Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Sampled-Data Set Stabilization of Switched Boolean Control Networks

Amol Yerudkar * **Carmen Del Vecchio** * **Luigi Glielmo** *

* *Department of Engineering, University of Sannio, Piazza Roma 21, Benevento 82100, Italy (e-mail: ayerudkar, c.delvecchio, glielmo@unisannio.it).*

Abstract: In this paper, the set stabilization of switched Boolean control networks (SBCNs) under sampled-data feedback control is addressed. Here, the control input is switching signal-dependent, and SBCNs can switch only at the sampling instants. First, the sampled point control invariant subset (SPCIS) of SBCNs is defined, and an algorithm is provided to obtain the largest SPCIS under arbitrary switching signal. Based on the largest SPCIS, some necessary and sufficient conditions are presented for the set stabilization of SBCNs by switching signal-dependent sampled-data (SSDSD) state feedback control. Furthermore, a constructive procedure is given to design all possible SSDSD state feedback controllers. Finally, some examples are presented to illustrate the effectiveness of the obtained results.

Keywords: Switched Boolean control networks; Sampled-data control; Set stabilization; All possible controllers; Systems biology.

1. INTRODUCTION

Boolean Networks (BNs) were first proposed by Kauffman for the analysis of metabolic stability of gene regulatory networks (GRNs) (Kauffman, 1969). Since then there has been a widespread interest in BNs, as an effective model for intricate GRNs. Moreover, topological structure analysis of BNs plays significant role in the treatment of various diseases. For instance, treatment of breast cancer and leukemia (Choi et al., 2012). Exogenous perturbations have also been observed in many biological systems, which can be characterized as *control* (Fauré and Thieffry, 2009). Thus, by adding Boolean inputs to BNs we can develop control strategies; BNs with inputs and outputs are called Boolean control networks (BCNs).

In the last decade, D. Cheng and co-authors proposed an algebraic framework for BNs and BCNs that resorts to the semi-tensor product (STP), and allows to cast the logical dynamics of BNs (BCNs) into discrete-time linear (bilinear) systems (Cheng and Qi, 2010; Cheng et al., 2010). Thus, one can study BCNs with methods resembling classical control theory. Many control-theoretic problems on BCNs have been investigated, including but not limited to, controllability and observability (Laschov and Margaliot, 2012; Cheng et al., 2016; Zhang and Johansson, 2020), stability and stabilization (Cheng et al., 2011; Li et al., 2013; Bof et al., 2015; Li and Wang, 2017), optimal control (Laschov and Margaliot, 2013; Fornasini and Valcher, 2014; Wu et al., 2019).

In reality, most biological systems exhibit switching behavior compared with the purely discrete dynamics of the typical BN model (El-Farra et al., 2005). Switching can be triggered by inherent system mechanisms or by uncertainties and exogenous disturbances, which are very often observed in biological systems. Such biological systems can be modeled using a class of BNs, namely switched Boolean networks (SBNs). The research on SBNs and switched Boolean control networks (SBCNs) has spurred significant interest from the control community. And, indeed, within this setting, controllability and control design (Li and Wang, 2015), synchronization (Chen et al., 2015), set stability (Guo et al., 2017) of SBNs, disturbance decoupling (Li et al., 2014), output tracking control (Yerudkar et al., 2019b), and stabilization (Li and Tang, 2017; Yerudkar et al., 2020) of SBCNs, have been successfully investigated.

Stabilization of SBCNs is one of the crucial issues. It is of great importance in medical practice to modify system behavior to make it desirable by using control inputs. This can help to decide therapeutic intervention strategies to achieve a healthy and robust state in specific GRNs. For example, a disease progression modeling in terms of the tumor (diseased cell) growth and designing treatment methods to eradicate diseases cells. In fact, SBCNs facilitate such modeling and offer a natural choice to model the development of cellular systems from a practical perspective. Therapeutic efficacy requires an optimal control policy for administrating an adequate dose of medicine that causes maximum tumor damage while reducing the associated side effects (Vahedi et al., 2009). If one can design all possible feedback controllers for treating the disease under consideration, it can be advantageous to identify optimal control policies.

In this paper, we address the problem of designing all possible switching signal-dependent sampled-data (SSDSD) state feedback controllers (i.e., a complete solution) to set stabilize SBCNs under arbitrary switching signal. The sampled-data control technique has been extensively studied in recent decades (Oishi and Fujioka, 2010; Hauroigne et al., 2011; Fujioka, 2009; Hetel and Fridman, 2013). In practice, where the aim is to achieve stabilization or synchronization, the traditional state feedback controller is updated at every instant resulting in higher energy consumption. On the other hand, the sampled-data state feedback controller is updated during each sampling period. The sampled-data control can, therefore, reduce the number of controller updates while achieving the same desired effect (Zhu et al., 2020).

Recently, the sampled-data state feedback control technique is studied in (Zhu et al., 2020) and (Xu et al., 2019; Liu et al., 2019) to set stabilize BCNs and probabilistic BCNs, respectively. In (Liu et al., 2018), the synchronization problem of BCNs is investigated by the sampled-data state feedback control. Nonetheless, the sampled-data control for SBCNs has not been investigated yet. Although SBCNs represent the generalized version of BCNs with switching, results on BCNs do not apply to SBCNs directly, and further research is required to address the issue of sampled-data state feedback control for SBCNs.

By following this stream of research, in this paper the problem of finding a complete solution to the set stabilization problem of SBCNs under SSDSD state feedback control considering arbitrary switching signal is studied. By resorting to the algebraic state-space representation of SBCNs, a new algorithm is presented to obtain the largest sampled point control invariant subset (SPCIS) followed by some necessary and sufficient conditions for the sampled-data set stabilizability of SBCNs. Here, the switching can occur only at sampling instants. Further, an algorithm is presented to obtain all possible complete families of controllable sets thereby all possible SSDSD state feedback controllers are designed under arbitrary switching signal through a constructive method.

The paper is organized as follows. Preliminaries and problem formulation are stated in Section 2. In Section 3, we present a new algorithm to obtain the largest SPCIS, and investigate how to design all possible SSDSD state feedback controllers based on all the complete families of controllable sets of SBCNs. Some illustrative examples are given in Section 4 to validate the presented results. Finally, a brief conclusion is provided in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, some necessary preliminaries which will be used in the sequel are given.

$\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}_+$ denote the sets of real numbers, natural numbers and nonnegative integers, respectively. Given $k,\ n \in \mathbb{Z}_+$, with $k \leq n$, we denote the integer set $\{k,\ k+1,\ \ldots,\ n\}$ by the symbol $[k,\ n]$. $\mathcal{B} := \{0,1\}$, and $\mathcal{B}^n := \underbrace{\mathcal{B} \times \ldots \times \mathcal{B}}_{n}$. $\Delta_n := \{\delta_n^i \mid i = 1,\ldots,n\}$, where δ_n^i denotes the ith canonical vector of size k. A matrix $L \in \mathbb{R}_{n\times r}$ is called a *logical matrix*, if $L = \begin{bmatrix}\delta_n^{i_1} & \delta_n^{i_2} & \ldots & \delta_n^{i_r}\end{bmatrix}$. It is shortly denoted by $L = \delta_n\,[i_1\ \ i_2\ \ \ldots\ \ i_r]$. Denote the set of all $m \times n$ logical matrices by $\mathcal{L}_{m\times n}$. A $n \times m$ matrix $A = (a_{ij})$ is called a Boolean matrix, if $a_{ij} \in \mathcal{B}$, $\forall i, j$. Denote the set of $n \times m$ Boolean matrices by $\mathcal{B}_{n\times m}$. $\mathrm{Col}_i(A)$ and $(A)_{i,j}$ denote the ith column and the (i,j)th element of matrix A, respectively. $|\cdot|$ represents the number of elements in a set.

Definition 1. (Cheng et al., 2010) Let $A \in \mathbb{R}_{m\times n}$, $B \in \mathbb{R}_{p\times q}$. Denote the least common multiple of n and p by $t = \mathrm{lcm}\{n, p\}$. Then the STP of A and B is defined as $A \ltimes B := (A \otimes I_{\frac{t}{n}})(B \otimes I_{\frac{t}{p}})$, where $\otimes$ is the Kronecker product of matrices.

Remark 2. Henceforth, Boolean state, input and output variables are identified by uppercase letters and their equivalent vector form by lowercase letters. We omit the symbol "$\ltimes$" throughout this paper.

Lemma 3. (Cheng et al., 2010) Let $f(X_1,\ldots,X_n) : \mathscr{D}^n \to \mathscr{D}$ be a logical function. Then there exists a unique logical matrix $M_f \in \mathscr{L}_{2\times 2^n}$, called the structure matrix of f, representing f so that $f(X_1,\ldots,X_n) \longleftrightarrow M_f \ltimes_{i=1}^n x_i$, $x_i \in \Delta_2$, where $\ltimes_{i=1}^n x_i = x_1 \ltimes x_2 \ltimes \ldots \ltimes x_n$.

A SBCN with n nodes, m control inputs, and a switching signal with p values leading to p sub-networks is described as follows:

$$\begin{cases} X_1(t+1) = f_1^{\sigma(t)}\left(U(t),\ X(t)\right), \\ \quad\vdots \\ X_n(t+1) = f_n^{\sigma(t)}\left(U(t),\ X(t)\right), \end{cases} \tag{1}$$

where $\sigma : \mathbb{Z}_+ \to P := \{1,\ 2,\ \ldots,\ p\}$ is the switching signal, $X(t) := \big(X_1(t),\ldots,X_n(t)\big) \in \mathcal{B}^n$ is the state, $U(t) := \big(U_1(t),\ldots,U_m(t)\big) \in \mathcal{B}^m$ is the control input, and $f_i^j : \mathcal{B}^{n+m} \to \mathcal{B}$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,p$, is logical function. Given a switching signal $\sigma(t)$ and a switching signal-dependent control sequence $U_{\sigma(t)}(t) : \mathbb{Z}_+ \times P \to \mathcal{B}^m$, $t \in \mathbb{Z}_+$, starting from an initial state $X_0 = X(0) \in \mathcal{B}^n$, denote the state trajectory of the system (1) by $X\big(t;\ X(0),\ U_\sigma\big)$.

Based on Lemma 3, by setting $x(t) = \ltimes_{i=1}^n x_i(t)$, $u(t) = \ltimes_{i=1}^m u_i(t)$, $y(t) = \ltimes_{i=1}^q y_i(t)$, the algebraic form of SBCN (1) can be given as

$$x(t+1) = L_{\sigma(t)} u(t) x(t), \tag{2}$$

where $x(t) \in \Delta_{2^n}$, $u(t) \in \Delta_{2^m}$, and $L_{\sigma(t)} \in \mathcal{L}_{2^n\times 2^{n+m}}$ is the network transition matrix of (1).

The aim of this paper is to design all possible SSDSD state feedback controllers of the form

$$U_{i,\sigma(t)}(t) = K_{i,\sigma(t_\iota)}\left(X_1(t_\iota),\ldots,X_n(t_\iota)\right),\ t \in [t_\iota,\ t_{\iota+1}), \tag{3}$$

where $K_{i,\sigma(t_\iota)}$, $i = 1,2,\ldots,m$ are logical functions, constant sampling period is denoted by $\tau := t_{\iota+1} - t_\iota \in \mathbb{Z}_+$, and $t_\iota = \iota\tau \geq 0$, $\iota = 0, 1\ldots$ are sampling instants.

The algebraic form of (3) leads to

$$u_{\sigma(t)}(t) = K_{\sigma(t_\iota)} x(t_\iota),\ t \in [t_\iota,\ t_{\iota+1}), \tag{4}$$

where with a slight abuse of notation, $K_{\sigma(t_\iota)} \in \mathcal{L}_{2^m\times 2^n}$ is the state feedback gain matrix.

Let $\mathcal{S} = \{\delta_{2^n}^{l_1},\ \delta_{2^n}^{l_2},\ldots,\ \delta_{2^n}^{l_s}\}$ be a subset of Δ_{2^n}.

Definition 4. The SBCN (2) is said to be $\mathcal{S}$-stabilizable if, for any switching signal $\sigma(t)$ and any initial state $x(0) \in \Delta_{2^n}$, there exists $u_{\sigma(t)}(t)$ of the form (4), and an integer $T \in \mathbb{Z}_+$ such that $x(t;\ x(0),\ u_\sigma) \in \mathcal{S}$, for every $t \geq T$.

Definition 5. A subset $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ is called a sampled point control invariant subset (SPCIS) of $\mathcal{S}$ for system (2) with sampling period τ, if for any switching signal $\sigma(t)$, there exists a control sequence of the form (4) such that for any initial state $x(t_\iota) \in \tilde{\mathcal{S}}$, $x(t;\ x(t_\iota),\ u_\sigma) \in \mathcal{S}$, $\forall t \in [1,\ \tau-1]$, and $x(\tau;\ x(t_\iota),\ u_\sigma) \in \tilde{\mathcal{S}}$. The union of two control invariant subsets is still invariant. The union of all SPCISs contained in $\mathcal{S} \subseteq \Delta_{2^n}$ is called the largest SPCIS of $\mathcal{S}$ for SBCN (2), denoted by $\tilde{\mathcal{S}}_I$.

3. MAIN RESULTS

This section investigates designing all possible SSDSD state feedback controllers to globally feedback stabilize the SBCNs under arbitrary switching signal.

3.1 SPCIS and family of controllable sets

In this subsection, we present a new algorithm to obtain the largest SPCIS included in $\mathcal{S}$, followed by necessary and sufficient conditions for the global feedback stabilization of SBCNs. Further, we provide a constructive algorithm to find all possible families of controllable sets under arbitrary switching signal, which play a crucial role in the control design.

For every choice of the switching signal-dependent control, namely for every $\delta_{2^m}^{c^\theta}$, $c^\theta \in [1,\ 2^m]$ we denote by $L_{\theta_{c^\theta}} = L_\theta \delta_{2^m}^{c^\theta} \in \mathcal{L}_{2^n \times 2^n}$, where $\theta_{c^\theta} \in [1,\ p]$. We present **Algorithm 1** to find the largest SPCIS included in $\mathcal{S}$.

Algorithm 1 To find the largest sampled point control invariant subset

Input: $\mathcal{S} = \left\{\delta_{2^n}^{l_1}, \delta_{2^n}^{l_2}, \ldots, \delta_{2^n}^{l_s}\right\} \subseteq \Delta_{2^n}$ with $1 \le l_1 < \cdots < l_s \le 2^n$, τ, and the network transition matrices
Output: The largest SPCIS, $\tilde{\mathcal{S}}_I$, contained in $\mathcal{S}$
1: Initialization: $\eta \leftarrow 0$, $\mathcal{S}_\eta \leftarrow \mathcal{S}$
2: **while** $\eta \le |S_0|$ **do**
3: $\quad \eta \leftarrow \eta + 1$
4: $\quad \mathcal{S}_\eta = \left\{\delta_{2^n}^{l_i} \in \mathcal{S}_{\eta-1} \mid \forall \theta_{c^\theta} \in P,\ L_{\theta_{c^\theta}}^{\kappa} \delta_{2^n}^{l_i} \in \mathcal{S}\ \forall \kappa \in [1,\ \tau-1] \text{ and } L_{\theta_{c^\theta}}^{\tau} \delta_{2^n}^{l_i} \in \mathcal{S}_{\eta-1}, \text{ where } c^\theta \in [1,\ 2^m]\right\}$
5: $\quad$ **if** $\mathcal{S}_\eta = \mathcal{S}_{\eta-1}$ **then break;**
6: $\quad$ **end if**
7: **end while**
8: $\tilde{\mathcal{S}}_I \leftarrow \mathcal{S}_\eta$

From **Algorithm 1** the largest SPCIS can be obtained, and according to Definition 4 and Definition 5 we consider the $\mathcal{S}$-stabilization problem for SBCN (2) as: for any switching signal $\sigma(t)$, any initial state $x(0) \in \Delta_{2^n}$ can be driven to $\tilde{\mathcal{S}}_I$ in $T\tau$ steps and it always stays in $\tilde{\mathcal{S}}_I$ after $T\tau$ steps only when there exists an integer T.

Here, we present the following example to verify **Algorithm 1** and to show the need for calculating SPCIS.

Example 6. Consider a SBCN of the form (2), with $n = 3$, $m = 2$ and $p = 4$, and suppose that

$$\begin{aligned} L_1 =& \delta_8[1\ 1\ 1\ 2\ 5\ 8\ 2\ 3\ 1\ 5\ 4\ 1\ 7\ 1\ 1\ 6 \\ & 4\ 1\ 8\ 1\ 5\ 5\ 1\ 1\ 3\ 1\ 1\ 7\ 1\ 7\ 3\ 5], \\ L_2 =& \delta_8[1\ 2\ 2\ 2\ 5\ 8\ 4\ 3\ 1\ 5\ 4\ 1\ 7\ 2\ 1\ 6 \\ & 4\ 2\ 8\ 1\ 5\ 6\ 1\ 1\ 3\ 2\ 1\ 7\ 1\ 7\ 3\ 5], \\ L_3 =& \delta_8[1\ 3\ 5\ 3\ 7\ 8\ 6\ 3\ 1\ 7\ 4\ 5\ 7\ 5\ 1\ 8 \\ & 8\ 5\ 8\ 1\ 7\ 7\ 3\ 1\ 3\ 3\ 1\ 8\ 5\ 8\ 4\ 6], \\ L_4 =& \delta_8[1\ 4\ 6\ 3\ 7\ 8\ 8\ 3\ 1\ 7\ 4\ 5\ 7\ 6\ 1\ 8 \\ & 8\ 6\ 8\ 1\ 7\ 8\ 3\ 1\ 3\ 4\ 1\ 8\ 5\ 8\ 4\ 6]. \end{aligned} \tag{5}$$

Given $\mathcal{S} = \{\delta_8^1,\ \delta_8^4,\ \delta_8^8\} \subseteq \Delta_8$ and $\tau = 2$, we aim to find the largest SPCIS included in $\mathcal{S}$.
By following **Algorithm 1**, for $\delta_8^1 \in \mathcal{S}$ we get $c^\theta = 1$ such that $\forall \theta_1 \in P$, $L_{\theta_1}\delta_8^1 \in \mathcal{S}$ and $L_{\theta_1}^2 \delta_8^1 \in \mathcal{S}$ holds. Similarly, for both δ_8^4 and δ_8^8 we get $c^\theta = 3$ such that the conditions in Step 4 of **Algorithm 1** hold. Thus, we obtain the largest SPCIS as $\tilde{\mathcal{S}}_I = \mathcal{S}$.
Further, for δ_8^6 we get $c^\theta = 1$ such that $\forall \theta_1 \in P$, $L_{\theta_1}\delta_8^6 \in \mathcal{S}$, but $L_{\theta_1}^2 \delta_8^6 \notin \mathcal{S}$. Here, the state δ_8^6 enters the set $\mathcal{S}$ but does not remain there for a given sampled period. Thus, it is crucial to design the largest SPCIS to study the state stabilization problem of SBCNs using sampled-data technique.

In the following, we define the set $E_k(\tilde{\mathcal{S}}_I)$ consisting of all the states that can be steered to $\tilde{\mathcal{S}}_I$ in $k\tau$ steps under any switching signal sequence.

Lemma 7. For SBCN (2), we have

i. $E_1(\tilde{\mathcal{S}}_I) = \Big\{x_0 \in \Delta_{2^n} : \exists c^\theta \in [1,\ 2^m] \text{ such that } \big(L_{\theta_{c^\theta}}^{\tau} x_0 \in \tilde{\mathcal{S}}_I\big) \text{ holds for every } \theta_{c^\theta} \in P\Big\}.$
ii. $E_{k+1}(\tilde{\mathcal{S}}_I) = \Big\{x_0 \in \Delta_{2^n} : \exists c^\theta \in [1,\ 2^m] \text{ such that } \big(L_{\theta_{c^\theta}}^{\tau} x_0 \in E_k(\tilde{\mathcal{S}}_I)\big) \text{ holds for every } \theta_{c^\theta} \in P\Big\}.$

Proof. Statement (i) is clear from the algebraic state-space form (2) of SBCN and (ii) follows immediately by induction. □

Next, based on Lemma 7, the following basic properties of the set $E_k(\tilde{\mathcal{S}}_I)$ can be derived straightforwardly.

Lemma 8. i. If $\tilde{\mathcal{S}}_I \subseteq E_1(\tilde{\mathcal{S}}_I)$, then $E_k(\tilde{\mathcal{S}}_I) \subseteq E_{k+1}(\tilde{\mathcal{S}}_I)$, for all $k \ge 1$;
ii. If $E_1(\tilde{\mathcal{S}}_I) = \tilde{\mathcal{S}}_I$, then $E_k(\tilde{\mathcal{S}}_I) = \tilde{\mathcal{S}}_I$, for all $k > 1$;
iii. If $E_{k+1}(\tilde{\mathcal{S}}_I) = E_k(\tilde{\mathcal{S}}_I)$ for some $k \ge 1$, then $E_j(\tilde{\mathcal{S}}_I) = E_k(\tilde{\mathcal{S}}_I)$, for all $j \ge k$.

Then, we have the following theorem.

Theorem 9. The set stabilization problem to the set $\tilde{\mathcal{S}}_I$ is solvable for the SBCN (2) by SSDSD state feedback control (4) if and only if the following conditions hold

(1) $\tilde{\mathcal{S}}_I \neq \emptyset$;
(2) there exists an integer T such that $E_T(\tilde{\mathcal{S}}_I) = \Delta_{2^n}$.

Proof. [Sufficiency] Assume that $\tilde{\mathcal{S}}_I \neq \emptyset$ and there exists an integer T such that $E_T(\tilde{\mathcal{S}}_I) = \Delta_{2^n}$. Consequently, for any switching signal $\sigma(t)$, $\forall x(0) \in \Delta_{2^n}$, one can obtain an input $u_{\sigma(t)}(t)$, $t \in [0,\ T-1]$ steering the state trajectory from $x(0)$ to $x(T\tau) \in \tilde{\mathcal{S}}_I$. Since $\tilde{\mathcal{S}}_I$ is nonempty, for any switching signal $\sigma(t)$, $\exists u_{\sigma(t)}(t)$, $t \in [T,\ +\infty)$, such that $x(t\tau) \in \tilde{\mathcal{S}}_I$, $\forall t \ge T$. Thus, the system (2) is set stabilizable to $\tilde{\mathcal{S}}_I$ under arbitrary switching signal by control law (4).
[Necessity] Assume that the set stabilization problem is solvable under any switching signal by a state feedback (4). Then, there exists an integer T such that for $t \ge T$, we have $x(t\tau) \in \tilde{\mathcal{S}}_I$.

Necessity is proved by contradiction. If $\tilde{\mathcal{S}}_I = \emptyset$, then the condition (2) is not satisfied. Hence, it suffices to assume that only condition (2) is not satisfied. This means that, for any integer T, $E_T(\tilde{\mathcal{S}}_I) \neq \Delta_{2^n}$. Then, there exists an initial state $x(0)$ and a switching sequence $\sigma(t)$, such that for every input sequence $u_{\sigma(t)}(t)$, $t \in [t_\iota,\ t_{\iota+1})$ the state $x\big(T\tau;\ x(0),\ u_\sigma\big)$ does not belong to $\tilde{\mathcal{S}}_I$ for any T. Thus, the set stabilization problem is not solvable under arbitrary switching signal, which is a contradiction to the solvability of the set stabilization by (4). Therefore, condition 2) must hold. This completes the proof. □

To this end, for any subset Γ_k of Δ_{2^n}, $k \in \mathbb{Z}_+$, we define the one-step controllable set as follows:

$$R_1(\Gamma_k) = \big\{x_0 \in \Delta_{2^n} \ : \ \exists c^\theta \in [1,\ 2^m] \text{ such that } \big(L^\tau_{\theta_{c^\theta}} x_0 \in \Gamma_k\big) \text{ holds for every } \theta_{c^\theta} \in P\big\}. \tag{6}$$

We now present **Algorithm 2**, the aim of which is to obtain a sequence of sets $\Gamma^j_0,\ \Gamma^j_1,\ \ldots,\ \Gamma^j_{T_j}$ for SBCNs under arbitrary switching signal, where j represents the jth sequence of sets. In the sequel, we call this sequence of sets, $\{\Gamma^j_0,\ \Gamma^j_1,\ \ldots,\ \Gamma^j_{T_j}\}$, as a family of controllable sets. Some additional comments are provided at the end of the algorithm.

Algorithm 2 To find a family of controllable sets for SBCNs under arbitrary switching signal

Input: The network transition matrix $L_{\sigma(t)}$
Output: A family of controllable sets

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Initialization: j ← 1, k_j ← 0, Γ^j_{k_j} ← S̃_I,
while Γ^j_{k_j} ≠ ∅ do
    k_j ← k_j + 1
    Choose any non-empty subset
        Γ^j_{k_j} ⊆ R_1( ⋃_{i=0}^{k_j−1} Γ^j_i ) \ ( ⋃_{i=0}^{k_j−1} Γ^j_i )
end while
T_j ← k_j − 1
if ⋃_{i=0}^{T_j} Γ^j_i = Δ_{2^n} then
    {Γ^j_0, Γ^j_1, ..., Γ^j_{T_j}} is a valid solution
else The solution is invalid
end if
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Definition 10. A set $\{\Gamma^j_0, \Gamma^j_1, \ldots, \Gamma^j_{T_j}\}$ is called a family of controllable sets, if $\Gamma^j_{i_j} \cap \Gamma^j_{k_j} = \emptyset$, $\forall i_j \neq k_j$, $\Gamma^j_{k_j} \subseteq R_1\big(\bigcup_{i=0}^{k_j-1} \Gamma^j_i\big) \setminus \big(\bigcup_{i=0}^{k_j-1} \Gamma^j_i\big)$, $k_j = 1, \ldots, T_j$, and $R_1\big(\bigcup_{i=0}^{T_j} \Gamma_i\big) \setminus \Big(\bigcup_{i=0}^{T_j} \Gamma_i\Big) = \emptyset$.
A family of controllable sets is complete if $\bigcup_{i=0}^{T_j} \Gamma^j_i = \Delta_{2^n}$. Denote all possible complete families of controllable sets by $\Theta_j,\ j = 1, \ldots, \omega$.

From **Algorithm 2** it is clear that all the states in $\Gamma^j_{k_j}$ *can take maximum* $k_j\tau$ *steps to reach* $\tilde{\mathcal{S}}_I$ under arbitrary switching signal. Since a family comprises all disjoint sets, $\Gamma^j_{k_j},\ k_j = 1, 2, \ldots, T_j$, one can find an integer T_j satisfying $T_j \leq 2^n - |\tilde{\mathcal{S}}_I|$ and $\Gamma^j_i \neq \emptyset$, $i \leq T_j$ such that $R_1\left(\bigcup_{i=0}^{T_j} \Gamma^j_i\right) \setminus \left(\bigcup_{i=0}^{T_j} \Gamma^j_i\right) = \emptyset$. Hence, **Algorithm 2** always provides a family of controllable sets $\{\Gamma^j_i : i = 0, 1, \ldots, T_j\}$, which describes a backward path to $\tilde{\mathcal{S}}_I$.

It is worthwhile noticing that, by iterating the algorithm several times and considering all possible combinations of $R_1\left(\bigcup_{i=0}^{k_j-1} \Gamma^j_i\right) \setminus \left(\bigcup_{i=0}^{k_j-1} \Gamma^j_i\right)$ in the selection of $\Gamma^j_{k_j}$, one can obtain all possible families of controllable sets $\{\Gamma^j_i : i = 0, 1, \ldots, T_j\}$, $j \in [1,\ \omega]$, where T_j depends on the selection of non-empty subset $\Gamma^j_i,\ i = 1, \ldots, T_j - 1$.

With this setting, in the next subsection we discuss the design of all possible SSDSD state feedback controllers.

3.2 All possible SSDSD state feedback control design

From Definition 10, it can be noted that for each $\Theta_j = \{\Gamma^j_0, \Gamma^j_1, \ldots, \Gamma^j_{T_j}\}$ and each $\delta^\upsilon_{2^n} \in \Delta_{2^n}$, since $\bigcup_{k_j=0}^{T_j} \Gamma^j_{k_j} = \Delta_{2^n}$ and $\Gamma^j_{k_j} \subseteq R_1\big(\bigcup_{i=0}^{k_j-1} \Gamma^j_i\big) \setminus \big(\bigcup_{i=0}^{k_j-1} \Gamma^j_i\big)$, $k_j = 1, \ldots, T_j$, one can find a unique integer $k_\upsilon \in [0,\ T_j]$ such that $\delta^\upsilon_{2^n} \in \Gamma^j_{k_\upsilon}$. In other words, each state $\delta^\upsilon_{2^n}$ belongs to a unique set $\Gamma^j_{k_\upsilon}$, $k_\upsilon \in [0,\ T_j]$. Further, for each complete family of controllable sets we denote by Φ_i, $i = 1, \ldots, \omega$, the sets of switching signal-dependent state feedback controllers, $K_\theta = \delta_{2^m}[c^\theta_1\ c^\theta_2\ \ldots\ c^\theta_{2^n}]$, i.e., $\Phi_i := \big\{K_\theta = \delta_{2^m}[c^\theta_1\ c^\theta_2\ \ldots\ c^\theta_{2^n}]\big\}$, where $\theta \in [1,\ p]$ and $c^\theta_\upsilon \in [1,\ 2^m]$. Now, we present the following algorithm to design all possible SSDSD state feedback gain matrices to set stabilize the SBCN (2) to $\tilde{\mathcal{S}}_I$ under arbitrary switching signal. The algorithm is based on (Yerudkar et al., 2019a, Th. 3.5).

Algorithm 3 To find all possible SSDSD state feedback gain matrices K_θ

Input: $L_{\sigma(t)}$ and all complete families of controllable sets $\Theta_j = \{\Gamma^j_0,\ \Gamma^j_1,\ \ldots,\ \Gamma^j_{T_j}\}$, $j \in [1,\ \omega]$
Output: All possible K_θ

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Initialization: K_θ = δ_{2^m}[c^θ_1 c^θ_2 ... c^θ_{2^n}], Φ_j := {K_θ},
    c^θ_υ ∈ [1, 2^m], θ ∈ [1, p]
for j ← 1 to ω do
    for θ ← 1 to p do
        for υ ← 1 to 2^n do
            if δ^υ_{2^n} ∈ S̃_I then
                Find c^θ_υ ∈ [1, 2^m] such that
                    (L_θ δ^{c^θ_υ}_{2^m})^τ δ^υ_{2^n} ∈ S̃_I
            else Find c^θ_υ ∈ [1, 2^m] such that
                    (L_θ δ^{c^θ_υ}_{2^m})^τ δ^υ_{2^n} ∈ ⋃_{l=1}^{k_υ} Γ^j_{l−1}
            end if
        end for
    end for
end for
```

The algorithm tries to set a value in $[1,\ 2^m]$ for each integer $c^\theta_1, c^\theta_2, \ldots, c^\theta_{2^n}$, that amounts to a possible choice of input. Such inputs, corresponding to the specific value of switching sequence and of the state, allow the desired transitions. In particular, for each complete family of controllable sets, we obtain control inputs $\delta^{c^\theta_\upsilon}_{2^m} \in \Delta_{2^m}$ such that under arbitrary switching signal and given sampling period τ, (i) $\delta^\upsilon_{2^n} \in \Gamma_0 = \tilde{\mathcal{S}}_I$, will remain at $\tilde{\mathcal{S}}_I$, (ii) the states in Γ_{k_υ}, $k_\upsilon = 1, \ldots, T_j$ will be steered to Γ_0 in at most $k_\upsilon\tau$ steps. Thus, we obtain the set of all possible switching signal-dependent state feedback controllers as $\bigcup_{j=1}^{\omega} \Phi_j$, where $\Phi_j = \big\{K_\theta = \delta_{2^m}[c^\theta_1\ c^\theta_2\ \ldots\ c^\theta_{2^n}]\big\}$.

Example 11. We continue with Example 6 to calculate all possible SSDSD state feedback controllers. By iterating **Algorithm 2** once we get the following complete family of controllable sets: $\big\{\Gamma^1_0 = \{\delta^1_8,\ \delta^4_8,\ \delta^8_8\},\ \Gamma^1_1 =$

$\{\delta_8^2,\ \delta_8^3,\ \delta_8^5,\ \delta_8^6,\ \delta_8^7\}\}$. By iterating the same algorithm for system (5) one can obtain all possible complete families of controllable sets, which are not listed in the paper due to space limitation.

Next, by utilizing all complete families of controllable sets and following **Algorithm 3** we get all possible SSDSD state feedback controllers as follows: $K_1 = \delta_4[c_1^1\ c_2^1\ c_3^1\ c_4^1\ c_5^1\ c_6^1\ c_7^1\ 3]$: $c_1^1 = 1, 2, 3$; $c_2^1 = c_3^1 = c_7^1 = 1, 2, 3, 4$; $c_4^1 = 2, 3$; $c_5^1 = 2, 4$; $c_6^1 = 1, 2, 4$. $K_2 = \delta_4[c_1^2\ 2\ c_3^2\ c_4^2\ c_5^2\ c_6^2\ c_7^2\ 3]$: $c_1^2 = 1, 2, 3$; $c_3^2 = 2, 3, 4$; $c_4^2 = 2, 3$; $c_5^2 = 2, 4$; $c_6^2 = 1, 2, 4$; $c_7^2 = 1, 2, 3, 4$. $K_3 = \delta_4[c_1^3\ c_2^3\ c_3^3\ 3\ c_5^3\ c_6^3\ c_7^3\ c_8^3]$: $c_1^3 = 1, 2, 3$; $c_2^3 = c_3^3 = c_6^3 = c_7^3 = 1, 2, 3, 4$; $c_5^1 = 1, 2, 3$; $c_8^1 = 2, 3$. $K_4 = \delta_4[c_1^4\ c_2^4\ c_3^4\ c_4^4\ c_5^4\ c_6^4\ c_7^4\ \delta_8^4]$: $c_1^4 = c_5^4 = 1, 2, 3$; $c_2^4 = c_3^4 = c_7^4 = 1, 2, 3, 4$; $c_4^4 = 3, 4$; $c_6^1 = 1, 3, 4$; $\delta_8^4 = 2, 3$.

Remark 12. When $\tau = 1$, the problem reduces to the set stabilization of SBCNs under arbitrary switching signal, and all possible switching signal-dependent state feedback controllers can be obtained straightforwardly. Further, when both τ and p are 1, the problem boils down to the set stabilization of BCNs and one can easily design all possible state feedback controllers. In this case, compared to the method presented in (Li and Wang, 2017) less computational efforts are needed. In particular, one need not find the reachable sets $E_l(x_e)$ while obtaining all possible families of controllable sets by **Algorithm 2**, where x_e is the given equilibrium point. Finding $E_l(x_e)$, $l \in [1,\ T]$, involves calculating the series of matrices M^l, where $M := \sum_{i=1}^{2^m} Col_i(L)$ and $M \in \mathcal{L}_{2^n \times 2^n}$. Thus, **Algorithm 2** reduces the computational complexity by $O((T-1)2^{3n})$.

4. EXAMPLES

In this section we show with the help of two biological examples, how the SSDSD control technique introduced in Section 3 can be applied to control the dynamics of GRNs.

Example 13. Consider a GRN with four genes, namely WNT5A, pirin, S100P and STC3, which was derived in (Pal et al., 2004) to study metastatic melanoma. Assuming that WNT5A and S100P are states $X_1(t)$ and $X_2(t)$, respectively; pirin and STC3 are control inputs $U_1(t)$ and $U_2(t)$, respectively, we consider the GRN dynamics as:

$$\begin{cases} X_1(t+1) = f_1^{\sigma(t)}(U_1(t),\, U_2(t),\, X_1(t),\, X_2(t)), \\ X_2(t+1) = f_2^{\sigma(t)}(U_1(t),\, U_2(t),\, X_1(t),\, X_2(t)), \\ X_3(t+1) = f_3^{\sigma(t)}(U_1(t),\, U_2(t),\, X_1(t),\, X_2(t)), \end{cases} \tag{7}$$

where $\sigma : \mathbb{Z}_+ \to P := \{1,\ 2\}$ and we have: $f_1^1 = U_1(t) \wedge [U_2(t) \wedge (X_1(t) \to X_2(t)) \vee (\neg U_2(t) \wedge (X_1(t) \wedge X_2(t)))] \vee [\neg U_1(t) \wedge U_2(t) \wedge (X_1(t) \longleftrightarrow X_2(t))]$, $f_1^2 = U_1(t) \wedge (\neg X_1(t) \wedge X_2(t)) \vee U_2(t) \wedge (\neg X_2(t))$, $f_2 = \neg U_1(t) \wedge U_2(t) \wedge X_2(t) \vee (\neg X_1(t)) \vee X_2(t)$.

The state transition matrices are given as:

$$\begin{aligned} L_1 =& \delta_4[1\ 2\ 3\ 2\ 1\ 2\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4], \\ L_2 =& \delta_8[1\ 2\ 1\ 2\ 3\ 2\ 3\ 4\ 3\ 4\ 1\ 2\ 3\ 4\ 3\ 4]. \end{aligned} \tag{8}$$

Here, $\mathcal{S} = \{\delta_4^1,\ \delta_4^3,\ \delta_4^4\}$, $\tau = 2$, and the aim is to design all possible SSDSD state feedback controllers to set stabilize the system (8) to the largest SPCIS included in $\mathcal{S}$ under arbitrary switching signal.

By following **Algorithm 1** we obtain that $\tilde{\mathcal{S}}_I = \mathcal{S}$. Further, by following **Algorithm 2** and **Algorithm 3**, we design all possible SSDSD state feedback controllers as follows: $K_1 = \delta_4[c_1^1\, c_2^1\, c_3^1\, c_4^1]$: $c_1^1 = c_3^1 = 1, 2, 3, 4$; $c_2^1 = 3, 4$ $c_4^1 = 2, 3, 4$. $K_2 = \delta_4[c_1^2\, c_2^2\, c_3^2\, c_4^2]$: $c_1^1 = c_3^1 = 1, 2, 3, 4$; $c_2^1 = 3, 4$ $c_4^1 = 2, 4$.

Example 14. Consider the following SBCN of apoptosis network derived in (Chaves, 2009),

$$\begin{cases} X_1(t+1) = f_1^{\sigma(t)}(U_1(t),\, U_2(t),\, X_1(t),\, X_2(t),\, X_3(t)), \\ X_2(t+1) = f_2^{\sigma(t)}(U_1(t),\, U_2(t),\, X_1(t),\, X_2(t),\, X_3(t)), \\ X_3(t+1) = f_3^{\sigma(t)}(U_1(t),\, U_2(t),\, X_1(t),\, X_2(t),\, X_3(t)), \end{cases} \tag{9}$$

where $X_1(t)$, $X_2(t)$ and $X_3(t)$ denote the concentration levels of the inhibitor of apoptosis proteins (IAP), the active caspase 3 (C3a), and the active caspase 8 (C8a), respectively. Here, we assume $U_1(t)$, $U_2(t)$ as two control inputs and $\sigma : \mathbb{Z}_+ \to P := \{1,\ 2,\ 3,\ 4\}$.

In the algebraic form (2), we give the network transition matrices for four sub-networks as follows:

$$\begin{aligned} L_1 =& \delta_8[1\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2 \\ & 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 6], \\ L_2 =& \delta_8[3\ 1\ 3\ 2\ 3\ 1\ 3\ 1\ 3\ 4\ 3\ 4\ 3\ 2\ 3\ 2 \\ & 4\ 4\ 4\ 2\ 4\ 2\ 4\ 2\ 4\ 4\ 4\ 4\ 4\ 2\ 4\ 6], \\ L_3 =& \delta_8[5\ 3\ 5\ 4\ 5\ 1\ 5\ 5\ 5\ 6\ 5\ 6\ 5\ 6\ 5\ 6 \\ & 6\ 6\ 6\ 4\ 6\ 2\ 6\ 2\ 6\ 6\ 6\ 6\ 6\ 6\ 6\ 8], \\ L_4 =& \delta_8[7\ 3\ 7\ 4\ 7\ 1\ 7\ 5\ 7\ 8\ 7\ 8\ 7\ 6\ 7\ 6 \\ & 8\ 8\ 8\ 4\ 8\ 2\ 8\ 2\ 8\ 8\ 8\ 8\ 8\ 6\ 8\ 8]. \end{aligned} \tag{10}$$

We aim to design all possible SSDSD state feedback controllers to set stabilize the apoptosis network (10), given that $\mathcal{S} = \{\delta_8^2,\ \delta_8^4,\ \delta_8^6,\ \delta_8^8\}$ and $\tau = 2$.

We get $\tilde{\mathcal{S}}_I = \mathcal{S}$ from **Algorithm 1**. Then, by following **Algorithm 2** one can obtain all possible complete families of controllable sets, which are not listed here due to space restriction. We obtain all possible SSDSD state feedback controllers as follows: $K_1 = K_2 = K_3 = K_4 = \delta_4[c_1^1\ c_2^1\ c_3^1\ c_4^1\ c_5^1\ c_6^1\ c_7^1\ c_8^1]$: $c_1^1 = c_3^1 = c_5^1 = c_7^1 = 3, 4$; $c_2^1 = c_6^1 = c_8^1 = 2, 3, 4$; $c_4^1 = 1, 2, 3, 4$.

5. CONCLUSION

In this paper, we have investigated the design of all possible SSDSD state feedback stabilizers for SBCNs under arbitrary switching signal. A new algorithm has been presented to find the largest SPCIS followed by necessary and sufficient conditions to ensure the sampled-data set stabilizability of SBCNs to the largest SPCIS. Further, an algorithm to obtain all possible complete families of controllable sets has been presented. A constructive procedure has been presented to design all possible SSDSD state feedback controllers by utilizing all the complete families of controllable sets. In case the complete solution is superfluous, one can also design a subset of controllers by using the presented results. Finally, some illustrative examples have been presented to demonstrate applications of the proposed approach.

Computational load is a major challenge in the STP-based techniques for controlling SBCNs. Future research includes finding efficient methods to resolve the time and space complexities and scalability issues.

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