each other. We suggest a direct Lyapunov approach to the full-order closed-loop system and provide linear matrix inequalities (LMIs) for finding N and the exponential decay rate of the closed-loop system. Different from some existing qualitative methods, we prove that the LMIs are always feasible for large enough N leading to easily verifiable conditions. A numerical example demonstrates the efficiency of our method that gives non-conservative bounds on N and δ.

Keywords: Distributed parameter systems, heat equation, observer-based control, modal decomposition, Lyapunov method.

1. INTRODUCTION

Observer-based controllers for linear PDEs have been constructed by the modal decomposition approach Curtain (1982); Lasiecka and Triggiani (2000); Orlov et al. (2004); Katz et al. (2020), the backstepping method Krstic and Smyshlyaev (2008) and by the spatial decomposition (sampling) method Selivanov and Fridman (2018), where the observer is found in the form of a PDE. A PDE observer (which can be implemented via approximations Lasiecka and Triggiani (2000)) usually leads to separation of the controller and observer designs. Finite-dimensional observers and the resulting controllers, which are very attractive in applications, generally do not allow such separation. Therefore, design of the latter controllers is a very challenging control problem. Recently, a delayed finite-dimensional boundary observer was introduced for the 1-D heat eqaution in Selivanov and Fridman (2019). Sampled-data observers for the 1-D heat equation under non-local outputs have been considered in Karafyllis et al. (2019).

Finite-dimensional observer-based controllers for parabolic systems were designed by modal decomposition approach Curtain (1982); Balas (1988); Christofides (2001); Harkort and Deutscher (2011). In particular, for bounded control and observation operators, it was shown in Balas (1988) that the closed-loop system is stable provided the dimension of the controller is large enough. A singular perturbation approach that reduces the controller design to a finite-dimensional slow system was suggested in Christofides (2001), without giving constructive and rigorous conditions for finding the dimension of the slow system which guarantees a desired closed-loop performance of the full-order system. However, a constructive method for finding this dimension was not provided. In Harkort and Deutscher (2011), modal decomposition was combined with cascaded output observers to design a finite-dimensional observer-based controller in the case of bounded observation and control operators. Nevertheless, as mentioned therein, the obtained bound on the number of required output observers may be difficult to compute and is highly conservative.

The objective of the present paper is finite-dimensional observer-based control of 1-D heat equation with constructive and easily implementable design conditions. We apply the proposed method to the case of bounded observation and control operators (i.e, non-local measurement and actuation). However, our approach can be modified to deal with boundary sensing together with non-local actuation (See Remark 3.2). We use a modal decomposition approach. The dimension of the controller, N_0, is equal to the number of modes which decay slower than a given decay rate $\delta > 0$. The observer may have a larger dimension $N \geq N_0$. The observer and controller gains are found separately.

Inspired by Coron and Trélat (2004); Karafyllis et al. (2019); Prieur and Trélat (2018), we suggest a direct Lyapunov approach to the full-order closed-loop system and provide LMIs for finding N and the exponential decay

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rate of the closed-loop system. The main challenge in the proposed finite-dimensional observer-based control is to prove that the obtained LMIs are always feasible for large enough N (see e.g. proof of Theorem 2 below). As seen in the example, our LMIs are not conservative.

Notation. We denote by $L^2(0,1)$ the Hilbert space of Lebesgue measurable and square integrable functions $f: [0,1] \to \mathbb{R}$ with the corresponding inner product $\langle f,g\rangle := \int_0^1 f(x)g(x)dx$ and induced norm $\|f\|_{L^2}^2 := \langle f,f\rangle$. $H^1(0,1)$ is the Sobolev space of functions $f: [0,1] \to \mathbb{R}$ with a square integrable weak derivative. The norm defined on $H^1(0,1)$ is $\|f\|_{H^1}^2 := \|f\|_{L^2}^2 + \|f'\|_{L^2}^2$. The standard Euclidean norm on $\mathbb{R}^n$ will be denoted by $\|\cdot\|$. $H^2(0,1)$ is the Sobolev space of functions $f: [0,1] \to \mathbb{R}$ with a square integrable weak derivative of the second order. For $A \in \mathbb{R}^{n\times n}$, the operator norm of A, induced by $\|\cdot\|$, is denoted by $\|\cdot\|_2$. For $P \in \mathbb{R}^{n\times n}$, the notation $P > 0$ means that P is symmetric and positive definite. The sub-diagonal elements of a symmetric matrix will be denoted by $*$. For $U \in \mathbb{R}^{n\times n}$, $U > 0$ and $x \in \mathbb{R}^n$ we denote $\|x\|_U^2 := x^T U x$.

2. MATHEMATICAL PRELIMINARIES

Recall that the regular Sturm-Liouville eigenvalue problem

$$\begin{aligned} \phi'' + \lambda\phi = 0, \quad x \in [0,1], \\ \phi'(0) = \phi(1) = 0, \end{aligned} \tag{2.1}$$

induces a sequence of eigenvalues $\lambda_n = (n - \frac{1}{2})^2\pi^2, n \geq 1$ with corresponding eigenfunctions

$$\phi_n(x) = \sqrt{2}\cos\left(\sqrt{\lambda_n}x\right), n \geq 1.$$

Moreover, the eigenfunctions form a complete orthonormal system in $L^2(0,1)$ Boyce et al. (2017). We will require the following Lemma. We omit the proof due to space constraints.

Lemma 1. Let $h \in L^2(0,1)$ be a function such that

$$h(x) = \sum_{n=1}^{\infty} h_n\phi_n(x) = \sum_{n=1}^{\infty} h_n\sqrt{2}\cos(\sqrt{\lambda_n}x), \tag{2.2}$$

where the equality is in $L^2(0,1)$. Then, $h \in H^1(0,1)$ and satisfies $h(1) = 0$ iff $\sum_{n=1}^{\infty}\lambda_n h_n^2 < \infty$. Moreover,

$$\|h'\|_{L^2}^2 = \sum_{n=1}^{\infty}\lambda_n h_n^2. \tag{2.3}$$

3. OBSERVER-CONTROLLER DESIGN

3.1 Non-local measurement and actuation: L^2-stability

Consider the reaction-diffusion system

$$\begin{aligned} &z_t(x,t) = z_{xx}(x,t) + qz(x,t) + b(x)u(t), \\ &z_x(0,t) = 0, \quad z(1,t) = 0, \end{aligned} \tag{3.1}$$

where $t \geq 0$, $x \in [0,1]$, $z(x,t) \in \mathbb{R}$, $q \in \mathbb{R}$ is the reaction coefficient, $b \in L^2(0,1)$ and $u(t)$ is the control input. We consider non-local measurement

$$y(t) = \int_0^1 c(x)z(x,t)dx, \tag{3.2}$$

where $c \in L^2(0,1)$.

We begin by presenting the solution to (3.1) as

$$\begin{aligned} z(x,t) &= \sum_{n=1}^{\infty} z_n(t)\phi_n(x), \\ z_n(t) &= \langle z(\cdot,t),\phi_n\rangle = \int_0^1 z(x,t)\phi_n(x)dx. \end{aligned} \tag{3.3}$$

By differentiating under the integral sign, integrating by parts and using (2.1) we have

$$\begin{aligned} \dot{z}_n(t) &= \int_0^1 z_t(x,t)\phi_n(x)dx = \int_0^1 z_{xx}(x,t)\phi_n(x)dx \\ &+ \int_0^1 qz(x,t)\phi_n(x)dx + \int_0^1 b(x)u(t)\phi_n(x)dx \\ &= (-\lambda_n + q)z_n(t) + b_n u(t), \\ z_n(0) &= \langle z_0,\phi_n\rangle =: z_{0,n}, \quad b_n = \langle b,\phi_n\rangle. \end{aligned} \tag{3.4}$$

Let $\delta > 0$ be a desired decay rate. Since $\lim_{n\to\infty}\lambda_n = \infty$, there exists some $N_0 \in \mathbb{N}$ such that

$$-\lambda_n + q < -\delta, \quad n > N_0. \tag{3.5}$$

N_0 will define the dimension of the controller, whereas $N \geq N_0$ will be the dimension of the observer. We construct a finite-dimensional observer of the form

$$\hat{z}(x,t) := \sum_{n=1}^{N}\hat{z}_n(t)\phi_n(x), \tag{3.6}$$

where $\hat{z}_n(t)$ satisfy the ODEs

$$\begin{aligned} \dot{\hat{z}}_n(t) &= (-\lambda_n + q)\hat{z}_n(t) + b_n u(t) \\ &-l_n\left[\int_0^1 c(x)\left(\sum_{n=1}^{N}\hat{z}_n(t)\phi_n(x)\right)dx - y(t)\right], \\ \hat{z}_n(0) &= 0, \quad 1 \leq n \leq N. \end{aligned} \tag{3.7}$$

Here l_n are scalar observer gains. Denote

$$\begin{aligned} &A_0 = \mathrm{diag}\{-\lambda_1 + q, \ldots, -\lambda_{N_0} + q\}, \\ &L_0 = [l_1, \ldots, l_{N_0}]^T, \\ &C_0 = [c_1, \ldots, c_{N_0}], \\ &c_n = \langle c,\phi_n\rangle, \quad n \geq 1. \end{aligned} \tag{3.8}$$

Assume that

$$c_n \neq 0, \quad 1 \leq n \leq N_0. \tag{3.9}$$

Then, the pair (A_0, C_0) is observable by the Hautus lemma. We choose $l_1, \ldots, l_{N_0}$ such that L_0 satisfies the following Lyapunov inequality:

$$P_o(A_0 - L_0C_0) + (A_0 - L_0C_0)^T P_o < -2\delta P_o, \tag{3.10}$$

where $P_o \in \mathbb{R}^{N_0\times N_0}$ satisfies $P_o > 0$. Furthermore, we choose $l_n = 0, \ n > N_0$.

We assume

$$b_n \neq 0, \quad 1 \leq n \leq N, \tag{3.11}$$

where $b_n = \langle b,\phi_n\rangle$, and denote

$$B_0 := [b_1 \ \ldots \ b_{N_0}]^T. \tag{3.12}$$

By the Hautus lemma the pair (A_0, B_0) is controllable. Let $K_0 \in \mathbb{R}^{1\times N_0}$ satisfy

$$P_c(A_0 + B_0K_0) + (A_0 + B_0K_0)^T P_c < -2\delta P_c, \tag{3.13}$$

where $P_c \in \mathbb{R}^{N_0\times N_0}$ satisfies $P_c > 0$.

We propose a N_0-dimensional controller of the form

$$\begin{aligned} &u(t) = K_0\hat{z}^{N_0}(t) \\ &\hat{z}^{N_0}(t) = [\hat{z}_1(t), \ldots, \hat{z}_{N_0}(t)]^T, \end{aligned} \tag{3.14}$$

which is based on the N-dimensional observer (3.7).

Define $\mathcal{D}(\mathcal{A}) = \{w \in H^2(0,1) : w'(0) = w(1) = 0\}$. By applying Theorems 6.3.1 and 6.3.3 in Pazy (1983) (with $\alpha = 0$) it can be shown that system (3.1), (3.7) with $u(t) = K_0\hat{z}^{N_0}(t)$ and initial condition $z_0 = z(\cdot,0) \in L^2(0,1)$ has a unique solution

$$\xi \in C([0,\infty);\mathcal{H}), \quad \xi \in C^1((0,\infty);\mathcal{H}) \tag{3.15}$$

such that

$$\xi(t) \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}^N \quad \forall t > 0. \tag{3.16}$$

Let

$$e_n(t) = z_n(t) - \hat{z}_n(t), \ 1 \le n \le N \tag{3.17}$$

be the estimation error. By using (3.3) and (3.6), the last term on the right-hand side of (3.7) can be written as

$$\begin{aligned} &\int_0^1 c(x)\left[\sum_{n=1}^N \hat{z}_n(t)\phi_n(x) - \sum_{n=1}^\infty z_n(t)\phi_n(x)\right]dx \\ &= -\sum_{n=1}^N c_n e_n(t) - \zeta(t), \quad \zeta(t) = \sum_{n=N+1}^\infty c_n z_n(t). \end{aligned} \tag{3.18}$$

Then the error equation has the form

$$\begin{aligned} &\dot{e}_n(t) = (-\lambda_n + q)e_n(t) - l_n\left(\sum_{n=1}^N c_n e_n(t) + \zeta(t)\right), \\ &1 \le n \le N. \end{aligned} \tag{3.19}$$

Denote

$$\begin{aligned} &e^{N_0}(t) = [e_1(t),\ldots,e_{N_0}(t)]^T, \\ &e^{N-N_0}(t) = [e_{N_0+1}(t),\ldots,e_N(t)]^T, \\ &\hat{z}^{N-N_0}(t) = [\hat{z}_{N_0+1}(t),\ldots,\hat{z}_N(t)]^T, \\ &X(t) = \mathrm{col}\{\hat{z}^{N_0}(t), e^{N_0}(t), \hat{z}^{N-N_0}(t), e^{N-N_0}(t)\}, \\ &\mathcal{L} = \mathrm{col}\{L_0, -L_0, 0_{2(N-N_0)\times 1}\} \in \mathbb{R}^{2N}, \\ &\tilde{K} = [K_0, \ 0_{1\times(2N-N_0)}] \in \mathbb{R}^{1\times 2N}, \\ &F = \begin{bmatrix} A_0 + B_0K_0 & L_0C_0 & 0 & L_0C_1 \\ 0 & A_0 - L_0C_0 & 0 & -L_0C_1 \\ B_1K_0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}. \end{aligned} \tag{3.20}$$

From (3.4), (3.7), (3.14), (3.18) and (3.19) by using A_1, B_1, C_1 defined by

$$\begin{aligned} &A_1 = \mathrm{diag}\{-\lambda_{N_0+1} + q,\ldots,-\lambda_N + q\}, \\ &C_1 = [c_{N_0+1},\ldots,c_N], \\ &B_1 = [b_{N_0+1},\ldots,b_N]^T, \end{aligned} \tag{3.21}$$

we present the closed-loop system for $t \ge 0$ as follows:

$$\begin{aligned} &\dot{X}(t) = FX(t) + \mathcal{L}\zeta(t), \\ &\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n\tilde{K}X(t), \quad n > N. \end{aligned} \tag{3.22}$$

Note that the Cauchy-Schwarz inequality implies the following estimate

$$\zeta^2(t) \le \|c\|_{L^2}^2 \sum_{n=N+1}^\infty z_n^2(t). \tag{3.23}$$

For L^2-stability analysis of the closed-loop system (3.22), we define the Lyapunov function

$$V(t) = \|X(t)\|_P^2 + \sum_{n=N+1}^\infty z_n^2(t), \tag{3.24}$$

where $P \in \mathbb{R}^{2N\times 2N}$ satisfies $P > 0$. Since $z(\cdot,t) \in \mathcal{D}(\mathcal{A})$, the series $\sum_{n=N+1}^\infty z_n^2(t)$ can be differentiated term-by-term. Differentiation of $V(t)$ along (3.22) gives

$$\begin{aligned} &\dot{V} + 2\delta V = 2X^T(t)P\dot{X}(t) + \sum_{n=N+1}^\infty 2z_n(t)\dot{z}_n(t) \\ &+2\delta X^T(t)PX(t) + 2\delta\sum_{n=N+1}^\infty z_n^2(t) \\ &= X^T(t)\left[PF + F^TP + 2\delta P\right]X(t) + 2X^T(t)P\mathcal{L}\zeta(t) \\ &+2\sum_{n=N+1}^\infty (-\lambda_n + q + \delta)z_n^2(t) \\ &+2\sum_{n=N+1}^\infty z_n(t)b_n\tilde{K}X(t). \end{aligned} \tag{3.25}$$

Furthermore, the Cauchy-Schwarz inequality implies

$$\begin{aligned} &\sum_{n=N+1}^\infty 2z_n(t)b_n\tilde{K}X(t) \le \frac{1}{\alpha}\sum_{n=N+1}^\infty z_n^2(t) \\ &+\alpha\left(\sum_{n=N+1}^\infty b_n^2\right)\left\|\tilde{K}X(t)\right\|^2 \le \frac{1}{\alpha}\sum_{n=N+1}^\infty z_n^2(t) \\ &+\alpha\|b\|_{L^2}^2\left\|\tilde{K}X(t)\right\|^2, \end{aligned} \tag{3.26}$$

where $\alpha > 0$. Denote $\eta(t) = \mathrm{col}\{X(t), \zeta(t)\}$. By combining (3.25) with (3.26) and taking into account (3.23) we obtain for some $\beta > 0$

$$\begin{aligned} &\dot{V} + 2\delta V + \beta\left(\|c\|_{L^2}^2\sum_{n=N+1}^\infty z_n^2(t) - \zeta^2(t)\right) \\ &\le \eta^T(t)\Psi\eta(t) + 2\sum_{n=N+1}^\infty W_n z_n^2(t) \le 0 \end{aligned} \tag{3.27}$$

if $W_n = -\lambda_n + q + \delta + \frac{1}{2\alpha} + \frac{\beta\|c\|_{L^2}^2}{2} < 0, \ \ n > N$ and

$$\Psi = \begin{bmatrix} PF + F^TP + 2\delta P + \alpha\|b\|_{L^2}^2\tilde{K}^T\tilde{K} & P\mathcal{L} \\ * & -\beta \end{bmatrix} < 0. \tag{3.28}$$

Note that monotonicity of $\{\lambda_n\}_{n=1}^\infty$ and Schur's complement imply that $W_n < 0$ for all $n > N$ iff

$$\begin{bmatrix} -\lambda_{N+1} + q + \delta + \dfrac{\beta\|c\|_{L^2}^2}{2} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\alpha \end{bmatrix} < 0. \tag{3.29}$$

Summarizing, we arrive at:

Theorem 2. Consider (3.1) with $b \in L^2(0,1)$ satisfying (3.11), measurement (3.2) with $c \in L^2(0,1)$ satisfying (3.9), control law (3.14) and $z_0 \in L^2(0,1)$. Let $\delta > 0$ be a desired decay rate, $N_0 \in \mathbb{N}$ satisfy (3.5) and $N \in \mathbb{N}$ satisfy $N_0 \le N$. Assume that L_0 and K_0 are obtained using (3.10) and (3.13),respectively. If there exists a positive definite matrix $P \in \mathbb{R}^{2N\times 2N}$ and scalars $\alpha, \beta > 0$ which satisfy (3.28) and (3.29), then the solution $z(x,t)$ to (3.1) under the control law (3.14), (3.7) and the corresponding observer $\hat{z}(x,t)$ defined by (3.6) satisfy the following inequalities

$$\begin{aligned} &\|z(\cdot,t)\|_{L^2}^2 \le Me^{-2\delta t}\|z_0\|_{L^2}^2, \\ &\|z(\cdot,t) - \hat{z}(\cdot,t)\|_{L^2}^2 \le Me^{-2\delta t}\|z_0\|_{L^2}^2, \end{aligned} \tag{3.30}$$

with some constant $M > 0$. Moreover, LMIs (3.28) and (3.29) are always feasible for large enough N.

Proof. To show (3.30), we note that (3.27) implies

$$V(t) \le e^{-2\delta t} V(0), \quad t \ge 0. \tag{3.31}$$

By (3.24), for some $M_0 > 0$ we have

$$V(0) \le M_0 \left\| z_0 \right\|_{L^2}^2 . \tag{3.32}$$

Note that

$$\hat{z}_n^2 + e_n^2 = (z_n - e_n)^2 + e_n^2 \ge 0.5 z_n^2.$$

Then, by Parseval's equality,

$$\begin{aligned} V(t) &\ge \lambda_{min}(P) \sum_{n=1}^{N} \left[\hat{z}_n^2(t) + e_n^2(t) \right] + \sum_{n=N_0+1}^{\infty} z_n^2(t) \\ &\ge \min\left(\frac{\lambda_{min}(P)}{2}, 1 \right) \left\| z(\cdot,t) \right\|_{L^2}^2 , \quad t \ge 0, \\ V(t) &\ge \lambda_{min}(P) \sum_{n=1}^{N} e_n^2(t) + \sum_{n=N_0+1}^{\infty} z_n^2(t) \\ &\ge \min\left(\lambda_{min}(P), 1 \right) \left\| z(\cdot,t) - \hat{z}(\cdot,t) \right\|_{L^2}^2 . \end{aligned} \tag{3.33}$$

Finally, (3.31)-(3.33) imply (3.30).

For the proof of feasibility of LMIs (3.28) and (3.29) we will first show that the solution to the Lyapunov equation

$$P(F + \delta I) + (F + \delta I)^T P = -I. \tag{3.34}$$

has a norm, $\|P\|_2$, which is uniformly bounded in N. Note that this solution is given by

$$P = \int_0^\infty e^{(F+\delta I)^T t} e^{(F+\delta I)t} dt. \tag{3.35}$$

So, it is sufficient to show that for some independent on N constants $\kappa_0 > 0$ and $M_0 > 0$ the following inequality holds:

$$\left\| e^{(F+\delta I)t} \right\|_2 \le M_0 e^{-\kappa_0 t}, \quad t \ge 0. \tag{3.36}$$

To prove (3.36), we present F as $F = \tilde{F}_1 + \tilde{F}_2$, where

$$\tilde{F}_1 = \begin{bmatrix} A_0 + B_0 K_0 & L_0 C_0 & 0 & 0 \\ 0 & A_0 - L_0 C_0 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix} + \delta I,$$

$$\tilde{F}_2 = F + \delta I - \tilde{F}_1.$$

Since L_0 and K_0 satisfy (3.10) and (3.13), respectively, the block-diagonal matrix $\tilde{F}_1 = \mathrm{diag}\{F_{10}, F_{11}\}$ with $2N_0 \times 2N_0$ block F_{10} is Hurwitz. Moreover, for some N-independent $\kappa > 0$ and $M_1 > 1$

$$\begin{aligned} \left\| e^{\tilde{F}_{10} t} \right\|_2 &\le M_1 e^{-\kappa t}, \quad t \ge 0, \\ \left\| e^{\tilde{F}_1 t} \right\|_2 &\le \max\{ \| e^{\tilde{F}_{10} t} \|_2, e^{-\kappa t} \} \le M_1 e^{-\kappa t}. \end{aligned} \tag{3.37}$$

By Parseval's equality,

$$\begin{aligned} \| B_1 K_0 \|_2 &\le \| B_1 \| \, \| K_0 \| \le \| b \|_{L^2} \| K_0 \| , \\ \| L_0 C_1 \|_2 &\le \| L_0 \| \, \| C_1 \| \le \| c \|_{L^2} \| L_0 \| . \end{aligned} \tag{3.38}$$

Then, for some N-independent constant $M_2 > 0$

$$\begin{aligned} \left\| \tilde{F}_2 \right\|_2 &\le M_2 \max(\| B_1 K_0 \|_2 , \| L_0 C_1 \|_2) \\ &\le M_2 \max(\| b \|_{L^2} \| K_0 \| , \| c \|_{L^2} \| L_0 \|). \end{aligned} \tag{3.39}$$

From (3.37) and (3.39) it can be easily verified that for all $t_1 \ge 0$ and $t_2 \ge 0$ there exists N-independent $M_3 > 0$ such that

$$\begin{aligned} &\left\| \prod_{i=1}^{2} e^{\tilde{F}_1 t_i} \tilde{F}_2 \right\|_2 \\ &\le M_3 e^{-\kappa(t_1+t_2)} \cdot \| b \|_{L^2} \cdot \| K_0 \| \cdot \| c \|_{L^2} \cdot \| L_0 \| . \end{aligned} \tag{3.40}$$

Moreover, it can be shown that the block-diagonal matrix $\tilde{F}_1$ and nilpotent matrix $\tilde{F}_2$ satisfy

$$\prod_{i=1}^{3} (\tilde{F}_1^{n_i} \tilde{F}_2) = 0 \quad n_i = 0, 1, ...$$

Then for any $t_i \ge 0$ $(i = 1, 2, 3)$ we have

$$\prod_{i=1}^{3} \left(e^{\tilde{F}_1 t_i} \tilde{F}_2 \right) = 0. \tag{3.41}$$

For $t > 0$, we apply the following identity (see, e.g, Van Loan (1977)):

$$e^{(F+\delta I)t} = e^{\tilde{F}_1 t} + \int_0^t e^{\tilde{F}_1 (t-t_1)} \tilde{F}_2 e^{(F+\delta I)t_1} dt_1. \tag{3.42}$$

By using (3.42) again with t, t_1 replaced by t_1, t_2, respectively, and substituting back into (3.42), we obtain

$$\begin{aligned} e^{(F+\delta I)t} &= e^{\tilde{F}_1 t} + \int_0^t e^{\tilde{F}_1 (t-t_1)} \tilde{F}_2 e^{\tilde{F}_1 t_1} dt_1 \\ &+ \int_0^t \int_0^{t_1} e^{\tilde{F}_1 (t-t_1)} \tilde{F}_2 e^{\tilde{F}_1 (t_1 - t_2)} \tilde{F}_2 e^{(F+\delta I)t_2} dt_2 dt_1. \end{aligned}$$

Finally, repeating this step again and using (3.41) in the resulting triple integral leads to

$$\begin{aligned} e^{(F+\delta I)t} &= e^{\tilde{F}_1 t} + \int_0^t e^{\tilde{F}_1 (t-t_1)} \tilde{F}_2 e^{\tilde{F}_1 t_1} dt_1 \\ &+ \int_0^t \int_0^{t_1} e^{\tilde{F}_1 (t-t_1)} \tilde{F}_2 e^{\tilde{F}_1 (t_1 - t_2)} \tilde{F}_2 e^{\tilde{F}_1 t_2} dt_2 dt_1. \end{aligned} \tag{3.43}$$

From (3.43) and (3.40) we find

$$\left\| e^{(F+\delta I)t} \right\|_2 \le M_4 e^{-\kappa t} \left(1 + t + t^2 \right), \tag{3.44}$$

where $M_4 > 0$ is independent of N. Hence, (3.36) holds and $\|P\|_2$ is uniformly bounded in N.

We show next that (3.28) and (3.29) are feasible for large enough N with P that solves (3.34), $\alpha = N^{-1}$ and $\beta = N$, $\lambda_{N+1} = (N + \frac{1}{2})^2 \pi^2$. By Schur complement, (3.28) and (3.29) with the chosen decision variables are feasible iff

$$\begin{aligned} W_{N+1} &= -(N + \frac{1}{2})^2 \pi^2 + q + \delta + \frac{N \left(1 + \| c \|_{L^2}^2 \right)}{2} < 0, \\ \Xi &= -I + \frac{\| b \|_{L^2}^2}{N} \tilde{K}^T \tilde{K} + \frac{1}{N} P \mathcal{L} \mathcal{L}^T P < 0. \end{aligned}$$

It is clear that $W_{N+1} < 0$ holds for large N. Since $\|P\|_2, \|\tilde{K}\|_2, \|\mathcal{L}\|_2$ are uniformly bounded in N, all of the eigenvalues of Ξ approach -1 uniformly in N. Hence, $\Xi < 0$ for large enough N.

3.2 Non-local measurement and actuation: H^1-stability

In this section let $b \in H^1(0,1)$ with $b(1) = 0$. Then, by (2.3), we have $\|b'\|_{L^2}^2 = \sum_{n=1}^\infty \lambda_n b_n^2 < \infty$. Furthermore, we assume that $z_0 \in H^1(0,1)$ with $z_0(1) = 0$. We note that exponential H^1-convergence of the closed-loop system still holds under the assumption $z_0 \in L^2(0,1)$, due to the smoothing property of the heat equation (see Remark 3.1 below).

The observer and controller are defined as in Section 3.1. The closed-loop system is given by (3.22). Moreover, the estimate (3.23) continues to hold. For H^1-stability analysis, we modify $V(t)$, defined in (3.24), as follows

$$V(t) := \|X(t)\|_P^2 + \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t). \tag{3.45}$$

Note, that the series in (3.45) can be differentiated term by term since the solution to (3.1) satisfies $z(\cdot,t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$. Differentiating $V(t)$ along (3.22) gives

$$\begin{aligned} &\dot{V} + 2\delta V = X^T(t)\left[PF + F^T P + 2\delta P\right] X(t) \\ &+2X^T(t)P\mathcal{L}\zeta(t) + 2\sum_{n=N+1}^{\infty} \lambda_n(-\lambda_n + q + \delta) z_n^2(t) \\ &+ \sum_{n=N+1}^{\infty} 2 z_n(t)\lambda_n b_n \tilde{K} X(t). \end{aligned} \tag{3.46}$$

Furthermore, the Cauchy-Schwarz inequality and Lemma 1 imply

$$\begin{aligned} &\sum_{n=N+1}^{\infty} 2\lambda_n z_n(t) b_n \tilde{K} X(t) \le \frac{1}{\alpha}\sum_{n=N+1}^{\infty} \lambda_n z_n^2(t) \\ &+\alpha\left(\sum_{n=N+1}^{\infty} \lambda_n b_n^2\right)\left\|\tilde{K}X(t)\right\|^2 \le \frac{1}{\alpha}\sum_{n=N+1}^{\infty} \lambda_n z_n^2(t) \\ &+\alpha\left\|b'\right\|_{L^2}^2 \left\|\tilde{K}X(t)\right\|^2, \end{aligned} \tag{3.47}$$

Denote $\eta(t) = \mathrm{col}\{X(t), \zeta(t)\}$. By combining (3.46) with (3.47) and taking into account (3.23) we obtain for some $\beta > 0$

$$\begin{aligned} &\dot{V} + 2\delta V + \beta\left(\|c\|_{L^2}^2 \sum_{n=N+1}^{\infty} z_n^2(t) - \zeta^2(t)\right) \\ &\le \eta^T(t)\Psi^1\eta(t) + 2\sum_{n=N+1}^{\infty} \lambda_n W_n^{(1)} z_n^2(t) \le 0 \end{aligned} \tag{3.48}$$

if

$$\begin{aligned} &W_n^{(1)} = -\lambda_n + q + \delta + \frac{1}{2\alpha} + \frac{\beta\|c\|_{L^2}^2}{2\lambda_n} < 0, \quad n > N, \\ &\Psi^1 = \begin{bmatrix} PF + F^T P + 2\delta P + \alpha\|b'\|_{L^2}^2 \tilde{K}^T\tilde{K} & P\mathcal{L} \\ * & -\beta \end{bmatrix} < 0. \end{aligned} \tag{3.49}$$

Monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and Schur's complement imply that $W_n^{(1)} < 0$ for all $n > N$ iff

$$\begin{bmatrix} -\lambda_{N+1} + q + \delta + \dfrac{\beta\|c\|_{L^2}^2}{2\lambda_{N+1}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\alpha \end{bmatrix} < 0. \tag{3.50}$$

Summarizing, we arrive at:

Theorem 3. Consider (3.1) with $b \in H^1(0,1)$, $b(1) = 0$ satisfying (3.11), measurement (3.2) with $c \in L^2(0,1)$ satisfying (3.9), control law (3.14) and $z_0 \in H^1(0,1)$, $z_0(1) = 0$. Let $\delta > 0$ be a desired decay rate, $N_0 \in \mathbb{N}$ satisfy (3.5) and $N \in \mathbb{N}$ satisfy $N_0 \le N$. Assume that L_0 and K_0 are obtained using (3.10) and (3.13),respectively. If there exists a positive definite matrix $P \in \mathbb{R}^{2N\times 2N}$ and scalars $\alpha, \beta > 0$ which satisfy (3.49) and (3.50), then the solution $z(x,t)$ to (3.1) under the control law (3.14), (3.7) and the corresponding observer $\hat{z}(x,t)$ defined by (3.6) satisfy the following inequalities

$$\begin{aligned} &\|z(\cdot,t)\|_{H^1}^2 \le Me^{-2\delta t}\|z_0\|_{H^1}^2, \\ &\|z(\cdot,t) - \hat{z}(\cdot,t)\|_{H^1}^2 \le Me^{-2\delta t}\|z_0\|_{H^1}^2, \end{aligned} \tag{3.51}$$

with some constant $M > 0$. Moreover, LMIs (3.49) and (3.50) are always feasible for large enough N.

Remark 3.1. In the case where $z_0 \in L^2(0,1)$, Theorem 3 still implies exponential H^1-convergence (although not exponential stability) of the closed-loop system. Indeed, let $t_* > 0$ be small. Then $z(\cdot, t_*) \in \mathcal{D}(\mathcal{A})$. Therefore, by applying Theorem 3 we obtain

$$\begin{aligned} &\|z(\cdot,t)\|_{H^1}^2 \le Me^{-2\delta(t-t_*)}\|z(\cdot,t_*)\|_{H^1}^2, \\ &\|z(\cdot,t) - \hat{z}(\cdot,t)\|_{H^1}^2 \le Me^{-2\delta(t-t_*)}\|z(\cdot,t_*)\|_{H^1}^2, \end{aligned}$$

for all $t > t_*$, where $M > 0$ is some constant.

Remark 3.2. The method presented in this paper can be extended to the system (3.1) with boundary measurement

$$y(t) = z(0,t). \tag{3.52}$$

In this case, the innovation term (i.e, the output estimation error. See (3.18)) can be presented as

$$\begin{aligned} &\sum_{n=1}^{N} \hat{z}_n(t)\phi_n(0) - y(t) = -\sum_{n=1}^{N} c_n e_n(t) - \zeta(t) \\ &c_n = \phi_n(0) = \sqrt{2} \\ &\zeta(t) = z(0,t) - \sum_{n=1}^{N} c_n z_n(t). \end{aligned} \tag{3.53}$$

Using the Cauchy-Schwarz inequality and Lemma 1, it can be shown that

$$\begin{aligned} &\zeta^2(t) := \left[z(0,t) - \sum_{n=1}^{N} \phi_n(0) z_n(t)\right]^2 \\ &= \left[\int_0^1 \left(z_x(s,t) - \sum_{n=1}^{N} \phi_n'(s) z_n(t)\right) ds\right]^2 \\ &\le \left\|z_x(\cdot,t) - \sum_{n=1}^{N} \phi_n'(\cdot) z_n(t)\right\|_{L^2}^2 = \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t). \end{aligned} \tag{3.54}$$

Taking into account the estimate (3.54), exponential H^1-convergence of the closed-loop system can be obtained by considering the Lyapunov function (3.45) and applying arguments similar to (3.46)-(3.50).

4. NUMERICAL EXAMPLE

In the following example, we choose $q = 10$, which corresponds to an unstable open-loop system. The gains L_0 and K_0 are found from (3.10) and (3.13), respectively. The LMIs are verified by using the standard Matlab LMI toolbox.

Consider system (3.1) with measurement (3.2), where

$$\begin{aligned} &c(x) = \begin{cases} \sqrt{2}, & x \in [0.25, 0.75] \\ 0, & x \notin [0.25, 0.75] \end{cases}, \\ &b(x) = \begin{cases} \sqrt{2}(4x - 1), & x \in [0.25, 0.5] \\ \sqrt{2}(-4x + 3), & x \in [0.5, 0.75] \\ 0, & x \notin [0.25, 0.75] \end{cases}. \end{aligned} \tag{4.1}$$

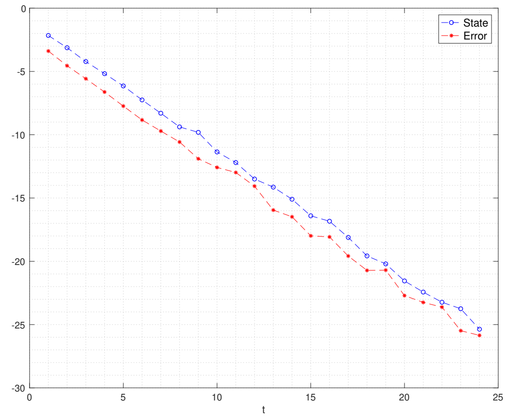

Fig. 1. Non-local measurement and actuation.

Note that $b(x) \in H^1(0,1)$, $b(1) = 0$ and $c(x) \in L^2(0,1)$. Let $N_0 = 1$ and $\delta = 1$. The obtained observer and controller gains are

$$K_0 = -57.6811, \quad L_0 = 29.217. \tag{4.2}$$

The LMIs of Theorem 3 are feasible for $N = 4$ (this is the minimal value of N that guarantees the LMIs feasibility).

For the simulation of solutions to the closed-loop system we chose

$$z_0(x) = x^2 - 1. \tag{4.3}$$

with $z_0(1) = 0$. The simulation was carried out for the corresponding PDE (3.1) with $u(t) = K_0\hat{z}_1(t)$ (using the finite-difference FTCS scheme) and ODEs (3.7) (using 4-th order Runge-Kutta scheme). The norms $\|z_x(\cdot,t)\|_{L^2}$ and $\|z_x(\cdot,t) - \hat{z}_x(\cdot,t)\|_{L^2}$ for $t > 0$ were estimated using (2.3) with $\|z_x\|^2_{L^2} = \sum_{n=1}^{40} \lambda_n z_n^2(t)$, whereas $z_n(t)$ were found from simulation of ODEs (3.4) (note that these ODEs are not part of the closed-loop system). The $H^1(0,1)$ norms of the state and estimation error $e = z - \hat{z}$ are presented, on a logarithmic scale, in Figure 1. The computed linear fits are given by

$$p_z(t) \approx -1.0031t - 1.1824, \;\; p_e(t) \approx -0.9873t - 2.0721,$$

which is consistent with a decay rate $\delta = 1$ up to numerical errors. Moreover, numerical simulations show that for $N = 3$, the closed-loop system is unstable.

5. CONCLUSION

The present paper has suggested the first constructive LMI-based method for finite-dimensional observer-based controller design in the case of 1-D linear heat equation. The method was applied for the case of bounded observation and control operators. However, the arguments presented can be modified for the heat equation with boundary measurement and non-local actuation (see Remark 3.2). Our method results in simple and constructive tools, which can be used for finite-dimensional observer-based control of other parabolic systems.

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