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Average density detectability in traffic networks using virtual road divisions ⋆

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Abstract: In this paper, we demonstrate the existence of a reduced-order open-loop observer to estimate the average density in a region of a large scale traffic network. We show that traffic networks are not generally average detectable, but that it is possible to find a virtual representation of the network using inhomogeneous road divisions such that the observer converges to the true values. We express the conditions for the required number of cells per road and their lengths such that the system is average detectable in terms of the network's topology and physical parameters. Moreover, we propose a method to calculate these divisions and give asymptotic bounds on the quality of the approximations.

Keywords: Average detectability, traffic state estimation, large-scale networks.

1. INTRODUCTION

Accurate traffic state estimation is important in modern intelligent transportation systems to implement control strategies. For large urban networks, it is a challenging task as sensor deployment can easily surpass budgetary constraints and road-based simulations become computationally expensive as the system size increases. One solution to these problems is the use of aggregated models that consider only the trajectory of the average density inside a given region. These frameworks have been used to establish the relationship between the average density of a region with its internal flow, as in Geroliminis and Daganzo (2008), and in the development of control strategies with low computational burden, as in Geroliminis et al. (2013). Methods to identify regions in a network for which aggregated models are best applicable are discussed in Lopez et al. (2017) and Ji and Geroliminis (2012).

This paper considers the scenario of predefined regions such that sensors (e.g. magnetic loop detectors) are available at the boundaries. We consider linear dynamics, i.e., the network is either in congestion or free-flow. In this simplified case, simple observers based on commonly used techniques (see Daganzo (1995)) track the full state, i.e., the density evolution of every road, which is computationally expensive and might be unnecessary. The estimation of aggregated states using low dimensional observers has been studied in Fernando et al. (2010); Sadamoto et al. (2017); Niazi et al. (2019). The established conditions for the existence of such observers result in the concept of average detectability. These conditions rely heavily on the network topology, and are not satisfied by traffic networks in practice.

In this paper, we study the applicability of average detectability conditions to traffic networks. We devise a method to carefully divide roads into virtual sections (or cells) to obtain a network representation that satisfy the average detectability conditions.

The paper is organized as follows: in Section 2 we introduce the model for traffic state evolution and the conditions required for the existence of an observer for the average density. Section 3 shows simple motivating examples that suggest how choosing specific road divisions yields average detectable networks. Section 4 states the problem and Section 5 presents the main results on how to select road divisions for general networks. Finally, in Section 6 the proposed methods are applied to a Manhattan grid network.

2. BACKGROUND

We represent a traffic network as a weighted directed graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, R, \boldsymbol{\ell}, \mathbf{v}\}$ such that the $\mathcal{N} = \{1, 2, \ldots, p\}$ are the nodes of the graph representing roads, and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ are the edges of the graph representing turns from one road to another. The adjacency matrix $R \in \mathbb{R}^{p\times p}$ has as elements the turning ratios between roads, i.e., $r_{i,j}$ is the fraction of vehicles that turn from road i to road j. The physical parameters $\boldsymbol{\ell} \in \mathbb{R}^p$ and $\mathbf{v} \in \mathbb{R}^p$ correspond to the road lengths and maximum velocities, respectively.

Let $\boldsymbol{\rho}(t) \in \mathbb{R}^p$ denote road densities at instant t. In this paper, we model the dynamic evolution of density as a linear system, which implies that only the cases where the traffic network is fully in free-flow or fully in congestion are considered. The mixed case implies non-linear dynamics, and is part of future research. We obtain the following linear system as described in Bianchin et al. (2019),

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$$\dot{\boldsymbol{\rho}}(t) = A\boldsymbol{\rho}(t) + L^{-1}B\mathbf{u}(t) \tag{1}$$

where $A = L^{-1}(R^\top - \mathbb{I})V$ with $L = \text{diag}(\boldsymbol{\ell})$, $V = \text{diag}(\mathbf{v})$, $B \in \mathbb{R}^{p\times q}$ is a selection matrix that indicates the boundary inflows, and $\mathbf{u}(t) \in \mathbb{R}^q$ contains the input demands.

Consider that sensors are located in a set of nodes $\mathcal{S} \subset \mathcal{N}$ corresponding to the boundaries (inflows and outflows) of the network. Without loss of generality, we index roads such that measured roads have the highest indexes, i.e., $\mathcal{S} = \{p - s + 1, \ldots, p\}$ with $q < s < p$. Thus, $\mathbf{y}(t) = C\boldsymbol{\rho}(t)$ where $C = [\mathbf{0}_{s\times m} \quad \mathbb{I}_s]$, and $m = p - s$ is the number of unmeasured nodes.

Consider a partition of the state vector as $\boldsymbol{\rho}(t) = [\boldsymbol{\rho}_1^\top(t) \quad \boldsymbol{\rho}_2^\top(t)]^\top$ such that $\boldsymbol{\rho}_1(t) \in \mathbb{R}^m$ correspond to the states of the unmeasured nodes, and $\boldsymbol{\rho}_2(t) \in \mathbb{R}^s$ to the states of the measured nodes. Note that $\boldsymbol{\rho}_2(t) = \mathbf{y}(t)$. The system matrices are partitioned accordingly,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{2}$$

where $A_{11} \in \mathbb{R}^{m\times m}$, $A_{12} \in \mathbb{R}^{m\times s}$, $A_{21} \in \mathbb{R}^{s\times m}$, $A_{22} \in \mathbb{R}^{s\times s}$, and $B_1 \in \mathbb{R}^{m\times q}$ and $B_2 \in \mathbb{R}^{s\times q}$. Analogously, let

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \; L = \begin{bmatrix} L_1 & \mathbf{0} \\ \mathbf{0} & L_2 \end{bmatrix}, \; V = \begin{bmatrix} V_1 & \mathbf{0} \\ \mathbf{0} & V_2 \end{bmatrix}. \tag{3}$$

We aim to estimate the *average* of the unmeasured states, i.e., $\rho_{av}(t) = \frac{1}{m}\mathbf{1}^\top\boldsymbol{\rho}_1(t)$ without requiring knowledge about the full vector $\boldsymbol{\rho}_1(t)$. Consider a lower-dimensional projected system in which the unmeasured states are aggregated. The average state follows

$$\begin{aligned} \dot{\rho}_{av}(t) = & \frac{1}{m}\mathbf{1}^\top A_{11}\mathbf{1}\rho_{av}(t) + \frac{1}{m}\mathbf{1}^\top A_{12}\boldsymbol{\rho}_2(t) \\ & + \frac{1}{m}\mathbf{1}^\top A_{11}\boldsymbol{\sigma}(t) + \frac{1}{m}\mathbf{1}^\top B_1\mathbf{u}(t) \end{aligned} \tag{4}$$

where $\boldsymbol{\sigma}(t)$ is the average deviation vector given by $\boldsymbol{\sigma}(t) = \boldsymbol{\rho}_1(t) - \mathbf{1}\rho_{av}(t)$. The problem of determining the convergence of the open-loop observer of the form

$$\dot{\hat{\rho}}_{av}(t) = \frac{1}{m}[\mathbf{1}^\top A_{11}\mathbf{1}\hat{\rho}_{av}(t) + \mathbf{1}^\top A_{12}\mathbf{y}(t) + \mathbf{1}^\top B_1\mathbf{u}(t)] \tag{5}$$

is studied by Functional Observability, as in Fernando et al. (2010), and more specifically by Average Observability, as in Niazi et al. (2019). Using results from these domains, we introduce the following theorem.

Theorem 1. [Niazi et al. (2019)] For systems of the form (4), the following statements are equivalent:

- The system is average detectable.
- $\mathbf{1}^\top A_{11} = -\gamma\mathbf{1}^\top$ with $\gamma > 0$.
- The open loop observer (5) converges, i.e., $\hat{\rho}_{av}(t) \to \rho_{av}(t)$ as $t \to \infty$.

In the following section we show that these conditions are not generally satisfied for traffic networks. Nevertheless, we will show by dividing each road into virtual cells it is possible to construct graphs which are average detectable.

3. MOTIVATING EXAMPLES

3.1 Highway: path graph

Consider a one way road as shown in Fig. 1.a. Sensors are located at the upstream and downstream boundaries of the road, represented by green strips in the figure. Let ℓ be the length of road between the sensors, and v be the maximum velocity. Assume that it is desired to divide this stretch into 3 virtual sections (cells), such that the sum of their lengths is ℓ, and all of them have maximum velocity v. Possible divisions are shown in Figs. 1.b and 1.c.

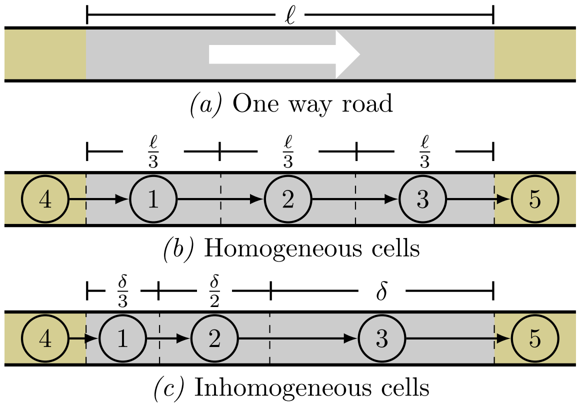

Fig. 1. One way road. Green strips represent sensors located and the upstream and downstream ends. Two different virtual divisions are shown.

First, consider the common approach of considering homogeneous cells (Fig. 1.b), such that cells 1-3 have each length $\ell/3$. The corresponding state matrix is,

$$A = \left[\begin{array}{ccc|cc} -3\ell^{-1} & 0 & 0 & 3\ell^{-1} & 0 \\ 3\ell^{-1} & -3\ell^{-1} & 0 & 0 & 0 \\ 0 & 3\ell^{-1} & -3\ell^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & -\ell_4^{-1} & 0 \\ 0 & 0 & \ell_5^{-1} & 0 & -\ell_5^{-1} \end{array}\right] v.$$

where ℓ_4 is the length of entry and ℓ_5 is the length of the exit. It can be seen that $\mathbf{1}^\top A_{11} = [0 \quad 0 \quad -3\ell^{-1}]v$. Thus, the condition $\mathbf{1}^\top A_{11} = -\gamma\mathbf{1}^\top$ from Theorem 1 is not satisfied, and thus, equal length divisions are not average detectable.

Now, consider a division such that cell 3 has length δ, cell 2 has length $\delta/2$ and cell 1 has length $\delta/3$, where $\delta = \frac{6}{11}\ell$ (see Fig. 1.c). The corresponding state matrix is

$$A = \left[\begin{array}{ccc|cc} -3\delta^{-1} & 0 & 0 & 3\delta^{-1} & 0 \\ 2\delta^{-1} & -2\delta^{-1} & 0 & 0 & 0 \\ 0 & \delta^{-1} & -\delta^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & -\ell_4^{-1} & 0 \\ 0 & 0 & \ell_5^{-1} & 0 & -\ell_5^{-1} \end{array}\right] v$$

and thus $\mathbf{1}^\top A_{11} = [-\delta^{-1} \quad -\delta^{-1} \quad -\delta^{-1}]v$. Note that all column sums are equal, and because of Theorem 1, this division is average detectable.

3.2 Circle road: networks

Consider a ring road as shown in Fig. 2. Suppose that sensors are located at the entry and the exit. Consider the graph representation in Fig. 3. The green nodes represent sensors in the network boundaries. For simplicity, we no longer index the nodes with sensors as they are not concerned with the average detectability conditions.

Denote by ℓ_1, ℓ_2, v_1, v_2 the lengths and max. velocities of the top and bottom sections of the circle, respectively. In

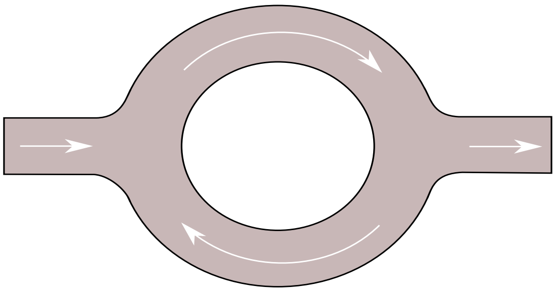

Fig. 2. Circle road with one entry and one exit.

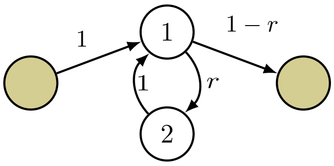

Fig. 3. Graph representation of a circle road.

this example, we are interested in expressing the required conditions for the lengths, so to simplify the writing of the equations let $v_1 = v_2 = 1$. The state matrix of the unmeasured partition is

$$A_{11} = \begin{bmatrix} -\ell_1^{-1} & \ell_1^{-1} \\ r\ell_2^{-1} & -\ell_2^{-1} \end{bmatrix}.$$

According to Theorem 1, to be able to reconstruct the average density this matrix must satisfy $\mathbf{1}^\top A_{11} = -\gamma \mathbf{1}^\top$, and therefore

$$\begin{aligned} -\ell_1^{-1} + r\ell_2^{-1} &= -\gamma \\ \ell_1^{-1} - \ell_2^{-1} &= -\gamma \end{aligned} \quad \Rightarrow \quad \ell_1 = \frac{2}{r+1}\ell_2.$$

As $r < 1$, the roads cannot be of equal length. Thus, we are interested in finding a way to modify the graph, such that physical parameters are conserved (i.e. lengths, velocities and turning ratios), but that the network is average detectable.

Let the physical lengths of the roads 1 and 2 be $\ell_1 = \ell_2 = \ell$. Consider a new graph where roads 1 and 2 are divided into n_1 and n_2 cells, respectively, as shown in Fig. 4. Let cells 1 to n_1 correspond to road 1, and cells $n_1 + 1$ to $n_1 + n_2$ correspond to road 2. Furthermore, let δ_i be the length of the i-th cell. The dimension and elements of this graph's state matrix, denoted by $A^{(n_1,n_2)}$, depend on the values of n_1, n_2 and the vector of cell lengths $\boldsymbol{\delta}$. The block matrix corresponding to the unmeasured states is

$$A_{11}^{(n_1,n_2)} = \begin{bmatrix} -\delta_1^{-1} & \delta_1^{-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\delta_2^{-1} & \delta_2^{-1} & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdots & & -\delta_{n_1}^{-1} & \delta_{n_1}^{-1} & \cdots & 0 \\ 0 & \cdots & & 0 & -\delta_{n_1+1}^{-1} & \cdots & 0 \\ \vdots & & & & & \ddots & \vdots \\ r\delta_{n_1+n_2}^{-1} & 0 & \cdots & 0 & 0 & \cdots & -\delta_{n_1+n_2}^{-1} \end{bmatrix}.$$

The average detectability condition requires $-\delta_1^{-1} + r\delta_{n_1+n_2}^{-1} = -\gamma$ and $-\delta_i^{-1} + \delta_{i-1}^{-1} = -\gamma$ for $i = 2, 3, \ldots, n_1 + n_2$. Using these equations, we can calculate section lengths as

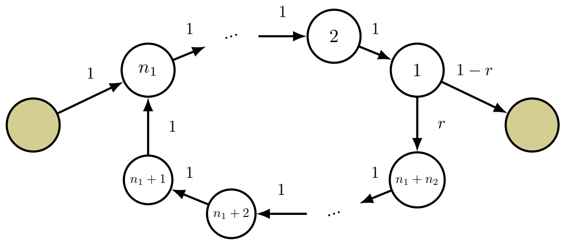

Fig. 4. Graph of a circle road with virtual partitions. Road 1 is divided into n_1 sections, and road 2 into n_2.

$$\delta_i = \frac{1}{[i + \frac{r}{1-r}(n_1 + n_2)]\gamma}. \tag{6}$$

The specific values of n_1 and n_2 must be such that the physical parameters of the network are conserved, this is,

$$\ell = \sum_{i=1}^{n_1} \delta_i = \sum_{i=1}^{n_2} \delta_{n_1+i}. \tag{7}$$

By substituting (6) into (7), we obtain

$$\sum_{i=1}^{n_1} \frac{1}{i + \frac{r}{1-r}(n_1 + n_2)} = \sum_{i=1}^{n_2} \frac{1}{i + \frac{1}{1-r}(n_1 + rn_2)}.$$

The values of n_1 and n_2 that satisfy this equation yield a network partition that is average detectable. Note that as the summands on both side of the equation are different, then it must be $n_1 \neq n_2$.

4. PROBLEM STATEMENT

In the previous section, we discussed how some simple traffic networks can be given an average detectable representation by dividing each road into several inhomogeneous virtual cells.

For a given traffic network $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, R, \boldsymbol{\ell}, \mathbf{v}\}$ and vector $\mathbf{n} \in \mathbb{N}^m$, we introduce the following notation:

Definition: Road division. Consider an arbitrary road $i \in \mathcal{N}$, and n_i the corresponding element of $\mathbf{n}$. A division of road i is a directed path graph whose nodes $\{i^{(1)}, i^{(2)}, \ldots, i^{(n_i)}\}$ are virtual cells of i. The downstream cell is denoted $i^{(1)}$, whereas $i^{(n_i)}$ denotes the upstream cell. Additionally, the length and velocity of the k-th cell of road i are denoted by $\delta_i^{(k)}$ and $v_i^{(k)}$, respectively.

Defintion: Virtual graph. A graph $\mathcal{G}^{(\mathbf{n})} = \{\mathcal{N}^{(\mathbf{n})}, \mathcal{E}^{(\mathbf{n})}, R^{(\mathbf{n})}, \boldsymbol{\delta}^{(\mathbf{n})}, \mathbf{v}^{(\mathbf{n})}\}$ is called a virtual graph of $\mathcal{G}$ according to $\mathbf{n}$ if its nodes $\mathcal{N}^{(\mathbf{n})}$ correspond to the road divisions of the nodes $\mathcal{N}$, and the adjacency matrix $R^{(\mathbf{n})}$ satisfies $r^{(\mathbf{n})}_{i^{(1)},j^{(n_j)}} = r_{i,j}$ and $r^{(\mathbf{n})}_{i^{(j)},i^{(k)}} = 1$ if $k = j - 1$. Additionally, $\boldsymbol{\delta}^{(\mathbf{n})} = [\delta_1^{(1)} \cdots \delta_1^{(n_1)}\ \delta_2^{(1)} \cdots \delta_2^{(n_2)}\ \cdots\ \delta_m^{(1)} \cdots \delta_m^{(n_m)}]^\top$ and $\mathbf{v}^{(\mathbf{n})} = [v_1^{(1)} \cdots v_1^{(n_1)}\ v_2^{(1)} \cdots v_2^{(n_2)}\ \cdots\ v_m^{(1)} \cdots v_m^{(n_m)}]^\top$.

Defintion: Admissible virtual graph. A virtual graph $\mathcal{G}^{(\mathbf{n})}$ of $\mathcal{G}$ is said to be admissible if for every road $i \in \mathcal{N}$, the velocity of any cell is equal to the velocity of the road,

$$v_i^{(k)} = v_i, \tag{8}$$

and the sum of cell lengths is equal to the length of the road,

$$\ell_i = \sum_{k=1}^{n_i} \delta_i^{(k)}. \tag{9}$$

The postulated problem is as follows: for any given traffic network $\mathcal{G}$, find a vector $\mathbf{n}$ and constant $\gamma > 0$, such that the virtual graph $\mathcal{G}^{(\mathbf{n})}$ is admissible and average detectable.

5. VIRTUAL DIVISION FOR GENERAL NETWORKS

In this section, we present the conditions required for a virtual graph to be admissible and average detectable.

Theorem 2. Let $\mathcal{G} = \{\mathcal{N}, \mathcal{E}, R, \boldsymbol{\ell}, \mathbf{v}\}$ be a given traffic network. An admissible graph $\mathcal{G}^{(\mathbf{n})} = \{\mathcal{N}^{(\mathbf{n})}, \mathcal{E}^{(\mathbf{n})}, R^{(\mathbf{n})}, \boldsymbol{\delta}^{(\mathbf{n})}, \mathbf{v}^{(\mathbf{n})}\}$ is average detectable if and only if there exist $\mathbf{n} \in \mathbb{N}^m$, $\gamma > 0$, and $\boldsymbol{\delta}^{(\mathbf{n})}$ such that

$$\delta_i^{(k)} = \frac{v_i}{(v_i \mathbf{d}_i^\top \mathbf{n} + k)\gamma} \tag{10}$$

under the constraints (8) and (9), for all $i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n_i$; where $\mathbf{d}_i^\top$ is the i-th row of $D = (\mathbb{I} - R_{11})^{-1} R_{11} V_1^{-1}$.

Proof. Let $A^{(\mathbf{n})}$ be the state matrix for the unmeasured nodes of $\mathcal{G}^{(\mathbf{n})}$, such that

$$A^{(\mathbf{n})} = \mathrm{diag}(\boldsymbol{\delta}^{(\mathbf{n})})^{-1}(R^{(\mathbf{n})} - \mathbb{I})\mathrm{diag}(\mathbf{v}^{(\mathbf{n})}). \tag{11}$$

From Theorem 1, the graph is average detectable if and only if the column sums of $A_{11}^{(\mathbf{n})}$ are equal to $-\gamma$. Consider an arbitrary cell $i^{(k)}$, with $k \neq 1$ such that its downstream neighbor is cell $i^{(k-1)}$. The column sum of $A_{11}^{(\mathbf{n})}$ corresponding to this cell is

$$-\frac{v_i}{\delta_i^{(k)}} + \frac{v_i}{\delta_i^{(k-1)}} = -\gamma.$$

where we imposed the condition $v_i^{(k)} = v_i$. By induction, we can calculate the length of each cell from $\delta_i^{(1)}$,

$$\frac{1}{\delta_i^{(k)}} = \frac{1}{\delta_i^{(1)}} + \frac{k-1}{v_i}\gamma. \tag{12}$$

Cell $i^{(1)}$ has as out-neighbors all cells $j^{(n_j)}$ such that $(i, j) \in \mathcal{E}$. Thus, its corresponding column sum is,

$$-\frac{v_i}{\delta_i^{(1)}} + \sum_{j=1}^{m} \frac{r_{ij} v_i}{\delta_j^{(n_j)}} = -\gamma. \tag{13}$$

Define $\boldsymbol{\delta}_{(1)}^{-1} = [1/\delta_1^{(1)} \quad 1/\delta_2^{(1)} \quad \cdots \quad 1/\delta_m^{(1)}]$. By substituting (12) into (13), we obtain a system of linear equations,

$$(\mathbb{I} - R_{11})\boldsymbol{\delta}_{(1)}^{-1} = \gamma[R_{11} V_1^{-1}\mathbf{n} + (\mathbb{I} - R_{11})V_1^{-1}\mathbf{1}] \tag{14}$$

Thus,

$$\frac{1}{\delta_i^{(1)}} = \gamma\left(\mathbf{d}_i^\top \mathbf{n} + \frac{1}{v_i}\right) \tag{15}$$

are the solutions to (14) for the downstream cells of each road i. Substitution of (15) into (12) gives (10). □

5.1 Approximate solutions

Consider a virtual graph whose cell lengths are calculated according to (10). Define

$$f_i(\mathbf{n}, \gamma) = \ell_i - \frac{v_i}{\gamma} \sum_{k=1}^{n_i} \frac{1}{v_i \mathbf{d}_i^\top \mathbf{n} + k}. \tag{16}$$

such that it corresponds to the error in (9), i.e., the error between the sum of cell lengths and the length of road i. Thus, the problem of finding an average detectable and admissible division of a given graph is equivalent to finding a vector of integers $\mathbf{n}$ and a constant γ such that $f_i(\mathbf{n}, \gamma) = 0$ for all $i = 1, \ldots m$. However, this is difficult in practice, as it is a combinatorial problem. As a simplification, we can search for solutions that satisfy the constraints approximatively, that is, to find $\mathbf{n}$ and γ such that $|f_i(\mathbf{n}, \gamma)|$ is small.

In the following theorems, we propose an alternative system of equations used to calculate $\mathbf{n}$ and γ. To do this, we allow the values of $\mathbf{n}$ to take real (instead of only integer) values. Then, we approximate the sum in $f_i(\mathbf{n}, \gamma)$ using the natural logarithm. This results in a system of equations that is simpler to solve, but that results in approximation error. However, we show that this error is bounded and can be reduced by selecting different values of γ.

Theorem 3. Consider any given traffic network $\mathcal{G}$. Let $\mathbf{x} \in \mathbb{R}^m$ and $\gamma > 0$ such that,

$$[(K_\gamma - \mathbb{I})^{-1} K_\gamma - V(\mathbb{I} - R_{11})^{-1} V_1^{-1}]\mathbf{x} = \frac{1}{2}\mathbf{1}, \tag{17}$$

where $K_\gamma = \mathrm{diag}([e^{\gamma \ell_1 / v_1} \quad e^{\gamma \ell_2 / v_2} \quad \cdots \quad e^{\gamma \ell_m / v_m}])$. Let $\lfloor \cdot \rceil$ denote the nearest integer function. Then, $\mathbf{n} = \lfloor \mathbf{x} \rceil$ and γ satisfy

$$|f_i(\mathbf{n}, \gamma)| \sim O\left((v_i \mathbf{d}_i^\top \mathbf{n} + 1)^{-1}\right) \tag{18}$$

for $i = 1, 2, \ldots, m$.

Proof. Let ψ be the digamma function. Its definition and a list of properties can be found in Abramowitz and Stegun (1972). This function satisfies the following identity,

$$\sum_{k=1}^{n} \frac{1}{z+k} = \psi(z+n+1) - \psi(z+1).$$

Therefore, with $z = v_i \mathbf{d}_i^\top \mathbf{n}$, (16) can be rewritten as

$$f_i(\mathbf{n}, \gamma) = \ell_i - \frac{v_i}{\gamma}\left[\psi\left(v_i \mathbf{d}_i^\top \mathbf{n} + n_i + 1\right) - \psi\left(v_i \mathbf{d}_i^\top \mathbf{n} + 1\right)\right]. \tag{19}$$

Define $\epsilon(z) = \psi(z) - \ln(z - \frac{1}{2})$. It is known that for $z > \frac{1}{2}$, $\epsilon(z)$ is positive and monotonically decreasing. Furthermore, its asymptotic expansion is $\epsilon(z) = \frac{z^{-2}}{24} + \frac{z^{-3}}{24} + \ldots$ as $z \to \infty$. Thus, (19) becomes

$$f_i(\mathbf{n}, \gamma) = \ell_i - \frac{v_i}{\gamma}\left[\ln\left(v_i \mathbf{d}_i^\top \mathbf{n} + n_i + \tfrac{1}{2}\right) - \ln\left(v_i \mathbf{d}_i^\top \mathbf{n} + \tfrac{1}{2}\right) + \Delta_i(\mathbf{n})\right], \tag{20}$$

where $\Delta_i(\mathbf{n}) = \epsilon\left(v_i \mathbf{d}_i^\top \mathbf{n} + 1\right) - \epsilon\left(v_i \mathbf{d}_i^\top \mathbf{n} + n_i + 1\right)$, is the total error due to this approximation.

Using the Taylor expansion of the logarithm, it can be shown that for any non-negative vector $\mathbf{a}$ and $c > 0$, $\ln\left(\mathbf{a}^\top \lfloor \mathbf{x} \rceil + c\right) - \ln\left(\mathbf{a}^\top \mathbf{x} + c\right)$ is equal to

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left[\frac{\mathbf{a}^\top(\lfloor \mathbf{x} \rceil - \mathbf{x})}{\mathbf{a}^\top \mathbf{x} + c}\right]^k \sim O\left(\frac{1}{\mathbf{a}^\top \lfloor \mathbf{x} \rceil + 1}\right).$$

Thus, we can rewrite (20) as

$$f_i(\mathbf{n}, \gamma) = \ell_i - \frac{v_i}{\gamma}\left[\ln\left(v_i \mathbf{d}_i^\top \mathbf{x} + x_i + \tfrac{1}{2}\right) - \ln\left(v_i \mathbf{d}_i^\top \mathbf{x} + \tfrac{1}{2}\right) + \Delta_i(\mathbf{n}) + \eta_i(\mathbf{x})\right], \tag{21}$$

where $\eta_i(\mathbf{x})$ is the rounding error.

Now, consider the equation

$$0 = \ell_i - \frac{v_i}{\gamma}\left[\ln\left(v_i\mathbf{d}_i^\top\mathbf{x} + x_i + \tfrac{1}{2}\right) - \ln\left(v_i\mathbf{d}_i^\top\mathbf{x} + \tfrac{1}{2}\right)\right] \quad (22)$$

Using logarithm identities, this becomes

$$\gamma\frac{\ell_i}{v_i} = \ln\left(\frac{v_i\mathbf{d}_i^\top\mathbf{x} + x_i + \frac{1}{2}}{v_i\mathbf{d}_i^\top\mathbf{x} + \frac{1}{2}}\right),$$

which can be written as $x_i - (e^{\gamma\ell_i/v_i} - 1)v_i\mathbf{d}_i^\top\mathbf{x} = \frac{1}{2}(e^{\gamma\ell_i/v_i} - 1)$. Thus, we obtain a system of m equations,

$$[\mathbb{I} - (K_\gamma - \mathbb{I})V_1D]\mathbf{x} = \frac{1}{2}(K_\gamma - \mathbb{I})\mathbf{1}. \quad (23)$$

Substituting the expression for D into (23) and rearranging terms we obtain (17), and thus, (22) is satisfied for the considered $\mathbf{x}$ and γ. Substituting (22) into (21), we get

$$|f_i(\mathbf{n}, \gamma)| = \frac{v_i}{\gamma}|\Delta_i(\mathbf{n}) + \eta_i(\mathbf{x})| \quad (24)$$

Note that $|\Delta(\mathbf{n})| < \epsilon(v_i\mathbf{d}_i^\top\mathbf{n} + 1)$, and so $\Delta(\mathbf{n}) \sim O[(v_i\mathbf{d}_i^\top\mathbf{n}+1)^{-2}]$. Additionally, $\eta_i(\mathbf{x}) \sim O[(v_i\mathbf{d}_i^\top\mathbf{n}+1)^{-1}]$.

Thus $|f_i(\mathbf{n}, \gamma)| \sim O\left((v_i\mathbf{d}_i^\top\mathbf{n} + 1)^{-1}\right)$, completing the proof. □

Theorem 4. There exists γ_{max} such that for every $0 < \gamma < \gamma_{max}$, the solution to (17) is positive. Moreover, as γ approaches γ_{max} the magnitude of $\mathbf{x}$ grows arbitrarily big.

Proof. Let $M = (K_\gamma - \mathbb{I})^{-1}K_\gamma - V_1(\mathbb{I} - R_{11})^{-1}V_1^{-1}$. Assume that M is invertible. Using Woodbury's identity, we can write M^{-1} as

$$(\mathbb{I} - K_\gamma^{-1}) + (\mathbb{I} - K_\gamma^{-1})V_1K_\gamma(\mathbb{I} - R_{11}K_\gamma)^{-1}V_1^{-1}(\mathbb{I} - K_\gamma^{-1}).$$

Therefore, M is invertible only if $\mathbb{I} - R_{11}K_\gamma$ is invertible.

Let $\lambda(R_{11})$ denote the spectral radius of R_{11}. It can be shown that $\mathbb{I} - R_{11}$ is an invertible M-matrix, Rodriguez-Vega et al. (2019), and thus, $\lambda(R_{11}) < 1$. For sufficiently small γ, K_γ can be made arbitrarily close to $\mathbb{I}$, such that $\lambda(R_{11}K_\gamma) < 1$.

Let γ_{max} be such that $\lambda(R_{11}K_{\gamma_{max}}) = 1$. Thus, for every $\gamma < \gamma_{max}$, $\mathbb{I} - R_{11}K_\gamma$ is an invertible M-matrix such that $(\mathbb{I} - R_{11}K_\gamma)^{-1} = \mathbb{I} + \sum_{k=1}^{\infty}(R_{11}K_\gamma)^k$. As $\gamma \to \gamma_{max}$, the nonzero elements of $(R_{11}K_\gamma)^k$ increase exponentially. For $\gamma = \gamma_{max}$, the sum diverges and the matrix is not invertible. Finally, for $0 < \gamma < \gamma_{max}$, $(\mathbb{I} - R_{11}K_\gamma)^{-1}$ and $(\mathbb{I} - K_\gamma^{-1})$ are non-negative, which implies that M^{-1} is non-negative. □

6. SIMULATED EXAMPLE AND RESULTS

Consider the example traffic network shown in Fig. 5, which corresponds to the line-graph for a Manhattan Grid of 4×4 intersections. Assume that all roads have the same length of $\ell = 500$m, and the same free-flow velocity of $v = 30$ km·h^{-1}. As all speeds and lengths are equal, $K_\gamma = \exp(\gamma\frac{\ell}{v})\mathbb{I}$, and $\gamma_{max} = -(v/\ell)\ln[\lambda(R_{11})]$.

It was shown in the previous section that for a given value of $\gamma \in (0, \gamma_{max})$, there exists one vector $\mathbf{n}$ that is an approximate solution to the problem. Figure 7-Left shows $\mathbf{n}$ that solves (17) for different values of γ. For

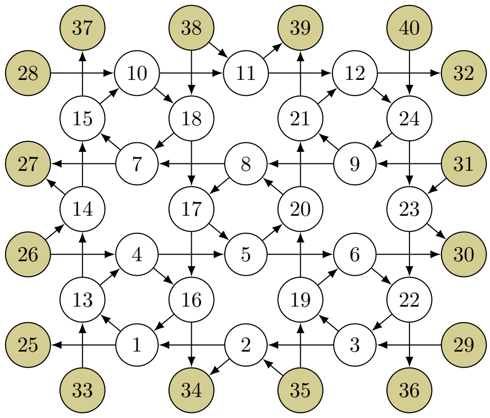

Fig. 5. Line-graph for a 4×4 Manhattan grid (nodes correspond to roads). All turning ratios are set to 50%. Green nodes symbolize sensor locations.

$\gamma < 0.88\gamma_{max}$, the elements of $\mathbf{n}$ are all 1. As γ approaches the maximum value, $\mathbf{n}$ increases quickly.

However, using (17) to calculate $\mathbf{n}$ induces an error $f_i(\mathbf{n}, \gamma)$ in the admissibility constraint. Consider the total root mean square error (RMSE) for all roads $i = 1, \ldots, m$, as shown in Fig. 7-Middle. As γ increases, the upper limit of this error decreases, approaching 0 as $\gamma \to \gamma_{max}$. This is because the number of cells per road is also increasing rapidly, so Theorem 3 is applicable. In this sense, the lowest error is obtained by choosing γ very close to γ_{max}.

Consider the observer $\dot{\hat{\rho}}_{av}(t) = -\gamma\hat{\rho}_{av}(t) + \mathbf{b}^\top\mathbf{y}(t)$ where $\mathbf{b}^\top = \frac{1}{\mathbf{1}^\top\mathbf{n}}\mathbf{1}^\top A_{12}^{(\mathbf{n})}$. It can be shown that the elements of this vector are given by

$$b_i = \frac{1}{\mathbf{1}^\top\mathbf{n}}\sum_{j=1}^{m} r_{i,j}\frac{v_i}{\delta_j^{(n_j)}}.$$

Note that even if the dimension of $A_{12}^{(\mathbf{n})}$ grows with $\mathbf{n}$, the entries of $\mathbf{b}^\top$ are easily computed. As γ increases, $\mathbf{n}$ goes to infinity, but the lengths $\boldsymbol{\delta}^{(\mathbf{n})}$ go to 0. Figure 7-Right shows the values of $\mathbf{b}^\top$ as a function of γ. For the considered network, the limit of $\mathbf{b}^\top$ as $\gamma \to \gamma_{max}$ exists.

As a specific case, let $\gamma = 0.95\gamma_{max}$, which corresponds to a vector $\mathbf{n}$ with elements 2, 3 and 4, and a RMS error below 3%. The corresponding virtual graph is shown in Fig. 6. Using the virtual graph as an input, we performed a

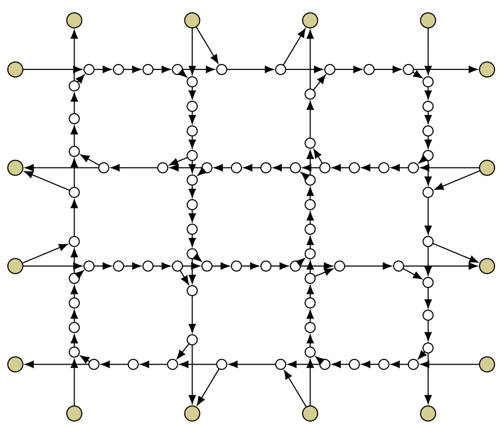

Fig. 6. Virtual graph using $\gamma = 0.95\gamma_{max}$.

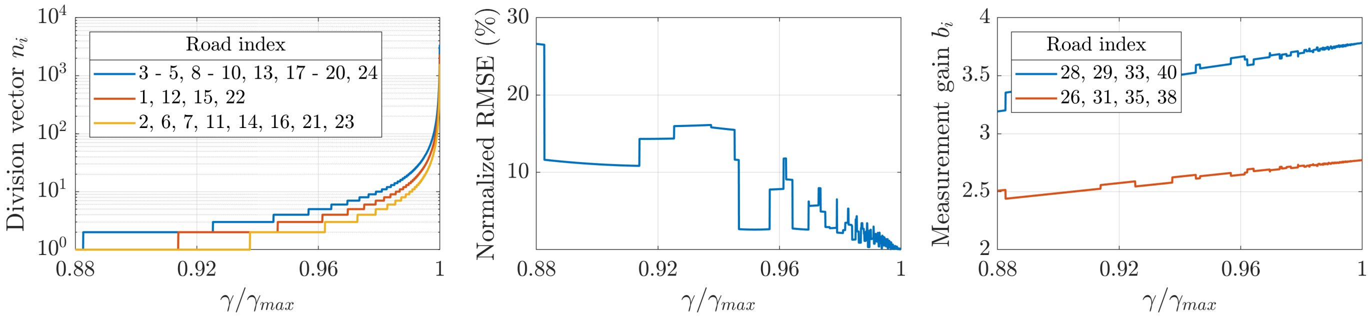

Fig. 7. **Left:** Approximate solutions of (17) for $\mathbf{n}$ for all unmeasured nodes. **Middle:** Normalized root mean square error, $\sqrt{\sum_i f_i(\mathbf{n}, \gamma)^2}/\ell$. **Right:** Measurement gain for the open-loop observer.

simulation using random initial conditions, and sinusoidal inputs with additive noise. The trajectory of the real average density is shown in blue in Fig. 8.

Using the measurements $\mathbf{y}$ as an input, we used the open-loop observer to estimate the average density. The trajectory of the estimate is shown in red in Fig. 8. This observer converges to the real solution as expected. Note that to deploy the observer, the virtual graph of Fig. 6 is not needed; only γ, $\mathbf{n}$, and the lengths of the upstream cells $\delta_j^{(n_j)}$ are required, and can be calculated off-line. Therefore, the on-line deployment of the observer requires little computational power and is applicable for large-scale networks.

The computational cost of the off-line calculations consist of a matrix inversion for each γ, which require $O(p^3)$ operations. In addition, γ needs to be iterated to obtain the desired precision. To find the value of γ_{max} we can use the fact that $\mathbf{n}$ (and its rate of change) is non-decreasing for $\gamma < \gamma_{max}$, and has negative values for $\gamma > \gamma_{max}$. Hence, a modified version of Newton's algorithm can be used to find this value. More efficient methods will be studied in future work.

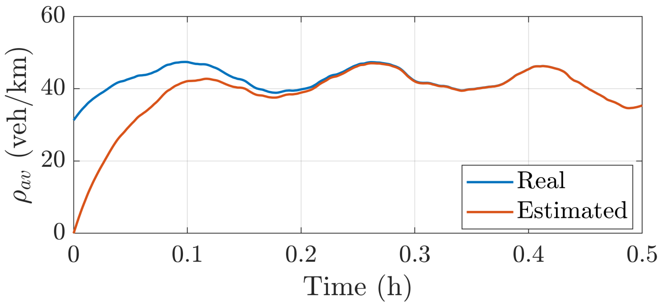

Fig. 8. Real and estimated average density.

7. CONCLUDING REMARKS

In this work, we propose a strategy to modify a given traffic network by dividing each road into cells, such that the modified system is average detectable, i.e., there exists a one-dimensional open loop observer that estimates the system's average density. The strategy consists on calculating the number of divisions per road, and the length of each cell. Exact conditions for these variables were found, as well as a procedure to find approximate solutions. These techniques have as degree of freedom the observer gain γ, for which an upper bound was found.

With simulations we show that as γ approaches the upper bound, the approximation error for satisfying the conditions approaches zero, at the cost of incrementing the number of divisions per road. Nevertheless, in the limit, the observer remains well defined. Also, we show that the modified graph is not required to the deployment of the observer: only the number of cells and their lengths are needed.

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