Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Network-based deployment of the second-order multi agents: a PDE approach ⋆

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Abstract: Deployment of a second-order nonlinear multi agent system over a desired open smooth curve in 2D or 3D space is considered. We assume that the agents have access to their velocities and to the local information of the desired curve and their displacements with respect to their closest neighbors, whereas in addition a leader is able to measure his absolute position. We assume that a small number of leaders transmit their measurements to other agents through a communication network. We take into account the following network imperfections: the variable sampling, transmission delay and quantization. We propose a static output-feedback controller and model the resulting closed-loop system as a disturbed (due to quantization) nonlinear damped wave equation with delayed point state measurements, where the state is the relative position of the agents with respect to the desired curve. To manage with the open curve we consider Neumann boundary conditions. We derive linear matrix inequalities (LMIs) that guarantee the input-to-state stability (ISS) of the system. The advantage of our approach is in the simplicity of the control law and the conditions. Numerical example illustrates the efficiency of the method.

Keywords: Distributed parameter systems, multi-agent systems, network-based control, time-delay.

1. INTRODUCTION

Deployment of large-scale multi agent system (MAS), where a group of agents rearrange their positions into a target spatial configuration in order to achieve a common goal, has attracted attention of many researchers in the recent years Mesbahi and Egerstedt (2010), Oh et al. (2015). This is due to their vast applications, such as cooperative movement of robots or vehicles Ren et al. (2007), biochemical reaction networks, animal flocking behavior (see Olfati-Saber (2006)), search-and-rescue, environmental sensing and monitoring Dunbabin and Marques (2012), etc. The majority of the existing work in the field of MAS is concentrated on deploying of interconnected agents, modeled by ordinary differential equations (ODEs) that provides efficient methods when the number of agents is low.

When the number of agents is large, a methodology based on partial differential equations (PDEs) becomes efficient. In Frihauf and Krstic (2010); Meurer (2012), the agents were treated as a continuum, and the collective dynamics was modeled by a reaction-diffusion PDE, under the boundary control. Feedforward control combined with backstepping-based boundary controller was implemented in Freudenthaler and Meurer (2016), where the collective dynamics was modeled by a modified viscous Burger's equation. Finite-time transitions between desired deployment formations along predefined spatial-–temporal paths by means of boundary control were contemplated in Meurer and Krstic (2011). Pilloni et al. (2015) address the problem of driving the state of a network of agents, modeled by boundary controlled heat equations, toward a common steady state profile. Formation tracking control of a MAS, where the collective dynamics was modeled by a wave PDE was studied in Tang et al. (2017). Qi et al. (2019) considered the control of collective dynamics of a large scale MAS moving in a 3D space under the occurrence of arbitrarily large boundary input delay.

In the case of measurements of the leaders' absolute positions, the majority of PDE-based results employ the PDE observer for output-feedback control. The latter may be difficult for implementation. Recently a simple static output-feedback controller was suggested in Wei et al. (2019), where it was proposed to transmit the leader absolute displacement with respect to the desired curve to all the agents by using communication network. The network-based results of Wei et al. (2019) were confined to the first-order integrators and to deployment onto the closed curves. Among the network imperfection, the quantization effects were neglected. Moreover, in the case of several leaders a common delay (i.e. synchronized transmissions in the same time with the same network-induced delay) was considered that may be restrictive.

⋆ This work was supported by Israel Science Foundation (grant no. 673/19) and by C. and H. Manderman Chair at Tel Aviv University.

In this paper we study deployment of the second-order nonlinear agents onto the open curves. We assume that the agents have access to their velocities and to the local information of the desired curve and their displacements with respect to their closest neighbors, whereas in addition a leader is able to measure his absolute displacement with respect to the desired curve. As in Wei et al. (2019) we propose to transmit the leaders absolute displacements to other agents by using communication network. However, these transmissions are not synchronous that leads to multiple delays in the closed-loop system. Moreover, we take into account the quantization effect (see Liberzon (2003)).

By applying the time-delay approach to networked control systems (see Chapter 7 of Fridman (2014)), we model the resulting closed-loop system as a disturbed (due to quantization) nonlinear damped wave equation with the delayed point state measurements under the Neumann boundary conditions. The state is the relative position of the agents with respect to the desired curve. Note that the existing results under the point delayed measurements are confined to one delay and to unperturbed systems (see Fridman and Blighovsky (2012); Kang and Fridman (2019); Terushkin and Fridman (2019)). Here, for the first time for such systems, we analyze the ISS of the closed-loop system by combining the Lyapunov-Krasovskii method with the generalized Halanay's inequality (Wen et al. (2008)). Moreover, we treat the case of multiple delays. We derive LMIs that guarantee ISS. The advantage of our approach is in the simplicity of the control law and the conditions. Numerical example of deployment onto smooth open curve in a 3D space illustrates the efficiency of the method.

Notation Throughout the paper the notation $P > 0$ with $P \in \mathbb{R}^{n\times n}$ stands for a symmetric and positive definite matrix, with the symmetric elements denoted by $*$. Functions, continuous (continuously differentiable) in all arguments, are referred to as of class $\mathcal{C}$ (of class $\mathcal{C}^1$). $L^2(0,L)$ is the Hilbert space of square integrable functions $z(\xi)$, $\xi \in [0,L]$ with the norm $\|z\|^2_{L^2} = \int_0^L z^2(\xi)d\xi$. $\mathscr{H}^1(0,L)$ is the Sobolev space of absolutely continuous scalar functions $z:[0,L]\to\mathbb{R}$ with $\frac{dz}{d\xi}\in L^2(0,L)$. $\mathscr{H}^2(0,L)$ is the Sobolev space of scalar functions $z:[0,L]\to\mathbb{R}$ with absolutely continuous $\frac{dz}{d\xi}$ and with $\frac{d^2z}{d\xi^2}\in L^2(0,L)$.

1.1 Mathematical preliminaries

The following inequalities will be useful:

Lemma 1.1. Wirtinger's inequality Hardy et al. (1988). Let $z\in\mathscr{H}^1[a,b]$ be a scalar function, with the boundary values stated below. Then

$$\sigma\int_a^b z^2(\xi)d\xi \le \frac{(b-a)^2}{\pi^2}\int_a^b \left[\frac{dz(\xi)}{d\xi}\right]^2 d\xi \qquad (1)$$

where

$$\sigma = \begin{cases} 1, & \text{if} \quad z(a)=z(b)=0; \\ 1/4, & \text{if} \quad z(a)=0 \text{ or } z(b)=0. \end{cases}$$

Lemma 1.2. Generalized Halanay's inequality Wen et al. (2008). Let $V:[t_0-\tau_M,\infty)\longrightarrow\mathbb{R}^+$ be a locally absolutely continuous function, and $w:[t_0,\infty)\longrightarrow\mathbb{C}$ be a bounded continuous function satisfying $\|w(t)\|\le\Delta_w$, where $\Delta_w > 0$ is given. If there exists $0<\alpha_1<\alpha_0$ and γ^2 such that for all $t\ge t_0$ the following holds

$$\dot V(t) + 2\alpha_0 V(t) - 2\alpha_1 \sup_{-\tau_M\le\theta\le 0} V(t+\theta) - \gamma^2|w(t)|^2 \le 0,\ t\ge t_0,$$

then

$$V(t)\le\exp\left(-2\alpha(t-t_0)\right)\sup_{-\tau_M\le\theta\le 0} V(t_0+\theta) + \frac{\gamma^2}{\varepsilon}\Delta_w^2,\ t\ge t_0, \qquad (2)$$

where $\varepsilon = 2(\alpha_0-\alpha_1)>0$, and $\alpha>0$ is a unique positive solution of

$$\alpha = \alpha_0 - \alpha_1\exp(2\alpha\tau_M).$$

2. MAIN RESULTS

2.1 Problem formulation

Consider a group of N agents, described by the second-order dynamics, that can move in space $\mathbb{R}^n, n\in\{1,2,3\}$. Our aim is to deploy N agents onto a $\mathcal{C}^2$ curve $\Gamma:[0,L]\longrightarrow\mathbb{R}^n$. If $\Gamma(0)\neq\Gamma(L)$, the curve Γ is open. For simplicity, we assume that the desired curve does not evolve over time. We neglect collision avoidance as we assume agents of zero volume operating within a large workspace. Furthermore, we assume that no static or moving obstacles are present in the operating workspace.

The dynamics of each agent, in each dimension n, $n=\{1,2,3\}$, is given by

$$\ddot z_i(t) = u_i(t) + f(z_i,t), \quad i=1,\dots,N, \quad t\ge t_0. \qquad (3)$$

where $z_i(t)\in\mathbb{R}$, $u_i\in\mathbb{R}$ are components of the position and control of agent i respectively, and the acceleration nonlinearities f are of class $\mathcal{C}^1$. For brevity, the notation of dimension is omitted. We assume that the derivative f_z is uniformly bounded by a constant $\rho_1>0$:

$$|f_z(z,t)|\le\rho_1, \quad \forall(z,t)\in\mathbb{R}\times[0,L]\times[t_0,\infty). \qquad (4)$$

N points are assigned on the desired curve with constant spacing $h=\frac{L}{N}$, namely $\Gamma(h),\dots,\Gamma(hN)$ which will give the final desired position of each agent.

The leader-enabled deployment of mobile agents is considered under the following assumptions:

(1) Agents $i=2,\dots,N-1$ measure their displacements with respect to the closest neighbors $i-1,i+1$. They have access to $\Gamma((i-1)h),\Gamma(ih)$ and $\Gamma((i+1)h)$. The boundary agents with $i=1$ and $i=N$ measure the relative positions of the agents 2 and $N-1$ and have access to $\Gamma(h),\Gamma(2h)$ and $\Gamma((N-1)h),\Gamma(Nh)$ respectively.
(2) All the agents measure their own velocity $\dot z_i$ with respect to the global coordinate system. However, no agent can measure its global position z_i except for the leader agents labeled z_{i_m}, $m\in\{1,\dots,M\}$.
(3) The agents form a chain topology, and the adjacent agents keep their order.

Our objective is to deploy the agents onto the desired curve Γ by exploiting $M << N$ leaders.

2.2 Controller design and PDE model

We propose a leader-follower displacement-based control, where the position measurements of the leader agents are transmitted through communication network to other agents. Define

$$\begin{array}{ll} z_0(t) = z_2(t), & z_{N+1}(t) = z_{N-1}(t), \\ \Gamma(0) = \Gamma(2h), & \Gamma((N+1)h) = \Gamma((N-1)h). \end{array} \tag{5}$$

Consider the following static output-feedback controller:

$$\begin{aligned} u_i(t) =& \frac{\upsilon^2}{h^2}\left[z_{i+1}(t) - 2z_i(t) + z_{i-1}(t)\right] \\ &- \frac{\upsilon^2}{h^2}\left[\Gamma\big((i+1)h\big) - 2\Gamma\big(ih\big) + \Gamma\big((i-1)h\big)\right] \\ &- \beta\dot{z}_i(t) - f(\Gamma(ih), t) + \bar{u}_i(t), \quad i = 1, \ldots, N. \end{aligned} \tag{6}$$

Here $\bar{u}_i$ will be found below as the product of a constant gain $K > 0$ on the corresponding leaders' position measurements. Note that the proposed controller (6) contains 3 gains: υ^2 (larger υ^2 allows to reduce the number of the leaders Wei et al. (2019)), K (compensates the destabilizing effect of the nonlinearity f) and β (improves the convergence). Given υ^2, we aim to achieve the deployment with as small as possible number M of leaders.

Denote the error

$$e_i(t) = z_i(t) - \Gamma(ih), \quad u = 0, \ldots N+1, \ t \geq t_0.$$

We have

$$\begin{aligned} f(z_i, t) - f(\Gamma(ih), t) &= \rho(e_i, t)e_i(t), \\ \rho(e_i, t) &= \int_0^1 f_z\left(\theta e_i + \Gamma(ih), t)\right) d\theta, \end{aligned}$$

where due to (4)

$$|\rho| \leq \rho_1 \quad \forall (e_i, t) \in \mathbb{R} \times [t_0, \infty).$$

Then the closed-loop system (3), (6) can be presented as:

$$\begin{aligned} \ddot{e}_i(t) =& \frac{\upsilon^2}{h^2}\left[e_{i+1}(t) - 2e_i(t) + e_{i-1}(t)\right] \\ &- \beta\dot{z}_i(t) + \rho(e_i, t)e_i(t) + \bar{u}_i(t), \quad i = 1, \ldots, N. \end{aligned} \tag{7}$$

We further treat the large-scale MAS (3) as a continuum with a spatial domain $x \in [0, L]$. Following Fridman and Blighovsky (2012); Wei et al. (2019), we divide $x \in [0, L]$ into M sampling intervals

$$0 = x_0 < x_1 < \ldots < x_M = L, \tag{8}$$

where the length of the interval is bounded

$$x_m - x_{m-1} = \Delta_m < \Delta, \quad m = 1, ..., M. \tag{9}$$

Since we aim to minimize M, we can always assume that $\Delta \leq \frac{L}{M+1}$. Similar to Terushkin and Fridman (2019), we place the leader agent approximately in the middle of each interval (see Fig. 1) such that

$$\hat{x}_m - x_{m-1} \leq \frac{\Delta}{2}, \quad x_m - \hat{x}_m \leq \frac{\Delta}{2}.$$

Note that if in discretization, the number N_m of agents located on $[x_{m-1}, x_m]$ is even, then leader may be located in such a way that he has $0.5N_m - 1$ agents on $[x_{m-1}, x_m]$ from the left (or right) and $0.5N_m$ from the right (or left).

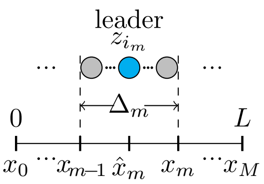


Fig. 1. MAS: leader location

The leader z_{i_m} sends his absolute (relative to Γ) position $z_{i_m} - \Gamma(i_m h)$ to all the agents z_i located on $[x_{m-1}, x_m)$ through communication network. The measurements are subject to quantization effect Liberzon (2003), Fridman and Dambrine (2009). A quantizer is a piecewise constant function $q : \mathbb{R} \to \mathbb{R}$ such that

$$|q(y) - y| \leq \Delta_q, \tag{10}$$

where Δ_q is the quantization error bound. Here for simplicity we assume that the quantizer has no constraints on the quantization range.

Let $s_0^m < s_1^m < ...$ be the sampling times of the measurements with $\lim_{k\to\infty} s_k^m = \infty$ and η_k^m are network-induced delays. We assume that $s_k + \eta_k < s_{k+1} + \eta_{k+1}$ for all k. The agents z_i from the interval $[x_{m-1}, x_m]$ employ the controller

$$\bar{u}_i(t) = Kq(z_{i_m}(s_k^m) - \Gamma(i_m h)), \quad t \in [s_k + \eta_k, s_{k+1} + \eta_{k+1}), \tag{11}$$

where $\bar{u}_i(t) = 0$ for $t < t_0$.

By using the time-delay approach to network-controlled systems (see Chapter 7 of Fridman (2014)), denote

$$\tau^m = \tau^m(t) = t - s_k^m, \ t \in [s_k^m + \eta_k^m, s_{k+1}^m + \eta_{k+1}^m), \ k = 0, 1, \ldots$$

where $\tau^m(t) \leq \tau_M \ \forall m = 1, ..., M$ and τ_M is the sum of maximum transmission interval and maximum allowable delay. Then the controller (11) can be presented as

$$\begin{array}{l} \bar{u}_i(t) = Ke_{i_m}(t - \tau^m) + Kw_m(t - \tau^m), \quad t \geq t_0, \\ w_m(t - \tau^m) = q(e_{i_m}(t - \tau^m)) - e_{i_m}(t - \tau^m) \end{array} \tag{12}$$

with

$$|w_m(t - \tau^m)| \leq \Delta_q, \quad \forall t \geq t_0. \tag{13}$$

System (7), (12) can be considered as the discretization in the spatial variable of the damped wave equation

$$\begin{aligned} e_{tt}(x, t) &= \upsilon^2 e_{xx}(x, t) - \beta e_t(x, t) + (\rho(e, t) - K)e(x, t) \\ &+ K\sum_{m=1}^{M} \chi_m\left[e(\hat{x}_m, t - \tau^m) + w_m(t - \tau^m)\right], \\ & x \in (0, L), \ t \geq t_0 \end{aligned} \tag{14}$$

under the Neumann boundary conditions

$$e_x(0, t) = e_x(L, t) = 0, \tag{15}$$

and initial conditions defined by the difference between of the initial agents positions and their target positions on Γ. Here χ_m are given by

$$\chi_m(x) = \begin{cases} 1, & x \in [x_{m-1}, x_m] \\ 0, & x \notin [x_{m-1}, x_m] \end{cases}, m = 1, \ldots, M. \tag{16}$$

Note that (5) with $e_0 = e_2$ and $e_{N+1} = e_{N-1}$ corresponds to spatial discretization under the Neumann boundary conditions. Due to (4)

$$|\rho| \leq \rho_1 \quad \forall (e, t)\mathbb{R} \times [t_0, \infty).$$

Consider initial conditions for (14), (15) as

$$e(x, t) \equiv e(x, t_0), \quad e_t(x, t) \equiv e_t(x, t_0), \quad t \leq t_0.$$

Well-posedness of (14), (15) can be proved similar to Terushkin and Fridman (2019). The state is presented as $\zeta(t) = [\zeta_0(t) \ \zeta_1(t)]^T = [e \ e_t(t)]^T$. Denote

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ \upsilon^2\frac{\partial^2}{\partial x^2} & -\beta I \end{bmatrix}, \quad F(\zeta, t) = \begin{bmatrix} 0 \\ F_1(\zeta_0, t) \end{bmatrix}.$$

Here $F_1 : \mathscr{H}^1(0, L) \times [t_0, \infty) \to L^2(0, L)$ is given by

$$F_1(\zeta_0,t) = f(z,t) - f(z-\zeta_0,t) - K\sum_{m=1}^{M}\chi_m(x)\left[\zeta_0(\hat{x}_m,t-\tau^m) + w_m(t-\tau^m)\right].$$

The resulting differential equation

$$\dot{\zeta}(t) = \mathscr{A}\zeta(t) + F(\zeta(t),t), \quad t \geq t_0 \tag{17}$$

is considered in the Hilbert space $\mathscr{H} = \mathscr{H}^1(0,L) \times L^2(0,L)$, where and $\|\zeta\|^2_{\mathscr{H}} = \|\zeta_{0x}\|^2_{L^2} + \|\zeta_1\|^2_{L^2}$.

The operator $\mathscr{A}$ with the dense domain

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{bmatrix}\zeta_0\\ \zeta_1\end{bmatrix} \in \mathscr{H}^2(0,L)\times\mathscr{H}^1(0,L) \middle| \zeta_{0x}(0) = \zeta_{0x}(L) = 0 \right\}$$

generates an exponentially stable semigroup (see Pazy (1983)). The times $s_k^m + \eta_k^m$, $k = 0,1,...,$ $m = 1,...,M$ are ordered as $t_0, t_1, ...$ By employing the step method for $t \in [t_0,t_1]$, $t \in [t_1,t_2]$ and applying Theorems 6.1.2 and 6.1.5 from Pazy (1983) (see Terushkin and Fridman (2019)), we find that a unique mild solution exists in $C([t_0,\infty),\mathscr{H})$ for (14), (15), initialized by $[e(\cdot,t_0)\, e_t(\cdot,t_0)]^T \in \mathscr{H}$. Moreover, if $[e(\cdot,t_0)\, e_t(\cdot,t_0)]^T \in \mathscr{D}(\mathscr{A}))$, then there exists a unique classical solution $C^1([t_0,\infty),\mathscr{H})$ with $\zeta(t) \in \mathscr{D}(\mathscr{A})$ for $t > t_0$.

2.3 ISS analysis

For the choice of the controller gains β and K we follow Remark 3.1 of Terushkin and Fridman (2019), where larger β and $K = \rho_1 + \frac{\beta^2}{4}$ lead to faster convergence.

Theorem 2.1. Consider the damped wave equation (14) under the Neumann boundary conditions (15) initialized for $t \leq t_0$ by $e(\cdot,t) \equiv e(\cdot,t_0) \in \mathscr{H}^1(0,L)$, $e_t(\cdot,t) \equiv e_t(\cdot,t_0) \in L^2(0,L)$ with the bounds $\tau_M, \rho_1, \Delta_q, \Delta$ in (13) and (9). Given $\alpha_0 > \alpha_1 > 0$, $\upsilon^2, \beta > 0, K = \rho_1 + \frac{\beta^2}{4}$, let there exist $\gamma, s > 0, r > 0, q_{12}$ and p_1, p_2, p_3 that satisfy the LMIs

$$P_0 = \begin{bmatrix} p_1 & p_2 \\ * & p_3 \end{bmatrix} > 0, \tag{18}$$

$$\alpha_0 p_3 - p_2 \leq 0, \tag{19}$$

$$R = \begin{bmatrix} r & q_{12} \\ * & r \end{bmatrix} \geq 0, \tag{20}$$

and

$$\Psi_{|\rho=\pm\rho_1} = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & Kp_2 & -2\alpha_M p_2 & -Kp_2 \\ * & \psi_{22} & Kp_2 + \tau_M^2 r & 0 & Kp_2 & 0 & -Kp_2 \\ * & * & \psi_{33} & \psi_{34} & 0 & 2\alpha_M p_2 & 0 \\ * & * & * & \psi_{44} & 0 & 0 & 0 \\ * & * & * & * & \psi_{55} & 0 & 0 \\ * & * & * & * & * & -2\alpha_M p_3 & 0 \\ * & * & * & * & * & * & -\gamma^2 \end{bmatrix} \leq 0, \tag{21}$$

where $\alpha_M = \frac{\alpha_1}{M}$ and

$\psi_{11} = 2p_2(\rho - K) + 2(\alpha_0 - \alpha_M)p_1 + s(1 - \exp(-2\alpha_0\tau_M))$
$\psi_{12} = p_1 + p_2(2\alpha_0 - \beta) + p_3(\rho - K)$
$\psi_{13} = Kp_2 + s\exp(-2\alpha_0\tau_M) + 2\alpha_M p_1$
$\psi_{14} = s\exp(-2\alpha_0\tau_M), \quad \psi_{22} = 2p_2 + 2p_3(\alpha_0 - \beta)$
$\psi_{33} = -(s + r)\exp(-2\alpha_0\tau_M) - 2\alpha_M p_1$
$\psi_{34} = -(s + q_{12})\exp(-2\alpha_0\tau_M)$
$\psi_{44} = -(s + r)\exp(-2\alpha_0\tau_M), \quad \psi_{55} = -2\alpha_M p_3 \upsilon^2(\pi^2/\Delta^2)$.

Then, (14) is ISS, namely there exists $c_0 > 0$ such that for all $t \geq t_0$

$$c_0(\|e_x(\cdot,t)\|^2_{L^2} + \|e_t(\cdot,t)\|^2_{L^2}) \leq \exp\left(-2\alpha(t-t_0)\right) \times \left[\|e_x(\cdot,t_0)\|^2_{L^2} + \|e_t(\cdot,t_0)\|^2_{L^2}\right] + \frac{\gamma^2}{\varepsilon}\Delta_q^2 L, \tag{22}$$

where $\varepsilon = 2(\alpha_0 - \alpha_1)$ and $\alpha > 0$ is a unique positive solution of $\alpha = \alpha_0 - \alpha_1\exp(2\alpha\tau_M)$.

Moreover, if the strict inequalities (18)-(21) are feasible with $\alpha_0 = \alpha_1 > 0$, then the error system (14), (15) is ISS with a small enough decay rate.

Proof Denote

$\nu_1 = e(x,t) - e(x,t-\tau^m), \quad \nu_2 = e(x,t-\tau^m) - e(x,t-\tau_M)$.

By employing the relations

$$e(x,t-\tau^m) = e(x,t) - \nu_1(x,t), \quad e(\hat{x}_m,t) = e(x,t) - \int_{\hat{x}_m}^{x} e_\zeta(\zeta,t)d\zeta \tag{23}$$

the error dynamics can be represented as

$$e_{tt} = \upsilon^2 e_{xx} - \beta e_t + (\rho - K)e + K\sum_{m=1}^{M}\chi_m\left[\nu_1 + \int_{\hat{x}_m}^{x} e_\zeta(\zeta,t-\tau^m)d\zeta - w_m(t-\tau^m)\right]. \tag{24}$$

For deriving the ISS conditions of (7), we use a Lyapunov functional similar to Terushkin and Fridman (2019)

$$V(t) = V_0(t) + V_s(t) + V_r(t), \quad t \in [t_k, t_{k+1}),\ k = 0,1,2,\ldots \tag{25}$$

where $V_0(t)$ is given by

$$V_0(t) = p_3\upsilon^2\int_0^L e_x^2 dx + \int_0^L [e\ e_t]P_0[e\ e_t]^T dx, \tag{26}$$

with P_0 given by (18), and

$$V_s(t) = s\sum_{m=1}^{M}\int_{x_{m-1}}^{x_m}\int_{t-\tau_M}^{t} e^{2\alpha_0(s-t)}e^2(x,s)ds\,dx, \tag{27}$$

$$V_r(t) = r\tau_M\sum_{m=1}^{M}\int_{x_{m-1}}^{x_m}\int_{-\tau_M}^{0}\int_{t+\theta}^{t} e^{2\alpha_0(s-t)}e_s^2(x,s)ds\,d\theta\,dx,$$

with some scalars $s, r \geq 0$. Here, V_s and V_r treat time-delay terms. This functional is defined on the mild solutions of (14), (15), and due to (18) it is positive definite with $V(t) \geq c'(\|e_x(\cdot,t)\|^2_{L^2} + \|e_t(\cdot,t)\|^2_{L^2})$ for some $c' > 0$.

As in Fridman (2013), we consider first

$$[e(\cdot,t) \equiv e(\cdot,t_0),\ e_t(\cdot,t) \equiv e_t(\cdot,t_0)]^T \in \mathscr{D}(\mathscr{A}).$$

Then we can differentiate $V(t)$ along the classical solutions of the wave equation. By differentiating V_s and V_r we have

$$\dot{V}_s + 2\alpha_0 V_s = s\sum_{m=1}^{M}\int_{x_{m-1}}^{x_m}\left(e^2(x,t) - e^{-2\alpha_0\tau_M}e^2(x,t-\tau_M)\right)dx \tag{28}$$

and

$$\dot{V}_r + 2\alpha_0 V_r \leq \tau_M^2 r\sum_{m=1}^{M}\int_{x_{m-1}}^{x_m} e_t^2 dx - \tau_M r e^{-2\alpha_0\tau_M}\sum_{m=1}^{M}\int_{x_{m-1}}^{x_m}\int_{t-\tau_M}^{t} e_s^2(x,s)ds\,dx. \tag{29}$$

Then, under (20) by Lemma 3.4 of Fridman (2014) we find

$$-\tau_M r\sum_{m=1}^{M}\int_{x_{m-1}}^{x_m}\int_{t-\tau_M}^{t} e_s^2 ds\,dx \leq -\sum_{m=1}^{M}\int_{x_{m-1}}^{x_m}[\nu_1\ \nu_2]R[\nu_1\ \nu_2]^T dx.$$

We proceed with differentiating (25) along (14). Note that integration by parts and substitution of boundary conditions leads to

$$2\upsilon^2\int_0^L (p_2 e + p_3 e_t)e_{xx}dx = -2\upsilon^2\int_0^L \left[p_2 e_x^2 + p_3 e_x e_{xt}\right]dx.$$

Then

$$\dot V_0 + 2\alpha_0 V_0 \le 2\upsilon^2(\alpha_0 p_3 - p_2)\int_0^L e_x^2 dx + \int_0^L [e\ e_t]G[e\ e_t]^T dx$$
$$+ 2\sum_{m=1}^M \int_{x_{m-1}}^{x_m} K(p_2 e + p_3 e_t)$$
$$\times\left[\nu_1 + \int_{\hat x_m}^x e_\zeta(\zeta, t-\tau^m)d\zeta - w_m(t-\tau^m)\right]dx$$

where

$$G \triangleq \begin{bmatrix} 2p_2(\rho - K) + 2\alpha_0 p_1 & p_1 + p_2(2\alpha_0 - \beta) + p_3(\rho - K) \\ * & 2p_2 + 2p_3(\alpha_0 - \beta)\end{bmatrix}.$$

By (13), we have

$$\sum_{m=1}^M \int_{x_{m-1}}^{x_m} |w_m(t-\tau^m)|^2 dx \le \Delta_q^2 L. \tag{30}$$

We will further apply the generalized Halanay's inequality (2) for some $0 < \alpha_1 < \alpha_0$. Note that

$$\sup_{-\tau_M \le \theta \le 0}\int_0^L e_x^2(x, t+\theta)dx \ge \frac{1}{M}\sum_{m=1}^M \sup_{-\tau_M\le\theta\le 0}\int_{x_{m-1}}^{x_m} e_x^2(x,t+\theta)dx$$
$$\ge \frac{1}{M}\sum_{m=1}^M\int_{x_{m-1}}^{x_m} e_x^2(x, t-\tau^m)dx. \tag{31}$$

Then

$$W \triangleq \dot V(t) + 2\alpha_0 V(t) - 2\alpha_1 \sup_{-\tau_M\le\theta\le 0} V(t+\theta) - \gamma^2\Delta_q^2 L$$
$$\le \dot V(t) + 2\alpha_0 V(t) - 2\alpha_M p_3 \upsilon^2 \sum_{m=1}^M \int_{x_{m-1}}^{x_m} e_x^2(x, t-\tau^m)dx$$
$$- 2\alpha_M \sum_{m=1}^M \int_{x_m}^{x_{m+1}} \begin{bmatrix} e(x,t-\tau^m) \\ e_t(x,t-\tau^m)\end{bmatrix}^T P_0 \begin{bmatrix} e(x,t-\tau^m) \\ e_t(x,t-\tau^m)\end{bmatrix} dx$$
$$- \gamma^2 \sum_{m=1}^M \int_{x_{m-1}}^{x_m} |w_m(t-\tau^m)|^2 dx. \tag{32}$$

By Wirtinger's inequality (1) with $(b-a) = (\Delta)/2$ and $\sigma = 1/4$ we have

$$\int_{x_{m-1}}^{x_m} e_x^2(x, t-\tau^m)dx =$$
$$\sum_{m=1}^M \Big[\int_{x_{m-1}}^{\hat x_m} e_x^2(x,t-\tau^m)dx + \int_{\hat x_m}^{x_m} e_x^2(x,t-\tau^m)dx\Big]$$
$$\ge \frac{\pi^2}{\Delta^2}\sum_{m=1}^M \Big[\int_{x_{m-1}}^{\hat x_m}[e(x,t-\tau^m) - e(\hat x_m, t-\tau^m)]^2 dx$$
$$+ \int_{\hat x_m}^{x_m}[e(x,t-\tau^m) - e(\hat x_m, t-\tau^m)]^2 dx\Big]$$
$$\ge \frac{\pi^2}{\Delta^2}\sum_{m=1}^M\int_{x_{m-1}}^{x_m}[e(x,t-\tau^m) - e(\hat x_m,t-\tau^m)]^2dx$$
$$= \frac{\pi^2}{\Delta^2}\sum_{m=1}^M \int_{x_{m-1}}^{x_m}\left(\int_{\hat x_m}^x e_\zeta(\zeta, t-\tau^m)d\zeta\right)^2 dx.$$

Denote

$$\eta = [e\ e_t\ \nu_1\ \nu_2\ \int_{\hat x_m}^x e_\zeta(\zeta, t-\tau^m)d\zeta\ e_t(x, t-\tau^m)\ w_m(t-\tau^m)].$$

Then, by taking into account (19),

$$W \le 2\upsilon^2(\alpha_0 p_3 - p_2)\int_0^L e_x^2 dx + \sum_{m=1}^M \int_{x_{m-1}}^{x_m} \eta\Psi\eta^T dx \le 0,$$

if $\Psi \le 0$, where Ψ is given by (21). Note that Ψ is affine in ρ. Thus, it is sufficient to verify (21) in the vertices $\pm\rho_1$.

Thus, under (18)-(21), due to Halanay's inequality, the classical solutions of the wave equation satisfy (22). Since $\mathscr{D}(\mathscr{A})$ is dense in $\mathscr{H}$, inequality (22) remains true (by continuous extension) for the mild solutions originated in $\mathscr{H}$.

The feasibility of strict LMIs with $\alpha_0 = \alpha_1 = 0$ implies their feasibility with a slightly larger $\bar\alpha_0 = \alpha_0 + \delta > 0$, where $\delta > 0$ is small, that completes the proof.

Remark 2.1. Differently from the existing works on distributed sampled-data control under point measurements Fridman and Blighovsky (2012); Wei et al. (2019); Terushkin and Fridman (2019), we consider the point measurements (12) under the different delays τ^m that leads to more restrictive conditions via Halanay's inequality with $\alpha_M = \frac{\alpha_1}{M}$ instead of α_1 for equal delays $\tau_1 = ... = \tau^M$. Note that still it is easier to satisfy the resulting conditions for $M >> 1$ than for $M = 1$ since the stabilizing term with the coefficient $-\frac{\alpha_1}{M}p_3\upsilon^2$ in (32) after application of Wirtinger's inequality in (33) leads to

$$-\frac{\alpha_1}{M}p_3\upsilon^2\frac{\pi}{\Delta^2} \le -\frac{\alpha_1(M+1)^2}{M}p_3\upsilon^2\frac{\pi}{L^2} \underset{M\to\infty}{\Longrightarrow} -\infty.$$

2.4 Numerical simulations

In the sequel, we validate the proposed control approach in a simulation. Consider a group of $N = 49$ agents, governed by (3) with a linear $f = z$, where $\rho_1 = 1$ in (4). Our objective is deployment from initial position of Γ_0 onto a smooth open curve Γ, parameterized in the interval $[0, \pi]$

$$\Gamma_0 = \left[\sin\left(\frac{\pi}{N}i\right), \cos\left(\frac{\pi}{N}i\right), 0\right], \quad i = 1, \dots, N \tag{33}$$
$$\Gamma = \left[\sin\left(\frac{\pi}{N}i\right) + 2\cos\left(\frac{2\pi}{N}i\right), \cos\left(\frac{\pi}{N}i\right) + 2\sin\left(\frac{\pi}{N}i\right), 2 + \cos\left(\frac{2\pi}{N}i\right)\right].$$

We design a controller with the gains

$$\upsilon^2 = 4.1,\ \beta = 3,\ K = 1 + \beta^2/4. \tag{34}$$

For the linear system, LMI (21) of Theorem 2.1 is verified only in one vertex $\rho = \rho_1 = 1$. We find that the LMIs of Theorem 2.1 are feasible for $M \ge 2$ leaders. By further verifying the LMIs of Theorem 2.1 with $\alpha_0 = \alpha_1 = 0.4$, we find that the system with two leaders ($M = 2$) is ISS provided $\tau_M \le 0.52$. Note that increasing υ^2 till 5.1, results in 1 leader being sufficient for deployment if $\tau_M \le 0.29$. We further show simulations of the deployment for $M = 2$, $\tau_M = 0.52$, where the network induced delays are bounded by $\eta_k^m \le 0.02$, and quantization error is bounded by $\Delta_q = 0.01$. The agents are divided into two groups: 1) agents $z_1, ..., z_{24}$ with the leader $z_{i_1} = z_{13}$ and 2) agents $z_{25}, ..., z_{49}$ with the leader $z_{i_2} = z_{37}$. Fig. 2 depicts the transitions of a system driven by two leaders from initial (marked blue) to final (marked pink) open curves, given by (33). Trajectories of the leaders are shown in cyan, whereas the trajectories of the followers are shown in grey. Fig.

3(a) demonstrates the convergence of the error energy for each dimension. The error e for one of the dimensions is illustrated by Fig. 3(b).

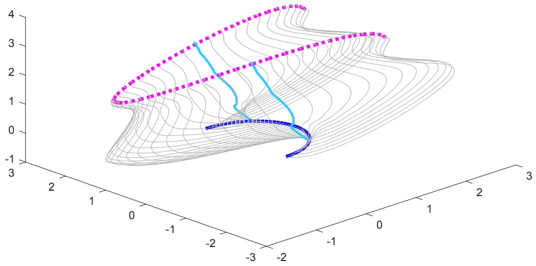
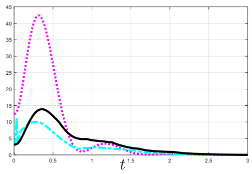
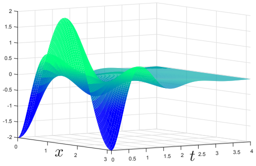


Fig. 2. Deployment of $N = 49$ agents from Γ_0 (blue) to Γ (magenta), with $M = 2$ leaders (trajectories in cyan)



(a)



(b)

Fig. 3. (a) Energy $\|e_x\|^2_{L^2} + \|e_t\|^2_{L^2}$ for each dimension (dotted - dim. 1, dashed - dim. 2, solid - dim. 3), (b) Convergence of e for dim. 1

3. CONCLUSIONS

We considered a network-based deployment of a large-scale second-order nonlinear MAS onto a smooth (open or closed) curve. Astatic output-feedback controller was designed, by employing the measurements of velocity by each agent, and of their displacements with respect to the closest neighbours, as well as the measurements of the leaders' absolute positions with respect to the curve. The leaders' positions are sent through communication network to other agents. The resulting closed-loop system was modeled as a disturbed (due to quantization) nonlinear damped wave equation with delayed point state measurements. As a by-product, for the first time for PDEs under delayed point measurements, the ISS conditions were derived by combining Lyapunov functionals with the generalized Halanay's inequality. LMI-based conditions were provided for finding the minimal number of leaders, and the maximal admissible network delays and sampling intervals that preserve the ISS.

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