Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Optimal Quantum Realization of a Classical Linear System

Rebbecca TY, Thien * **Shanon L., Vuglar** ** **Ian R., Petersen** ***

* *Research School of Electrical, Energy and Materials Engineering, The Australian National University, Canberra ACT 2601, Australia (e-mail: rebbecca.thien@anu.edu.au)*
** *Division of Engineering and Construction Management, John Brown University, Siloam Springs, AR 72761, United States of America (e-mail: shanonvuglar@gmail.com)*
*** *Research School of Electrical, Energy and Materials Engineering, The Australian National University, Canberra ACT 2601, Australia (e-mail: i.r.petersen@gmail.com)*

Abstract: Additional noise in a quantum system can be detrimental to the performance of a quantum coherent feedback control system. This paper proposes a Linear Matrix Inequality (LMI) approach to construct an optimal quantum realization of a given Linear Time Invariant (LTI) system. The quantum realization problem is useful in designing coherent quantum feedback controllers. An optimal method is proposed for solving this problem in terms of a finite horizon quadratic performance index, which is related to the amount of quantum noise appearing at the system's output. This cost function provides a measure of how much the additional quantum noise in the coherent controller will alter the feedback control system.

Keywords: Quantum system, quantum noise, physical realizability, linear matrix inequality, optimization, quadratic performance index.

1. INTRODUCTION

A quantum system is a system whose dynamics are described by the laws of quantum mechanics rather than classical mechanics. Specifically, in the Heisenberg picture of quantum mechanics the dynamics of observables (e.g., energy, position, momentum) are studied rather than the quantum states; e.g. see [Gennaro et al. (2009)]. Quantum linear systems are a class of quantum systems whose dynamics, take the specific form of a set of linear quantum stochastic differential equations (QSDEs). Such linear quantum systems are common to the area of quantum optics [Bachor and Ralph (2004); Gardiner and Zoller (2004); Walls and Milburn (2008)].

In general, a set of linear QSDEs need not correspond to a physically meaningful quantum system - it must satisfy additional constraints to represent a physical quantum system. The laws of quantum mechanics dictate that quantum systems evolve unitarily, implying that (in the Heisenberg picture) certain canonical commutation relations (CCR) are satisfied at all times. The notion of a physically realizable quantum linear stochastic system can be seen in [James et al. (2008)] where the authors also derive a necessary and sufficient characterization for such systems. The feedback control of quantum systems has attracted considerable interest in recent years [Wang and James (2015b,a); James and Nurdin (2015); Roy et al. (2017); Liu et al. (2019)] where the control system is designed to achieve closed loop properties such as stability, robustness, and entanglement. An important type of feedback control in which the feedback controller is itself a quantum system is referred as coherent quantum control [James et al. (2008); Nurdin et al. (2009); Vuglar (2015); Vuglar and Petersen (2017)]; see Figure 1. The use of this type of controller may lead to improved control system performance and may be preferable because of considerations of controller bandwidth and ease of implementation.

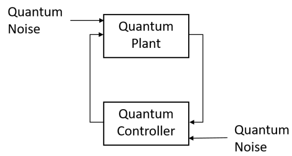

Fig. 1. Coherent Feedback Control Block Diagram

In [Vuglar and Petersen (2017)], the authors considered the minimum number of additional quantum noises needed to make a given, strictly proper, LTI system physically realizable. This system would correspond to the quantum controller in Figure 1 in which an LTI controller is first designed via a classical method such as LQG control and then additional quantum noises are added to make the controller physically realizable. In this work, we extend the results of [Vuglar and Petersen (2017)] by focusing on

⋆ This work was supported by the Australian Research Council under grant DP180101805. It was also supported by the Airforce Office of Scientific Research and the Office of Naval Research Global under agreement number FA2386-16-14065.

the extent to which the additional quantum noise affects the system output. This involves minimizing a specific performance index, related to the covariance of the system output. The difference between our work and [Vuglar and Petersen (2017)] is the authors in [Vuglar and Petersen (2017)] concentrate on minimizing the number of noises which enter the system (the dimension of the noise signal, v_2) whereas in this paper, we focus on evaluating the noise effect on the output signal y; see Figure 2.

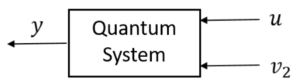

Fig. 2. Quantum System Block Diagram

Since our method involves minimizing a specified cost function, this cost function will be lower as compared to the previous method in [Vuglar and Petersen (2017)]. The main contribution of this work is twofold. First, we formulate an optimal physical realization problem and obtain an LMI solution. Second, we show that using our method, the performance of the system improves in terms of the specified cost function while maintaining physical realizability. Also, the algorithm proposed in this paper is simpler and gives better performance than the method in [Vuglar and Petersen (2017)].

The remainder of the paper proceeds as follows. In Section 2, we describe the quantum system model used, define physical realizability, and outline related previous results. Then, in Sections 3 and 4, we formulate our problem and present our algorithm for its solution, respectively. Two examples are given in Section 5 followed by a conclusion in Section 6.

2. BACKGROUND AND PREVIOUS RESULTS

2.1 Quantum Systems

In the Heisenberg picture of quantum mechanics, the dynamics of a quantum system are described by time-dependent operators acting on an appropriate Hilbert space. An important class of such systems can be described by the following linear quantum stochastic differential equations (LQSDE) [Hudson and Parthasarathy (1984); Belavkin (1992); Bouten et al. (2007); James et al. (2008)]

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t) \\ dy(t) &= Cx(t)dt + Ddw(t) \end{aligned} \tag{1}$$

where A, B, C and D are real matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n_y\times n}$ and $\mathbb{R}^{n_y\times n_w}$ (n, n_w, n_y are positive integers), respectively. Moreover, $x(t) = [x_1(t)...x_n(t)]$ is a column vector of self-adjoint, possibly non-commutative, system variables.

In order to represent the dynamics of a physically meaningful quantum system, equations (1) must also preserve certain *commutation relations* as follows:

$$[x_j(t), x_k(t)] = x_j(t)x_k(t) - x_k(t)x_j(t) = 2i\Theta_{jk} \tag{2}$$

where Θ is a real skew-symmetric matrix with components Θ_{jk} where $j, k = 1, ..., n$ and $i = \sqrt{-1}$.

The following theorem from [James et al. (2008)] provides an algebraic characterization of when the system (1) satisfies (2).

Theorem 1. (James et al. (2008)). The system in (1) will satisfy the *commutation relations* (2) for all $t \geq 0$ if and only if

$$iA\Theta + i\Theta A^T + BT_{\tilde{w}}B^T = 0. \tag{3}$$

The *commutation relations* (2) are said to be *canonical* (i.e., the system is fully quantum) if $\Theta = diag(J, J, ..., J)$ where J denotes the real skew-symmetric 2×2 matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the "*diag*" notation indicates a block diagonal matrix assembled from the given entries.

The vector quantity w describes the input signals and is assumed to admit the decomposition

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t)$$

where the self-adjoint, adapted process $\beta_w(t)$ is the signal part of $dw(t)$ and $d\tilde{w}$ is the noise part of $dw(t)$ [Hudson and Parthasarathy (1984); Parthasarathy (1992); Belavkin (1992)].The noise $\tilde{w}(t)$ is a vector of self-adjoint quantum noises with Ito table

$$d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}}dt$$

where $F_{\tilde{w}} = S_{\tilde{w}} + T_{\tilde{w}}$ is a nonnegative Hermitian matrix [Belavkin (1992); Parthasarathy (1992)] with $S_{\tilde{w}}$ and $T_{\tilde{w}}$ are real and imaginary, respectively.

In this work, we consider a special case of (1)

$$\begin{aligned} &dx(t) = Ax(t)dt + B_u du(t) + B_{v_1}dv_1(t) + B_{v_2}dv_2(t); \\ &dy(t) = Cx(t)dt + dv_1(t); \end{aligned} \tag{4}$$

see also [James et al. (2008); Vuglar and Petersen (2017)]. Here, $dw(t)$ from (1) has been partitioned into signal inputs, $du(t)$, direct feed through quantum vacuum noise inputs, $dv_1(t)$, and additional quantum vacuum noises, $dv_2(t)$.

2.2 Physically Realizability

In [James et al. (2008)], the authors introduced a rigorous notion of physical realizability based around the concept of an open quantum harmonic oscillator as the basic unit of a physically realizable quantum system. The following formally defines physical realizability.

Definition 1. The system (1) is said to be physically realizable if Θ is canonical and there exists a quadratic Hamiltonian operator $\mathcal{H} = (1/2)x(0)^T Rx(0)$, where R is real, symmetric, $n\times n$ matrix, and a coupling operator $\mathcal{L} = \Lambda x(0)$, where Λ is a complex-valued $(1/2)n_w \times n$ coupling matrix such that matrices $A, B, C,$ and D are given by

$$A = 2\Theta(R + \Im(\Lambda^\dagger\Lambda)) \tag{5a}$$

$$B = 2i\Theta[-\Lambda^\dagger \quad \Lambda^T]\Gamma \tag{5b}$$

$$C = P^T \begin{bmatrix} \Sigma_{n_y} & 0 \\ 0 & \Sigma_{n_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{bmatrix} \tag{5c}$$

$$D = [I_{n_y\times n_y} \quad 0_{n_y\times(n_w-n_y)}]. \tag{5d}$$

where

$$\Gamma = P_{N_w} diag_{N_w}(M);$$

$$M = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix};$$

$$\Sigma_{N_y} = [I_{N_y\times N_y} \quad 0_{N_y\times(N_w-N_y)}];$$

$$P_{N_w}(a_1, a_2, ..., a_{2N_w})^T = (a_1, ..., a_{2N_w-1}, a_2, ..., a_{2N_w})^T;$$

and diag(M) is an appropriately dimensioned square block diagonal matrix with each diagonal block equal to the matrix M. Note that the permutation matrix P has the unitary property $PP^T = P^T P = I$ and $N_w = (n_w/2)$ and $N_y = (n_y/2)$.

The following theorem [James et al. (2008)] gives necessary and sufficient conditions for physical realizability.

Theorem 2. [James et al. (2008)] The system (1) is physically realizable if and only if

$$iA\Theta + i\Theta A^T + BT_w B^T = 0 \tag{6a}$$

$$B \begin{bmatrix} I_{n_y \times n_y} \\ 0_{(n_w - n_y)\times n_y} \end{bmatrix} = \Theta C^T diag_{N_y}(J) \tag{6b}$$

and D satisfies

$$D = [I_{n_y \times n_y} \quad 0_{n_y \times (n_w - n_y)}].$$

The corresponding Hamiltonian and coupling matrices have explicit expressions as follows: the Hamiltonian matrix R

$$R = \frac{1}{4}(-\Theta A + A^T \Theta),$$

and the coupling matrix Λ

$$\Lambda = -\frac{1}{2} i [0_{N_w \times N_w} \quad I_{N_w \times N_w}](\Gamma^{-1})^T B^T \Theta.$$

2.3 Previous Results

The authors in [James et al. (2008)] considered the issue of physical realizability where necessary and sufficient conditions were derived for given controller state space matrices to be physical realizable as shown in **Section 2.2**. Particularly, the following theorem relating to physical realizability was proved.

Theorem 3. (James et al. (2008)). Let A, B_w, and C be real matrices in $\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n_w}$, and $\mathbb{R}^{n_y \times n}$ (n, n_w, n_y are positive integers), respectively. Also, let $F_{\tilde{u}}$ and Θ be *canonical*. Then, there exists an even integer $n_{v_2} \geq 0$ and matrices $B_{v_1} \in \mathbb{R}^{n\times n_{v_1}}, B_{v_2} \in \mathbb{R}^{n \times n_{v_2}}$, such that the corresponding system (4) is physically realizable.

This work was extended in [Vuglar and Petersen (2017)] where the minimum number of additional quantum noises n_{v_2} to make a system physically realizable was addressed through the following theorem:

Theorem 4. (Vuglar and Petersen (2017)). A system with given A, B_u and C matrices is considered. Then, there exist matrices B_{v_1} and B_{v_2} such that the system is physically realizable with $n_{v_2} = r$ (sufficient condition) and $n_{v_2} \geq r$ (necessary condition) where r is the rank of the matrix $\Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T \Theta - C^T \Theta_{n_y} C$.

In [James et al. (2008); Vuglar and Petersen (2017)], the matrices R, Λ, B_{v_1}, and B_{v_2} in (5) are constructed as follows by applying **Theorem 2** to the system (4)

$$R = -\frac{1}{4}(\Theta A + A^T \Theta^T); \tag{7a}$$

$$\Lambda = [\frac{1}{2} C^T P^T \begin{bmatrix} I \\ iI \end{bmatrix} \Lambda_{b_1}^T \quad \Lambda_{b_2}^T]^T; \tag{7b}$$

$$B_{v_1} = \Theta C^T \mathrm{diag}(J); \tag{7c}$$

$$B_{v_2} = 2i\Theta[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T] P \mathrm{diag}(M). \tag{7d}$$

Λ_{b_1} is any complex $(1/2)n_{v_2} \times n$ matrix such that

$$\Lambda_{b_1}^\dagger \Lambda_{b_1} = \Xi_1 + i\left(\frac{A^T\Theta^T - \Theta A}{4} - \frac{1}{4} C^T P^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} PC - \Im(\Lambda_{b_2}^\dagger \Lambda_{b_2})\right), \tag{8}$$

and

$$\Lambda_{b_2} = -i[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T] \; P\mathrm{diag}(M).$$

The matrix Ξ_1 in (8) is defined in [James et al. (2008); Vuglar and Petersen (2017)] as any real symmetric $n \times n$ matrix such that $\Lambda_{b_1}^\dagger \Lambda_{b_1}$ is nonnegative definite. According to [Vuglar and Petersen (2017)], equation (8) can be rewritten as

$$\Xi_2 = \Xi_1 + \frac{i}{4}\tilde{S} \geq 0 \tag{9}$$

where

$$\tilde{S} = \Theta B_u \Theta_{n_u} B_u^T \Theta - \Theta A - A^T \Theta - C^T \Theta_{n_y} C \tag{10}$$

and $\tilde{S}$ is real and skew symmetric which leads to

$$S = \frac{i}{4}\tilde{S} \tag{11}$$

being Hermitian.

Our main interest in this paper is to choose B_{v_2} to minimize the effect of the noise v_2 on the system's output. Therefore, B_{v_2} is chosen to minimize the associated finite horizon quadratic performance index which is given as

$$J_{tf}(t_f) = \int_0^{t_f} \langle (Cx)^T(t) Cx(t) \rangle dt \tag{12}$$

for the system (4). Here the term $Cx(t)$ in the system output in (4) is to be taken as the performance variable through which the cost function that will be optimized is defined. In [Shaiju et al. (2007)], the authors presented lemmas which provide a crucial link between this quadratic cost functional and the system matrices.

Lemma 5. (Shaiju et al. (2007)). The cost (12) can be expressed as

$$J_{tf}(t_f) = \int_0^{t_f} Tr(CQ(t)C^T) dt \tag{13}$$

where $Q(t)$ is defined as

$$Q(t) = \frac{1}{2}\langle (x(t)x^T(t) + (x(t)x^T(t))^T \rangle. \tag{14}$$

Indeed, using equation (14), equation (13) follows from

$$\begin{aligned} \langle (Cx)^T Cx \rangle &= \langle Tr((Cx)^T Cx) \rangle; \\ &= \langle Tr((x^T C^T Cx) \rangle; \\ &= \frac{1}{2}\langle Tr(C^T C[xx^T + (xx^T)^T]) \rangle; \\ &= Tr(C^T CQ); \\ &= Tr(CQC^T). \end{aligned}$$

Using the quantum Ito rule,

$$\begin{aligned} dQ(t) &= \frac{1}{2}(\langle (dx(t)x^T(t) \rangle + \langle (dx(t)x^T(t))^T \rangle) \\ &\quad + \langle x(t)dx^T(t) \rangle + \langle (x(t)dx^T(t))^T \rangle \\ &\quad + (B_{v_2} F_{v_2} B_{v_2}^T + (B_{v_2} F_{v_2} B_{v_2}^T)^T) dt; \\ &= (AQ(t) + Q(t)A^T + \frac{1}{2} B_{v_2}(F_{v_2} + F_{v_2}^T) B_{v_2}^T) dt; \\ &= (AQ(t) + Q(t)A^T + B_{v_2} B_{v_2}^T) \end{aligned}$$

where $\frac{1}{2}(F_{v_2} + F_{v_2}^T) = I$; i.e., all noises are canonical. Therefore, $Q(t)$ defined in equation (14) satisfies the differential equation

$$\dot{Q}(t) = AQ(t) + Q(t)A^T + B_{v_2}B_{v_2}^T. \tag{15}$$

We consider the infinite horizon case in which we define

$$\lim_{t_f \to \infty} Q(t) = Q; \tag{16}$$

and therefore

$$\limsup_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} Tr(CQ(t)C^T)dt = Tr(CQC^T) = J_{cf}. \tag{17}$$

Then equation (15) implies

$$A^TQ + QA + B_{v_2}B_{v_2}^T = 0. \tag{18}$$

In our work, we are interested in investigating the performance of the system by looking at the defined cost function while maintaining physical realizability. This differs from [Vuglar and Petersen (2017)] where the authors developed an algorithm to obtain a physically realizable system with a minimal number of additional quantum noises.

3. PROBLEM FORMULATION

By looking at equations (7) which follow from the requirement of physical realizability, it is straightforward to see that B_{v_1} is a fixed matrix that cannot be modified. Also, B_u from equations (4) is a fixed matrix and $dv_1(t)$ is the direct feed through quantum vacuum noise which therefore also cannot be avoided. Consequently, in choosing the matrix B_{v_2} we will concentrate only on the effect of the noise v_2 on the system output term Cx. Therefore, we consider the system

$$\begin{aligned} dx(t) &= Ax(t)dt + B_{v_2}dv_2(t); \\ dy(t) &= Cx(t)dt. \end{aligned} \tag{19}$$

Our problem can be formulated as given a fixed choice of Θ, find B_{v_2} that minimizes J_{cf} for the system (19) subject to the constraint (9) and that the system (4) is physically realizable; i.e., matrices A, B_u, B_{v_1}, B_{v_2} and C in equations (4) satisfy the corresponding conditions of **Theorem 2**.

To simplify the problem, we reformulate it as an optimization problem by first transforming (9) into an LMI constraint

$$\begin{bmatrix} \Xi_1 & \frac{\tilde{S}}{4} \\ -\frac{\tilde{S}}{4} & \Xi_1 \end{bmatrix} \geq 0. \tag{20}$$

Also, the term $B_{v_2}B_{v_2}^T$ from equations (18) can be rewritten as

$$\begin{aligned} B_{v_2}B_{v_2}^T &= 2i\Theta[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T]Pdiag(M)\times \\ &\quad diag(M^\dagger)P^T \begin{bmatrix} -\Lambda_{b_1} \\ \bar{\Lambda}_{b_1} \end{bmatrix} \Theta i2; \\ &= -2\Theta[-\Lambda_{b_1}^\dagger \quad \Lambda_{b_1}^T] \begin{bmatrix} -\Lambda_{b_1} \\ \bar{\Lambda}_{b_1} \end{bmatrix} \Theta; \\ &= -2\Theta(\Lambda_{b_1}^\dagger\Lambda_{b_1} + \Lambda_{b_1}^T\bar{\Lambda}_{b_1})\Theta. \end{aligned}$$

Meanwhile,

$$\begin{aligned} \mathrm{Re}(\Lambda_{b_1}^\dagger\Lambda_{b_1}) &= \frac{\Lambda_{b_1}^\dagger\Lambda_{b_1} + \Lambda_{b_1}^T\bar{\Lambda}_{b_1}}{2} \\ &= \Xi_1 \geq 0 \end{aligned}$$

which means $B_{v_2}B_{v_2}^T$ can be rewritten as

$$\begin{aligned} B_{v_2}B_{v_2}^T &= -4\Theta\Xi_1\Theta \\ &= 4i\Theta\Xi_1 i\Theta \geq 0. \end{aligned} \tag{21}$$

Finally, our problem can be reformulated as follows: We wish to minimize $J_{cf} = tr(CQC^T)$ with respect to Ξ_1 and Q subject to the constraints

$$\Xi_1 \geq 0; \tag{22a}$$
$$Q \geq 0; \tag{22b}$$
$$A^TQ + QA - 4\Theta\Xi_1\Theta = 0; \tag{22c}$$
$$\begin{bmatrix} \Xi_1 & \frac{\tilde{S}}{4} \\ -\frac{\tilde{S}}{4} & \Xi_1 \end{bmatrix} \geq 0. \tag{22d}$$

Once the optimal solution Ξ_1 is obtained from this LMI problem, we can construct the required matrix B_{v_2} from equation (21) using straightforward matrix factorization of a positive semi-definite matrix.

4. ALGORITHM

In this section, we describe our algorithm to obtain an optimal physically realizable (as per **Theorem (2)**) implementation of the system

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) \\ dy(t) &= Cx(t)dt, \end{aligned} \tag{23}$$

where A is taken to be stable. Furthermore, the algorithm optimizes B_{v_2} so as to minimize the defined cost function (17). The algorithm is as follows:

(1) Beginning with the matrices A, B_u, and C in (23), construct $\tilde{S}$ and S as in (10) and (11), respectively [see Vuglar and Petersen (2017) for further details].
(2) Find Ξ_1 by solving the LMI problem (22) using an optimization tool; e.g., CVX [Grant and Boyd (2014)].
(3) Evaluate the cost function (17).
(4) Find B_{v_1} using equation (7c).
(5) Find $\Lambda_{b_1}^\dagger\Lambda_{b_1}$ using equation (8).
(6) Find Λ_{b_1} using a positive semi-definite matrix factorization method; e.g., eigen-decomposition [Bernstein (2009)]. Then, find B_{v_2} using equation (7d).

5. EXAMPLE

In this section, we demonstrate our results with two examples. Furthermore, we compare the performance of our algorithm (in terms of the cost function (17)) with the method proposed by [Vuglar and Petersen (2017)].

5.1 Example 1

The example considered here was adapted from [Vuglar and Petersen (2017); James et al. (2008)] where

$$A = \begin{bmatrix} -1.389 & 0 & -0.447 & 0 \\ 0 & -1.389 & 0 & -0.447 \\ -0.200 & 0 & -0.250 & 0 \\ 0 & -0.200 & 0 & -0.250 \end{bmatrix};$$

$$B_u = \begin{bmatrix} -0.447 & 0 \\ 0 & -0.447 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};$$

$$C = \begin{bmatrix} -0.447 & 0 & 0 & 0 \\ 0 & -0.447 & 0 & 0 \end{bmatrix}.$$

We now apply the algorithm of the previous section.

(1) Construct the matrices

$$\tilde{S} = \begin{bmatrix} 0 & 2.379 & 0 & 0.647 \\ -2.379 & 0 & -0.647 & 0 \\ 0 & 0.647 & 0 & 0.500 \\ -0.647 & 0 & -0.500 & 0 \end{bmatrix};$$

$$S = \begin{bmatrix} 0 & 0.595i & 0 & 0.162i \\ -0.595i & 0 & -0.162i & 0 \\ 0 & 0.162i & 0 & 0.125i \\ -0.162i & 0 & -0.125i & 0 \end{bmatrix}.$$

(2) Solve the corresponding LMI problem (23) using CVX, we obtain

$$\Xi_1 = \begin{bmatrix} 0.595 & 0 & 0.162 & 0 \\ 0 & 0.595 & 0 & 0.162 \\ 0.162 & 0 & 0.125 & 0 \\ 0 & 0.162 & 0 & 0.125 \end{bmatrix}.$$

(3) Evaluate the optimal value of the cost function gives

$$J_{cf} = 0.339.$$

(4) Find the matrix B_{v_1} using equation (7c)

$$B_{v_1} = \begin{bmatrix} 0.447 & 0 \\ 0 & 0.447 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(5) Find $\Lambda_{b_1}^\dagger \Lambda_{b_1}$ using equation (8)

$$\Xi_1 = \begin{bmatrix} 0.595 & 0.595i & 0.162 & 0.162i \\ -0.595i & 0.595 & -0.162i & 0.162 \\ 0.162 & 0.162i & 0.125 & 0.125i \\ -0.162i & 0.162 & -0.125i & 0.125 \end{bmatrix}.$$

(6) Find Λ_{b_1} and B_{v_2} using eigen-decomposition and equation (7d) respectively to obtain

$$\Lambda_{b_1} = \begin{bmatrix} 0.081i & -0.081 & -0.261i & 0.261 \\ -0.767i & 0.767 & -0.239i & 0.239 \end{bmatrix};$$

$$B_{v_2} = \begin{bmatrix} 0 & -0.162 & 0 & 1.534 \\ 0.162 & 0 & -1.534 & 0 \\ 0 & 0.522 & 0 & 0.477 \\ -0.522 & 0 & -0.477 & 0 \end{bmatrix}.$$

This then defines the physically realizable system (4).

5.2 Example 2

For this example, the random matrices (A is Hurwitz)

$$A = \begin{bmatrix} -0.480 & -3.586 & 0.332 & 3.117 \\ 3.253 & -0.752 & -2.146 & -2.383 \\ -0.774 & 2.525 & -0.809 & -1.634 \\ -3.392 & 1.277 & 2.220 & -0.887 \end{bmatrix};$$

$$B_u = \begin{bmatrix} 0 & 0.318 \\ 0.536 & 0.163 \\ 0.693 & 1.159 \\ -0.688 & 0.128 \end{bmatrix};$$

$$C = \begin{bmatrix} 1.040 & -0.648 & 0 & -0.056 \\ 0 & -0.038 & 0.882 & -0.841 \end{bmatrix}$$

were considered.

(1) Construct the matrices

$$\tilde{S} = \begin{bmatrix} 0 & 1.442 & -2.344 & 4.540 \\ -1.442 & 0 & 2.399 & 0.270 \\ 2.344 & -2.399 & 0 & 0.761 \\ -4.540 & -0.270 & -0.761 & 0 \end{bmatrix};$$

$$S = \begin{bmatrix} 0 & 0.360i & -0.586i & 1.135i \\ -0.360i & 0 & 0.600i & 0.067i \\ 0.586i & -0.600i & 0 & 0.190i \\ -1.135i & -0.067i & -0.190i & 0 \end{bmatrix}.$$

(2) Solve the corresponding LMI problem (23) using CVX, we obtain

$$\Xi_1 = \begin{bmatrix} 3.045 & -1.917 & 2.549 & -1.375 \\ -1.917 & 2.136 & -1.956 & 1.496 \\ 2.549 & -1.956 & 2.656 & -1.155 \\ -1.375 & 1.496 & -1.155 & 1.755. \end{bmatrix}.$$

(3) Evaluate the optimal value of the cost function gives

$$J_{cf} = 7.211.$$

(4) Find the matrix B_{v_1} using equation (7c)

$$B_{v_1} = \begin{bmatrix} 0.038 & -0.648 \\ 0 & -1.040 \\ 0.841 & -0.056 \\ 0.882 & 0 \end{bmatrix}.$$

(5) Find $\Lambda_{b_1}^\dagger \Lambda_{b_1}$ using equation (8)

$$\Lambda_{b_1}^\dagger \Lambda_{b_1} = \begin{bmatrix} 3.045 & -1.917+0.360i & 2.549-0.586i & -1.375+1.135i \\ -1.917-0.360i & 2.136 & -1.956+0.600i & 1.496+0.067i \\ 2.549+0.586i & -1.956-0.600i & 2.656 & -1.155+0.190i \\ -1.375-1.135i & 1.496-0.067i & -1.155-0.190i & 1.754 \end{bmatrix}$$

(6) Find Λ_{b_1} and B_{v_2} using eigen-decomposition and equation (7d) respectively to obtain

$$\Lambda_{b_1} == \begin{bmatrix} 0.238-0.403i & 0.197-0.574i & 0.541+0.069i & 0.809 \\ -1.494-0.771i & 1.274+0.378i & -1.518-0.235i & 1.049 \end{bmatrix};$$

$$B_{v_2} = \begin{bmatrix} 1.149 & 0.394 & -0.757 & 2.548 \\ -0.805 & -0.475 & -1.543 & 2.987 \\ 0 & 1.617 & 0 & 2.098 \\ 0.139 & -1.082 & -0.469 & 3.035 \end{bmatrix}.$$

This then defines the physically realizable system (4).

5.3 Comparison with the algorithm from [Vuglar and Petersen (2017)]

We now compare the performance of the algorithm of this paper with the algorithm proposed in [Vuglar and Petersen (2017)] in terms of the cost function (17). The results are tabulated in Table (1). In our first example, both algorithms gave the same value of the cost function (17). However, in our second example, the algorithm proposed here performs better than the method of [Vuglar and Petersen (2017)].

	Example 1	Example 2
Our method	0.339	7.211
[Vuglar and Petersen (2017)] method	0.339	12.258

Table 1. Calculated cost function from both examples using two different methods.

6. CONCLUSION

Physical realizability is an essential element for realizing meaningful physical quantum systems. This is particularly important in designing coherent feedback control systems in which the controller is required to be a physically realizable quantum system, but the performance of the system is also crucial. In this work, we have proposed a method to find a physically realizable system that minimizes a cost function related to the amount the additional quantum noise affects the system output. An example shows that our proposed method can give a better cost function performance than a previous method.

Future work will consider whether examples can be found where our method does not give the minimum number of noises. In particular, we will investigate the conjecture that minimizing the cost function also minimizes the number of noises.

REFERENCES

Bachor, H. and Ralph, T.C. (2004). *A Guide to Experiments in Quantum Optics*. Wiley-VCH.

Belavkin, V.P. (1992). Quantum continual measurements and a posteriori collapse on CCR. *Communications in Mathematical Physics*, 146(3), 611–635.

Bernstein, D.S. (2009). *Matrix Mathematics: Theory, Facts, and Formulas (Second Edition)*. Princeton University Press.

Bouten, L., Handel, R.V., R., and James, M.R. (2007). An introduction to quantum filtering. *SIAM Journal on Control and Optimization*, 46(6), 2199–2241.

Gardiner, C. and Zoller, P. (2004). *Quantum Noise*. Springer-Verlag Berlin Heidelberg.

Gennaro, A., Mauro, F., and Giorgio, P. (2009). *Quantum Mechanics*. Cambridge University Press.

Grant, M. and Boyd, S. (2014). CVX: Matlab software for disciplined convex programming, version 2.1. `http://cvxr.com/cvx`.

Hudson, R.L. and Parthasarathy, K. (1984). Quantum ito's formula and stochastic evolutions. *Communications in Mathematical Physics*, 93(3), 301–323.

James, M.R. and Nurdin, H.I. (2015). A tutorial introduction to quantum feedback control. In *2015 IEEE Conference on Control Applications (CCA)*, 1–12.

James, M.R., Nurdin, H.I., and Petersen, I.R. (2008). H^∞ control of linear quantum stochastic systems. *IEEE Transactions on Automatic Control*, 53(8), 1787–1803.

Liu, Y., Dong, D., Petersen, I.R., Kuang, S., and Yonezawa, H. (2019). Feedback preparation of Bell states for two-qubit systems with time delay. In *2019 American Control Conference (ACC)*, 5008–5013.

Nurdin, H.I., James, M.R., and Petersen, I.R. (2009). Coherent quantum LQG control. *Automatica*, 45, 1837 – 1846.

Parthasarathy, K. (1992). *An Introduction to Quantum Stochastic Calculus*. Birkhäuser Basel.

Roy, S., Petersen, I.R., and Huntington, E.H. (2017). Coherent-classical estimation for linear quantum systems. *Automatica*, 82, 109–117.

Shaiju, A.J., Petersen, I.R., and James, M.R. (2007). Guaranteed cost LQG control of uncertain linear stochastic quantum systems. In *2007 American Control Conference*, 2118–2123.

Vuglar, S. (2015). *The Physical Realizability of Quantum Systems with Applications to Coherent Quantum Control*. Ph.D. thesis, University of New South Wales, Canberra, Australia.

Vuglar, S.L. and Petersen, I.R. (2017). Quantum noises, physical realizability and coherent quantum feedback control. *IEEE Transactions on Automatic Control*, 62, 998–1003.

Walls, D. and Milburn, G.J. (2008). *Quantum Optics*. Springer-Verlag Berlin Heidelberg.

Wang, S. and James, M.R. (2015a). H^∞ control of quantum feedback control systems with time delay. In *2015 10th Asian Control Conference (ASCC)*, 1–6.

Wang, S. and James, M.R. (2015b). Quantum feedback control of linear stochastic systems with feedback-loop time delays. *Automatica*, 52, 277 – 282.