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A Mixed Logical Dynamical Model of the Hegselmann–Krause Opinion Dynamics

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Abstract: The heterogeneous bounded confidence Hegselmann–Krause (HK) model has been widely considered in the literature for describing opinion dynamics. In this paper a mixed logical dynamical (MLD) representation of the HK model is proposed. The linear MLD model provides a compact representation where the different thresholds of the agents influence functions explicitly appear into the model inequalities. Numerical experiments of the proposed model are used to analyze how consensus and clustering are influenced by different agents confidence bounds.

Keywords: Opinion dynamics, social networks, multi-agent systems, Hegselmann–Krause model, mixed logical dynamical models, consensus.

1. INTRODUCTION

Opinion dynamics are widely used for the analysis of social networks, see Proskurnikov and Tempo (2017, 2018). In such models the state of each agent represents a measure of the intensity of his attitude toward a particular action or goal, see Friedkin (2015).

In Hegselmann and Krause (2002) the authors proposed a representation of opinion dynamics in which the time evolution of each agent attitude, also called opinion, depends on similar opinions through suitable influence functions which are activated within some connectivity thresholds. In particular, at each time-step the state of each agent is determined as the average of the opinions of the agents who are inside his interval of influence, i.e. his neighbors, see Lorenz (2005); Meng et al. (2016). The Hegselmann–Krause (HK) model is called homogeneous when all agents have the same connectivity thresholds, heterogeneous otherwise. The HK model is called symmetric when the connection between two agents depends on the absolute value of their distance, asymmetric if any agent may have different lower and upper thresholds which select lower and upper neighbors, respectively. In the homogeneous symmetric model, the highest (smallest) opinion is non-increasing (non-decreasing) and the agents attitudes preserve their order in time, as shown in Blondel et al. (2009); Motsch and Tadmor (2014). The convergence to the consensus for the homogeneous HK model has been analyzed in the literature, see Lorenz (2005); Blondel et al. (2010).

The heterogeneous HK model exhibits more complex behaviors. In Fu et al. (2015); Chazelle and Wang (2017) the agents are classified as open-minded, middle-minded and closed-minded based on different confidence thresholds; the number of final opinion clusters is determined by the closed-minded agents, see Fu and Zhang (2014); Fu et al. (2015). In Tangredi et al. (2017) specific influence functions have been chosen to model the interaction between two agents that cooperate and compete at the same time. Other variations of the heterogeneous HK model have been proposed by considering a limited connectivity in Parasnis et al. (2018) and opinions which are influenced by group pressure in Cheng and Yu (2019).

The analysis of consensus and clustering in heterogeneous asymmetric HK networks is an open issue far from trivial even for few agents, see Liang et al. (2013); Scafuti et al. (2015). In order to tackle this complex problem several restrictive assumptions are typically introduced, such as the connectivity preservation of the graph in Olfati-Saber et al. (2007); Etesami (2019) or a very limited number of agents in Iervolino et al. (2016); Tangredi et al. (2017). Special cases of heterogeneous HK model which reach a partial convergence are analyzed in Su et al. (2017). An heterogeneous symmetric continuous-time HK model has been considered in Frasca et al. (2019) where the thresholds of pairwise agents are adapted according to the average distances of the neighbors and a hybrid dynamical model is proposed by studying its stability properties too.

The possibility to include the heterogeneous HK model into the classical hybrid systems modeling framework can represent a useful preliminary step for future theoretical analysis aimed at achieving formal proofs of convergence. This paper provides a contribution in this direction, by proposing a reformulation of a quite general heterogeneous HK model in the mixed logical dynamical (MLD) form, see Bemporad and Morari (1999). MLD models have been shown to be a useful description for the analysis of a wide class of hybrid systems, see Heemels et al. (2001). By introducing the HK model based on the differences of the opinions between pairs of agents, in this paper we derive a corresponding MLD model which explicitly takes into account the thresholds of the confidence intervals. Simulation results show that the model is useful for capturing the consensus and clustering emergent behaviors of the network in terms of sensitivity with respect to the connectivity thresholds.

The rest of the paper is organized as follows. In Section 2 the discrete-time heterogeneous HK model is presented. In Section 3 the representation in the MLD framework of the logical conditions adopted for the model are recalled. In Section 4 it is shown how the proposed MLD model of the HK opinion dynamics is derived. In Section 5 a numerical steady state analysis of the network behavior for different connectivity thresholds is presented. Section 6 concludes the paper by proposing possible future directions for the exploitation of the proposed MLD model.

2. HEGSELMANN–KRAUSE MODEL

In this section we recall the discrete-time heterogeneous HK model described through the confidence thresholds that characterize the agents.

Let us consider a set of N agents whose attitudes are state variables $x_i \in [0,1]$, $i = 1,\dots,N$. For the sake of notation, we indicate $x_i \triangleq x_i(k)$ and $x_i^+ \triangleq x_i(k+1)$, where $k \in \mathbb{N}_0$ is the discrete-time variable. The corresponding dynamics is described by

$$x_i^+ = x_i + \frac{1}{\nu_i(x)} \sum_{j=1}^{N} \phi_{ij}(x_i,x_j)(x_j - x_i) \tag{1}$$

for any $i = 1,\dots,N$, where x is the vector of all attitudes and $\nu_i(x)$ represents the number of agents connected to the agent i at each time-step, called the neighbors of i. The term $1/\nu_i(x)$ is called *susceptibility* of the agent i, see Iervolino et al. (2018).

The function $\phi_{ij}(x_i,x_j)$ represents the influence of the agent j on the agent i and it includes a nonlinear dependence on the agents attitudes x_i and x_j. In particular, $\phi_{ij}(x_i,x_j) : [0,1]^2 \to \{0,1\}$ is equal to 1 when x_i^+ is influenced by the opinion x_j and 0 otherwise. The influence of the agent j on the dynamics of the agent i is characterized by a threshold policy depending on the difference $x_j - x_i$ as follows

$$\phi_{ij}(x_i,x_j) = \begin{cases} 1, & \text{if } -d_i^G \le x_j - x_i \le d_i^C \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The thresholds $d_i^G \in [0,1]$ and $d_i^C \in [0,1]$ are called *generosity* and *competition* thresholds of the agent i respectively, see Tangredi et al. (2017). The interval $[-d_i^G, d_i^C]$ is the confidence interval of the agent i. The so-called *cooperosity* and *coopetition* behaviors can be described by considering Fig. 1. If the attitude x_i is larger than x_j, the term $x_j - x_i$ of (1) in the dynamics of x_i is negative, i.e. the interaction of the agent i with the agent j contributes with a negative sign to the determination of the attitude x_i^+. This identifies a *generosity* behavior of i versus j if j is not influenced by i, i.e. $\phi_{ji}(x_j,x_i) = 0$. Moreover, if also $\phi_{ji}(x_j,x_i) = 1$, i.e. the agents i and j influence each other, the generosity is combined with the cooperation by obtaining a *cooperosity* behaviour of i versus j. Analogously, if $x_j - x_i > 0$. the interaction of the agent i with the agent j contributes with a positive sign to the determination of the attitude x_i^+. This identifies a *competitive* behaviour if $\phi_{ji}(x_j,x_i) = 0$ and a *coopetitive* behaviour if $\phi_{ji}(x_j,x_i) = 1$. Moreover, if $d_i^G = 0$ the agent i is a *pure selfish*, while if $d_i^C = 0$ the agent i is a *pure altruist*. If both thresholds of the agent i are equal to 0 he is a *stubborn*.

In the HK model the connectivity thresholds can be different from each other. The HK model is symmetric if $d_i^G = d_i^C$ for all $i = 1,\dots,N$, asymmetric otherwise.

From (2) it follows

$$\nu_i(x) = \sum_{j=1}^{N} \phi_{ij}(x_i,x_j) \tag{3}$$

for $i = 1,\dots,N$. From (2) it is $\phi_{ii}(x_i) = 1$ and then $\nu_i(x) \ge 1$ for all $i = 1,\dots,N$, i.e. any agent is self-interacting and it may happen that no agent $j \ne i$ exists that has an opinion within the confidence interval of the agent i. By substituting (3) in (1) one obtains

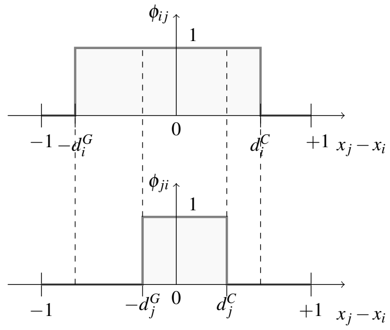

Fig. 1. Example of influence functions $\phi_{ij}(x_i,x_j)$ and $\phi_{ji}(x_j,x_i)$.

$$x_i^+ = \frac{1}{\sum_{j=1}^{N} \phi_{ij}(x_i,x_j)} \sum_{j=1}^{N} \phi_{ij}(x_i,x_j)\, x_j \tag{4}$$

for $i = 1,\dots,N$. Since the influence function $\phi_{ij}(x_i,x_j)$ can only be 0 (i is not influenced by j) or 1 (i is influenced by j), the model (4) has an interesting interpretation. Indeed, the right hand side of (4) is the average of the attitudes of the agents connected to i.

3. MODEL INEQUALITIES

The representation of the HK model in MLD form is based on the application of some typical expressions which involve logical and real variables in terms of linear inequalities. In this section we recall those formulations for the structures of interest for our model.

Consider a real vector $x \in \mathbb{R}^n$ and a logical variable $\delta \in \{0,1\}^\ell$. Say $f(x,\delta) : \mathbb{R}^n \times \{0,1\}^\ell \to \mathbb{R}$ an affine function and assume that $m \le f(x,\delta) \le M$ for all x and δ, where $m \in \mathbb{R}$ is a lower bound for the function and $M \in \mathbb{R}$ is an upper bound of the function. Define the logical scalar variable $\mu \in \{0,1\}$ as follows

$$[\mu = 1] \leftrightarrow [f(x,\delta) \le 0]. \tag{5}$$

This definition is equivalent to the set of inequalities

$$f(x,\delta) \le M(1-\mu) \tag{6a}$$
$$f(x,\delta) \ge \varepsilon + (m-\varepsilon)\mu \tag{6b}$$

where $\varepsilon > 0$ is a small scalar parameter. The equivalence between (5) and (6) can be easily verified. If $\mu = 0$ from (6b) it must be $f(x,\delta) > 0$, while if $\mu = 1$ from (6a) one obtains the right hand side of (5). Viceversa, if $f(x,\delta) \le 0$ from (6b) one obtains $\varepsilon + (m-\varepsilon)\mu \le 0$ which is verified only for $\mu = 1$. Finally, if $f(x,\delta) > 0$ from (6a) it follows $M(1-\mu) > 0$ which is verified only for $\mu = 0$.

The AND operator can be written in terms of inequalities too. Consider two logical scalar variables $\mu_1 \in \{0,1\}$ and $\mu_2 \in \{0,1\}$. Then the logical variable

$$\mu_3 = \mu_1 \mu_2 \tag{7}$$

which represents the AND operator between μ_1 and μ_2 can be equivalently written as the solution of the following set of linear inequalities

$$\mu_3 \le \mu_1 \tag{8a}$$
$$\mu_3 \le \mu_2 \tag{8b}$$
$$\mu_3 \ge \mu_1 + \mu_2 - 1. \tag{8c}$$

In particular, the first two determines that $\mu_3 = 0$ if either μ_1 or μ_2 are zero, while the third inequality ensures that if both μ_1 and μ_2 are equal to 1, then μ_3 cannot be zero.

By combining (5) and (7) it is easy to determine the inequality representation for

$$[\mu = 1] \leftrightarrow [f(x,\delta) = 0] \tag{9}$$

which can be obtained by considering the following set of conditions

$$[\mu_1 = 1] \leftrightarrow [f(x,\delta) \leq 0] \tag{10a}$$
$$[\mu_2 = 1] \leftrightarrow [-f(x,\delta) \leq 0] \tag{10b}$$
$$\mu = \mu_1\mu_2 \tag{10c}$$

where (10a) and (10b) can be expressed through inequalities in the form (6) while (10c) can be expressed through inequalities in the form (8).

Finally, the product between a logical variable μ and a scalar x can be written as a real variable

$$w = \mu x \tag{11}$$

which can be equivalent defined through the following set of inequalities

$$w \leq M\mu \tag{12a}$$
$$w \geq m\mu \tag{12b}$$
$$w \geq x - M(1-\mu) \tag{12c}$$
$$w \leq x + m(1-\mu) \tag{12d}$$

with $M \geq x$ and $m \leq x$. If $\mu = 0$ the inequalities (12a) and (12b) imply $w = 0$ and (12c) and (12d) are irrelevant. Otherwise, the inequalities (12a) and (12b) are irrelevant and (12c) and (12d) imply $w = x$.

4. MIXED LOGICAL DYNAMICAL HK MODEL

In this section, we obtain the MLD representation for the HK model (4).

4.1 Influence functions

Let us first consider how to represent the influence functions in the MLD framework. For each pair (i, j), $i = 1,\ldots,N$, $j = 1,\ldots,N$, the scalar inequality constraint in (2) can be equivalently rewritten with the two inequalities

$$x_j - x_i \leq d_i^C, \tag{13a}$$
$$x_j - x_i \geq -d_i^G. \tag{13b}$$

Let us introduce the logical variables $\delta_{ij}^C \in \{0,1\}$ and $\delta_{ij}^G \in \{0,1\}$, $i = 1,\ldots,N$, $j = 1,\ldots,N$, which are defined from (13) as follows

$$\left[\delta_{ij}^C = 1\right] \leftrightarrow \left[-x_i + x_j - d_i^C \leq 0\right], \tag{14a}$$
$$\left[\delta_{ij}^G = 1\right] \leftrightarrow \left[x_i - x_j - d_i^G \leq 0\right]. \tag{14b}$$

Each definition in (14) can be equivalently represented in terms of a set of linear inequalities by using (6). In particular, for (14a) the logical variable δ_{ij}^C takes the role of μ in (6) and the function $-x_i + x_j - d_i^C$ is used for $f(x,\delta)$. The parameters M and m in (6) can be chosen for (14a) such that $M \geq \max_{j=1,\ldots,N}\{-x_i + x_j - d_i^C\}$ and $m \leq \min_{j=1,\ldots,N}\{-x_i + x_j - d_i^C\}$ for $i = 1,\ldots,N$. Since all differences of the opinions belong to the set $[-1,1]$ and the thresholds belong to the set $[0,1]$, for the representation of (14a) in the form (6) one can choose $M = 1$ and $m = -2$.

Analogously for (14b) the logical variable δ_{ij}^G takes the role of μ in (6) and the function $x_i - x_j - d_i^G$ is used for $f(x,\delta)$. The values $M = 1$ and $m = -2$ are valid also for the representation of (14b) in the form (6).

In order to definite the influence function (2) in MLD form we need to define for each pair (i, j), $i = 1,\ldots,N$, $j = 1,\ldots,N$, a further auxiliary logical variable

$$\delta_{ij} = \delta_{ij}^C \delta_{ij}^G. \tag{15}$$

By using (7), the logical variable δ_{ij} can be equivalently defined in terms of a set of linear inequalities in the form (8) where δ_{ij} takes the role of μ_3, δ_{ij}^C for μ_1 and δ_{ij}^G for μ_2.

From (2) and (15) the influence functions are given by

$$\phi_{ij}(x_i,x_j) = 1\delta_{ij} + 0(1-\delta_{ij}) = \delta_{ij} \tag{16}$$

for $i = 1,\ldots,N$, $j = 1,\ldots,N$, and by the set of linear inequalities defined in terms of the state variables x_i and the boolean variables δ_{ij}^C, δ_{ij}^G, δ_{ij}. By looking at (6), each of the definitions (14a) and (14b) requires 2 linear inequalities in order to be represented in the MLD form. Moreover, by looking at (8), the representation of (15) for all pairs (i, j), $i = 1,\ldots,N$, $j = 1,\ldots,N$ requires $3N^2$ inequalities and then in total one requires $7N^2$ inequalities for the representation of the influence functions.

4.2 Susceptibility

As a further step towards the representation of the HK model (4) in the MLD form we need to represent the susceptibility

$$\frac{1}{\nu_i(x)} = \frac{1}{\sum_{j=1}^{N}\phi_{ij}(x_i,x_j)} \tag{17}$$

for $i = 1,\ldots,N$ in this hybrid framework. Since the number of agents is finite, the possible value assumed by the susceptibility belongs the set $\{1, \frac{1}{2}, \ldots, \frac{1}{N-1}, \frac{1}{N}\}$. For each $i = 1,\ldots,N$ we introduce N further logical variables μ_{ik}, $k = 1,\ldots,N$, defined as

$$[\mu_{ik} = 1] \leftrightarrow \left[\sum_{j=1}^{N}\delta_{ij} = k\right] \tag{18}$$

for $i = 1,\ldots,N$ and $k = 1,\ldots,N$. The meaning of each logical variable μ_{ik} is that it is equal to 1 if the i-th agent is influenced by $k-1$ neighbors (excluding himself) at some time-step. The equivalence of (18) in term of linear inequalities can be obtained by using (9) and (10) where the logical variable μ_{ik} takes the role of μ and the function $\sum_{j=1}^{N}\delta_{ij} - k$ is used for $f(x,\delta)$. The correspondent (10a) and (10b) can be written in form (6) with the parameters M and m such that

$$M \geq \max_{i=1,\ldots,N}\left\{\sum_{j=1}^{N}\delta_{ij} - k\right\}, \quad m \leq \min_{i=1,\ldots,N}\left\{\sum_{j=1}^{N}\delta_{ij} - k\right\} \tag{19}$$

respectively. Since $\sum_{j=1}^{N}\delta_{ij}$ belongs to the set $[1,N]$ and $k = 1,\ldots,N$, for the representation of (18) in the form (10) and (6) one can choose $M = N-1$ and $m = -M$.

Clearly at each time-step only one logical variable μ_{ik}, $k = 1,\ldots,N$, is equal to 1. Therefore the susceptibility of the agent i can be written as

$$\frac{1}{\sum_{j=1}^{N}\phi_{ij}(x_i,x_j)} = \sum_{k=1}^{N}\frac{1}{k}\mu_{ik} \tag{20}$$

for $i = 1,\ldots,N$, together with the inequalities required for (18). By looking at (10), each of the conditions (18) requires 7 linear inequalities in order to be represented in the MLD form and then in total one requires $7N^2$ inequalities for the representation of the susceptibility of the model.

4.3 Opinion dynamics

By substituting (16) and (20) in (4) one obtains

$$x_i^+ = \sum_{k=1}^{N} \frac{1}{k}\mu_{ik} \sum_{j=1}^{N} \delta_{ij} x_j = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{k}\mu_{ik}\delta_{ij}x_j \tag{21}$$

for $i = 1,\dots,N$. For each triple (i, j, k) with $i = 1,\dots,N$, $j = 1,\dots,N$, $k = 1,\dots,N$, one can define the logical variable

$$\gamma_{ijk} = \mu_{ik}\delta_{ij}. \tag{22}$$

The meaning of this logical variable is that γ_{ijk} is equal to 1 if the agent i is influenced by the agent j and he has $k-1$ neighbors excluding himself, and it is zero otherwise. By using (7), each logical variable γ_{ijk} can be equivalently defined in terms of a set of linear inequalities in the form (8) where γ_{ijk} takes the role of μ_3, μ_{ik} for μ_1 and δ_{ij} for μ_2. The representation of (22) for all triples (i, j, k) with $i = 1,\dots,N$, $j = 1,\dots,N$, $k = 1,\dots,N$ requires $3N^3$ inequalities.

By using (22) in (21) the HK model (4) can be rewritten as

$$x_i^+ = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{k}\gamma_{ijk}x_j \tag{23}$$

for $i = 1,\dots,N$. Let us define the scalar variables

$$w_{ijk} = \gamma_{ijk}x_j \tag{24}$$

for $i = 1,\dots,N$, $j = 1,\dots,N$, $k = 1,\dots,N$. Each of these variables is the product of a logical variable and a real variable. By using (11) each w_{ijk} can then be written as a set of linear inequalities in the form (12) where w_{ijk} takes the role of w, γ_{ijk} takes the role of μ and x_j takes the role of x. Moreover, one choose $M = 1$ and $m = 0$ for (24) in form (8), since it must be $M \geq x_j$ and $m \leq x_j$ for $j = 1,\dots,N$ and it is $x_j \in [0,1]$, $j = 1,\dots,N$. By looking at (12), each of the definitions (24) requires 4 linear inequalities in order to be represented in the MLD form. Therefore, by taking into account also the inequalities corresponding to the logical variable γ_{ijk} defined in (22), one requires in total $7N^3$ inequalities for the representation of the model dynamics.

By using (24) in (23) the model (4) can be rewritten as

$$x_i^+ = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{k} w_{ijk} \tag{25}$$

for $i = 1,\dots,N$. The MLD HK model consists of (25) for $i = 1,\dots,N$ together with the $7N^3 + 14N^2$ inequalities corresponding to (24), (22), (18), (15), (14).

For the sake of completeness we now report the inequalities of the model. By considering (24), from (12) with $M = 1$ and $m = 0$ the corresponding $4N^3$ inequalities can be rewritten as

$$0 \leq -w_{ijk} + \gamma_{ijk} \tag{26a}$$
$$0 \leq w_{ijk} \tag{26b}$$
$$-1 \leq -x_j + w_{ijk} - \gamma_{ijk} \tag{26c}$$
$$0 \leq x_j - w_{ijk} \tag{26d}$$

for $i = 1,\dots,N$, $j = 1,\dots,N$, $k = 1,\dots,N$. By considering (22), from (8) the corresponding $3N^3$ inequalities can be rewritten as

$$0 \leq -\gamma_{ijk} + \mu_{ik} \tag{27a}$$
$$0 \leq -\gamma_{ijk} + \delta_{ij} \tag{27b}$$
$$-1 \leq \gamma_{ijk} - \mu_{ik} - \delta_{ij} \tag{27c}$$

for $i = 1,\dots,N$, $j = 1,\dots,N$, $k = 1,\dots,N$. By considering (18), from (10) the corresponding $7N^2$ inequalities in the form (6) and (8) with the choice $M = N-1$ and $m = -M$ can be rewritten as

$$-k - N + 1 \leq -\sum_{j=1}^{N} \delta_{ij} - (N-1)\mu_{ik1} \tag{28a}$$
$$k + \varepsilon \leq \sum_{j=1}^{N} \delta_{ij} - (1 - N - \varepsilon)\mu_{ik1} \tag{28b}$$
$$k - N + 1 \leq \sum_{j=1}^{N} \delta_{ij} - (N-1)\mu_{ik2} \tag{28c}$$
$$-k + \varepsilon \leq -\sum_{j=1}^{N} \delta_{ij} - (1 - N - \varepsilon)\mu_{ik2} \tag{28d}$$
$$0 \leq -\mu_{ik} + \mu_{ik1} \tag{28e}$$
$$0 \leq \mu_{ik} + \mu_{ik2} \tag{28f}$$
$$-1 \leq \mu_{ik} - \mu_{ik1} - \mu_{ik2} \tag{28g}$$

for $i = 1,\dots,N$, $k = 1,\dots,N$, where μ_{ik1} and μ_{ik2} take the role of μ_1 and μ_2 in (10), respectively. By considering (15), from (8) the corresponding $3N^2$ inequalities can be rewritten as

$$0 \leq -\delta_{ij} + \delta_{ij}^C \tag{29a}$$
$$0 \leq -\delta_{ij} + \delta_{ij}^G \tag{29b}$$
$$-1 \leq \delta_{ij} - \delta_{ij}^C - \delta_{ij}^G \tag{29c}$$

for $i = 1,\dots,N$, $j = 1,\dots,N$. By considering (14), from (6) the corresponding $4N^2$ inequalities can be rewritten as

$$-d_i^C - 1 \leq x_i - x_j - \delta_{ij}^C \tag{30a}$$
$$d_i^C + \varepsilon \leq -x_i + x_j + (2+\varepsilon)\delta_{ij}^C \tag{30b}$$
$$-d_i^G - 1 \leq -x_i + x_j - \delta_{ij}^G \tag{30c}$$
$$d_i^G + \varepsilon \leq x_i - x_j + (2+\varepsilon)\delta_{ij}^G \tag{30d}$$

for $i = 1,\dots,N$, $j = 1,\dots,N$.

By collecting (25)–(30) the MLD HK model can be easily written in the matrix form

$$x^+ = Aw \tag{31a}$$
$$b \leq B_x x + B_w w + B_\gamma \gamma + B_\mu \mu + B_\delta \delta \tag{31b}$$

with the real variables $x \in [0,1]^N$ and $w \in [0,1]^{N^3}$, the logical variables $\gamma \in \{0,1\}^{N^3}$, $\mu \in \{0,1\}^{3N^2}$ and $\delta\{0,1\}^{3N^2}$, the constant matrices A and $B_\bullet$ with suitable dimensions, and the constant vector $b \in \mathbb{R}^{7N^3+14N^2}$ whose components are the left hand side of the inequalities (26)–(30). It should be noticed that the thresholds parameters $\{d_i^C\}_{i=1}^N$ and $\{d_i^G\}_{i=1}^N$ are part of the entries of the vector b and do not influence the other matrices of the MLD model (31). Clearly, all matrices of the model (31) are constant and do not depend on the particular initial conditions which can be arbitrary chosen provided that $0 \leq x_i(0) \leq 1$, $i = 1,\dots,N$.

The number of inequalities which define the model (31b) is quite large, i.e. $7N^3 + 14N^2$. This number depends only on the number of agents and does not depend on the fact that the thresholds may be different among the agents. On the other hand the model (31) represents the heterogeneous HK opinion dynamics for a quite general scenario where the influence functions of the agents can be characterized by different thresholds.

This compact matrix representation is linear in the MLD framework and this property might be exploited in future works for obtaining formal results on the convergence properties of the model.

Note that the MLD form of the HK model preserves the distributed nature of the original HK model, in the sense that if a new agent is added to the network, the corresponding model can be incrementally obtained form (25)–(30). In particular, for $N+1$ agents the dynamics (25) can be written as

$$x_i^+ = \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{k}w_{ijk} + \sum_{k=1}^{N+1}\frac{1}{k}w_{i,N+1,k} + \sum_{j=1}^{N}\frac{1}{N+1}w_{ij,N+1} \quad (32)$$

for $i = 1,\ldots,N+1$, the inequalities (26), (27), (29) and (30) remain the same and one must add the corresponding ones for $i = N+1$, $j = N+1$ and $k = N+1$, and the inequalities (28) must be rewritten by substituting N with $N+1$ for $i = 1,\ldots,N+1$, $k = 1,\ldots,N+1$.

5. SIMULATION RESULTS

In this section we provide a steady state analysis for different confidence thresholds. Consensus, clustering and time to convergence are considered. The term consensus indicates the steady state solution corresponding to all attitudes converging to the same opinion. For simplicity, in this section we consider the homogeneous model in which $d_i^C = d^C$ and $d_i^G = d^G$ for all $i = 1,\ldots,N$.

We now analyze the consensus value and the convergence time by varying the competition and generosity thresholds. In order to examine the relationship between the steady state and the bounds of the confidence interval we use a "walking along lines" approach in the plane (d^C, d^G), see Hegselmann and Krause (2002).

Let us consider the symmetric HK model, i.e. $d^C = d^G$. The distribution of agents for different confidence intervals in this symmetric HK model is shown in Fig. 2. The colored bar denotes the number of agents which are included within a range of opinions of amplitude equal to 0.05. The results are obtained from 100 agents characterized by a uniform initial distribution of opinions. Under these conditions, the consensus is reached for $d^C \geq 0.25$, i.e. for confidence intervals larger than about 0.5. Moreover, the preserving average condition is satisfied and then the consensus value is equal to the average of initial opinions. The number of steady state clusters increases by decreasing d^C.

The convergence time to the consensus can be evaluated from the time-step k_c in which the graph associated to the network becomes complete ($\phi_{ij}(x_i,x_j) = 1$ for all $i = 1,\ldots,N$, $j = 1,\ldots,N$). Indeed for a complete graph the dynamics (4) becomes $x_i^+ = \frac{1}{N}\sum_{j=1}^{N} x_j$, which corresponds to the same right hand side for all $i = 1,\ldots,N$, i.e. $x_i(k) = x_c = \frac{1}{N}\sum_{i=1}^{N} x_i(k_c)$ for all $k > k_c$ and for all $i = 1,\ldots,N$. Then the convergence time k_c to the consensus is $k_c + 1$. Figure 3 shows that k_c decreases by increasing the confidence interval.

Now we focus the analysis on the HK model with a constant confidence interval given by $d^G + d^C = \alpha$ with $\alpha \in [0,2]$. The results in Fig. 4 show the distributions of agents opinions for different values of the connectivity interval α by varying the competition threshold d^C. Obviously, the generosity threshold is given by $d^G = \alpha - d^C$. The results are obtained from 100 agents and a uniform initial distribution of opinions. The first

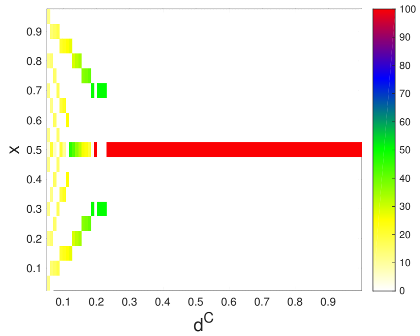

Fig. 2. Distribution of steady state values of the agents attitudes for different confidence thresholds in the symmetric HK model. Colors indicate the number of agents in each steady state group.

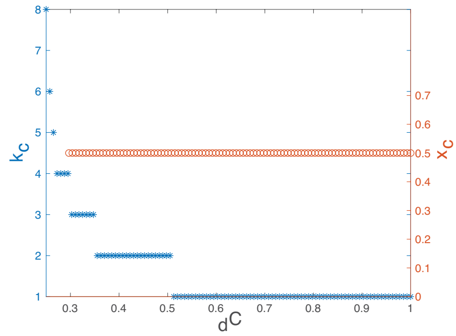

Fig. 3. Convergence time k_c and consensus value x_c in the symmetric HK model, i.e. $d^C = d^G$.

three cases indicate that the consensus is not reached because the confidence thresholds are too small. The number of clusters increases by decreasing α. Moreover, when the confidence threshold is equal to 0.3 and 0.4, the consensus is reached for high values of d^C or d^G. For $\alpha \geq 0.5$ the consensus is reached for all values of d^C and the strictly increasing curves show that x_c increases with the competition threshold d^C and then decreases with the generosity threshold d^G. Roughly speaking a more competitive behavior of the agents helps to reach a higher consensus value. This analysis can be useful for developing a consensus strategy based on the increase of the competition threshold in order to obtain a high attitude for all agents.

The convergence time k_c is shown in Fig. 5. The results highlight that the convergence is usually slower for a larger difference between the two thresholds. Analogously, asymmetric confidence intervals polarized towards the competition allow the agents to reach a larger consensus.

6. CONCLUSION

The heterogeneous HK opinion dynamics has been expressed in the MLD framework. The model takes into account the different agents confidence thresholds which explicitly appear into the system inequalities. Simulation results have shown the capability of the model to detect consensus and clustering for different values of the thresholds bounds. It is shown that by increasing the competition threshold of the agents the consensus corresponds to a larger attitude. The MLD model provides a compact representation of the HK model which could be

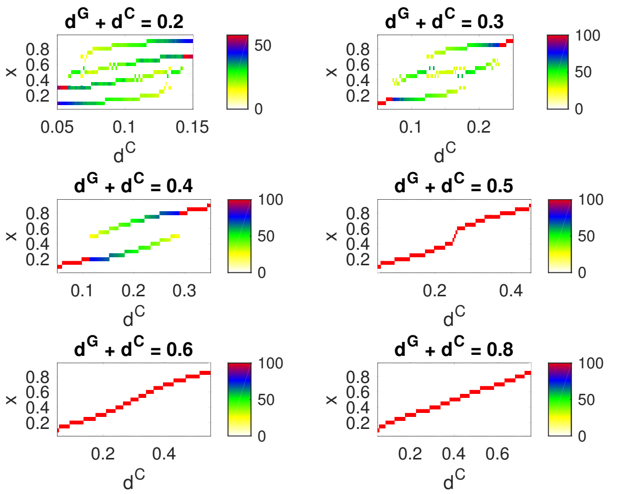

Fig. 4. Distribution of steady state values of the agents attitudes in the HK model with fixed confidence interval, i.e. $d^C + d^G = \alpha$ for different values of α.

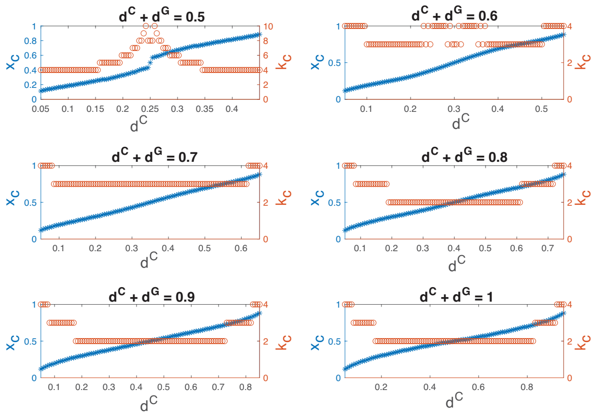

Fig. 5. Convergence time k_c and consensus value x_c in the scenarios where the consensus is reached, i.e. for confidence interval $\alpha \geq 0.5$.

exploited in future work in order to investigate for possible stability conditions for the opinion dynamics convergence to the consensus.

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