Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

On the decay rate for degenerate gradient flows subject to persistent excitation ⋆

Yacine Chitour * **Paolo Mason** * **Dario Prandi** *

* *Université Paris-Saclay, CNRS, CentraleSupelec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France (e-mail: {yacine.chitour, paolo.mason, dario.prandi}@centralesupelec.fr).*

Abstract: In this paper, we study the worst rate of exponential decay for degenerate gradient flows in $\mathbb{R}^n$ of the form $\dot{x}(t) = -c(t)c(t)^\top x(t)$, issued from adaptative control theory, under a persistent excitation (PE) condition. That is, there exists $a, b, T > 0$ such that, for every $t \geq 0$ it holds $a\,\mathrm{Id}_n \leq \int_t^{t+T} c(s)c(s)^\top ds \leq b\,\mathrm{Id}_n$. Our main result is an upper bound of the form $\frac{a}{(1+b)^2T}$, to be compared with the well-known lower bounds of the form $\frac{a}{(1+nb^2)T}$. As a byproduct, we also provide necessary conditions for the exponential convergence of these systems under a more general (PE) condition. Our techniques relate the worst rate of exponential decay to an optimal control problem that we study in detail.

Keywords: Persistent excitation, Degenerate gradient flow.

1. INTRODUCTION

In this paper, we consider the class of systems of the type

$$\dot{x}(t) = -c(t)c(t)^\top x(t), \qquad x \in \mathbb{R}^n, \tag{DGF}$$

where the signal $c : [0, \infty) \to \mathbb{R}^n$ is square integrable and verifies the *persistent excitation* condition. That is, there exist constants $a, b, T > 0$ such that,

$$\forall t \geq 0, \quad a\,\mathrm{Id}_n \leq \int_t^{t+T} c(s)c(s)^\top ds \leq b\,\mathrm{Id}_n\,. \tag{PE}$$

Here, Id_n denotes the $n \times n$ identity matrix and the inequalities are to be understood in the sense of symmetric matrices. Henceforth, we will denote by $\mathcal{C}_n(a, b, T)$ the set of signals satisfying (PE).

The above dynamics appears in the context of adaptive control and identification of parameters, and are usually referred to as *degenerate gradient flow systems* (DGF), since the Euclidean norm is decreasing along its trajectories (cf. Sondhi and Mitra (1976); Aeyels and Sepulchre (1994); Brockett (2000); Andersson and Krishnaprasad (2002), and references therein). As an immediate consequence, these trajectories are defined on $[0, +\infty)$, and it is well-known that (PE) condition is equivalent to global exponential stability of (DGF), see, e.g., Anderson (1977).

The rate of exponential decay for (DGF) is defined as

$$R(c) := -\limsup_{t\to\infty} \frac{\ln \|\Phi_c(t, 0)\|}{t},$$

where $\Phi_c(t, t_0)$ denotes the flow (or fundamental matrix) of (DGF) from t_0 to t. The main object of interest in this paper is the worst-rate of exponential decay

$$R(a, b, T, n) := \inf\{R(c) \mid c \in \mathcal{C}_n(a, b, T)\},$$

and, in particular, its estimation in terms of the parameters a, b, T, n. Let us recall the well-known estimate [1]

$$R(a, b, T, n) \geq \frac{Ca}{(1 + nb^2)T}. \tag{1}$$

for some universal constant $C > 0$. Our main result is then the following, which shows that (1) is optimal, for n fixed.

Theorem 1. There exists $C_0 > 0$ such that for every $n \in \mathbb{N}$, $T > 0$, and $0 < a \leq b$ it holds

$$R(a, b, T, n) \leq \frac{C_0 a}{(1 + b^2)T}.$$

Remark. The above implies that

$$\lim_{b\to+\infty} R(a, b, T, n) = 0.$$

This is in accordance with (Barabanov et al., 2005, Prop. 1), where it is proved that assuming only the first inequality in (PE), i.e., taking $b = +\infty$, is not sufficient to guarantee the convergence to the origin for trajectories of (DGF).

We now turn to a brief description of the strategy of proof, which, as in Barabanov and Ortega (2017), is based on a representation in polar coordinates of (DGF). Indeed, this representation allows us to introduce an optimal control problem (OCP), whose minimal value provides an upper bound for $R(a, b, T, n)$. We then show that, due to the monotonicity w.r.t. the dimension of the minimal value of (OCP), it is enough to bound it for $n = 2$. Finally, to obtain the result, we apply Pontryagin Maximum Principle to a convexification of this optimal control problem, and we explicitly integrate the resulting Hamiltonian system.

⋆ This research was partially supported by the iCODE Institute, research project of the IDEX Paris-Saclay, and by the Hadamard Mathematics LabEx (LMH) through the grant number ANR-11-LABX-0056-LMH in the “Programme des Investissements d’Avenir”. The first and last authors have been supported by the ANR project SRGI ANR-15-CE40-0018.

[1] Estimate (1) follows, e.g., by (Andersson and Krishnaprasad, 2002, Theorem 3.1), observing that, in their notations, $\beta^2 \leq 1$ and thus $-\ln(1 - \beta^2) \geq \beta^2$.

1.1 Generalized persistent excitation

Recently, there has been an increasing interest in considering more general types of persistent excitation conditions, cf. Barabanov and Ortega (2017); Praly (2017); Efimov et al. (2018). We focus on the following *generalised persistent excitation* condition:

$$a_\ell Id_n \leq \int_{\tau_\ell}^{\tau_{\ell+1}} c(t)c(t)^\top dt \leq b_\ell \operatorname{Id}_n, \tag{PEG}$$

where $(a_\ell)_{\ell\in\mathbb{N}}$, $(b_\ell)_{\ell\in\mathbb{N}}$ are sequences of positive numbers, and $(\tau_\ell)_{\ell\in\mathbb{N}}$ is a strictly increasing sequence of positive times such that $\tau_\ell \to +\infty$ as $\ell \to +\infty$.

An important question is then to determine under which condition (PEG) guarantees global asymptotic stability (GAS) for (DGF). In Praly (2017) (cf. also Barabanov and Ortega (2017)) the author proved the sufficient condition:

$$\sum_{\ell=0}^{\infty} \frac{a_\ell}{(1+b_\ell)^2} = +\infty. \tag{2}$$

As a byproduct of our analysis, in Appendix B we show that this condition is indeed necessary.

Theorem 2. All systems (DGF) that satisfy condition (PEG) are GAS if and only if (2) holds.

We stress that our interest lies in the study of systems satisfying (PEG) as a class. That is, the above theorem states that if (2) is not satisfied, then there exists an input signal satisfying (PEG) that is not GAS. However, for a fixed signal satisfying (PEG), condition (2) is not necessary for GAS, as shown in (Barabanov and Ortega, 2017, Prop. 7).

1.2 Notations

We denote by Sym_n the set of $n \times n$ symmetric real matrices, and by Sym_n^+ the subset of non-negative definite ones. Moreover, for $0 < a \leq b$, we let $\operatorname{Sym}_n(a,b)$ be the set of matrices $S \in \operatorname{Sym}_n$ such that $a\operatorname{Id}_n \leq S \leq b\operatorname{Id}_n$ in the sense of quadratic forms.

Observe that the class $\mathcal{C}_n(a,b,T)$ of signals that satisfy (PE) can be characterized as the class of those c such that $Q(t) := \int_t^{t+T} c(s)c(s)^\top ds \in \operatorname{Sym}_n(a,b)$ for $t \geq 0$. This can be relaxed to a persistent excitation condition on measurable functions $S : \mathbb{R}_+ \to \operatorname{Sym}_n^+$, by requiring the existence of positive constants a, b, T such that

$$\int_t^{t+T} S(s)\,ds \;\in\; \operatorname{Sym}_n(a,b),\;\; t \geq 0. \tag{3}$$

We let $\operatorname{Sym}_n^+(a,b,T)$ be the set of functions that satisfy (3). Clearly, $\mathcal{C}_n(a,b,T)$ can be identified with the rank one elements of $\operatorname{Sym}_n^+(a,b,T)$. Observe that if $S \in \operatorname{Sym}_n^+(a,b,T)$ (resp. $c \in \mathcal{C}_n(a,b,T)$), then the same is true for $USU^\top$ (resp. Uc), for any orthogonal matrix $U \in \mathrm{O}(n)$.

2. REDUCTION OF THEOREM 1 TO A 2D OPTIMAL CONTROL PROBLEM

On $\mathbb{R}^n$, we consider spherical coordinates $x = r\omega$, where $r \in \mathbb{R}_+$ and $\omega \in \mathbb{S}^{n-1}$. Then, (DGF) reads

$$\begin{aligned} \dot{r} &= -\left(c^\top \omega\right)^2 r, \\ \dot{\omega} &= -c^\top \omega\left(c - \left(c^\top \omega\right)\omega\right). \end{aligned}$$

Observe that the dynamics of ω do not depend on r. For any $x \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathcal{C}_n(a,b,T)$, and $t, T > 0$, it holds

$$\begin{aligned} \ln\left(\frac{\|\Phi_c(T+t,t)x\|}{\|x\|}\right) &= \ln\left(\frac{r(T+t)}{r(t)}\right) \\ &= -\int_t^{T+t} \left(c^\top \omega\right)^2 ds, \end{aligned} \tag{6}$$

where we let $\Phi_c(t,0)x = (r(t), \omega(t))$. We then consider the following optimal control problem

$$\min_{\substack{c\in\mathcal{C}_n^0(a,b,T),\\ \omega_0\in\mathbb{S}^{n-1}}} J(c,\omega_0), \quad J(c,\omega_0) := \int_0^T \omega^\top cc^\top \omega\, dt. \tag{OCP}$$

Here, $\mathcal{C}_n^0(a,b,T)$ denotes the set of restrictions of elements of $\mathcal{C}_n(a,b,T)$ to $[0,T]$ and $\omega : [0,T] \to \mathbb{S}^{n-1}$ is the trajectory of (5) corresponding to the control $c \in \mathcal{C}_n^0(a,b,T)$ and to the initial condition $\omega(0) = \omega_0$.

Let $\mu(a,b,T,n)$ be the infimum obtained for (OCP). In Proposition 4, we will relate the worst rate of exponential decay for (DGF) with this quantity. However, we first need to establish certain facts.

Since, in order to prove the existence of minimizers for optimal control problems, one usually needs to apply some compactness argument, it is useful to restate the optimal control problem on a closed space of admissible controls for a weak topology. For this purpose, we convexify the set of admissible controls. Observing that the convex hull of $\mathcal{C}_n^0(a,b,T)$ (identified with a set of rank-one matrix valued functions) is $\operatorname{Sym}_n^{+,0}(a,b,T)$, this yields the following optimal control problem:

$$\min_{\substack{S\in\operatorname{Sym}_n^{+,0}(a,b,T)\\ \omega_0\in\mathbb{S}^{n-1}}} \bar{J}(S,\omega_0), \quad \bar{J}(S,\omega_0) := \int_0^T \omega^\top S\omega\, dt. \tag{Conv}$$

Here, $\operatorname{Sym}_n^{+,0}(a,b,T)$ denotes the set of restrictions of elements of $\operatorname{Sym}_n^+(a,b,T)$ to $[0,T]$, and ω satisfies

$$\dot{\omega} = -S\omega + (\omega^\top S\omega)\omega, \quad \omega(0) = \omega_0. \tag{7}$$

Note that $\operatorname{Sym}_n^{+,0}(a,b,T)$ is the closure of $\mathcal{C}_n^0(a,b,T)$ in the L^1-weak topology of matrix-valued functions. Moreover it is easy to check that the input-output map associating with each (integrable) S the corresponding solution of (7) (taking values in the space of continuous functions from $[0,T]$ to $\mathbb{S}^{n-1}$) is continuous in the L^1-weak topology. Therefore one easily deduces that $\mu(a,b,T,n)$ coincides with the infimum obtained for (Conv).

The following crucial result is proved in the forthcoming paper Chitour et al. (2019).

Proposition 3. The optimal control problem (Conv) admits rank-one minimizers with constant trace. As a consequence the optimal control problem (OCP) admits minimizers. Moreover, there exists a $2T$-periodic control $c_* \in \mathcal{C}_n(2a,2b,2T)$ and an initial condition $\omega_0 \in \mathbb{S}^{n-1}$ such that

$$\omega_*(t) = \frac{\Phi_{c_*}(t,0)\omega_0}{\|\Phi_{c_*}(t,0)\omega_0\|},$$

is a $2T$-periodic trajectory and both $t \mapsto c_*|_{[0,T]}(t)$ and $t \mapsto c_*|_{[T,2T]}(t-T)$, together with the respective initial conditions ω_0 and $\omega_*(T)$, are minimizers for (OCP). Finally, the minimum $\mu(a,b,T,n)$ is independent of $T>0$.

Motivated by the previous result, from now on we drop the time dependence from $\mu(a,b,T,n)$, writing it as $\mu(a,b,n)$.

We now establish a link between μ and R, the worst rate of exponential decay for (DGF).

Proposition 4. It holds that

$$R(a,b,T,n) \leq \frac{2}{T}\mu\left(\frac{a}{2},\frac{b}{2},n\right).$$

Proof. Let $c_* \in \mathcal{C}_n(2a,2b,2T)$ be the $2T$-periodic control given by Proposition 3. It then follows from the latter and (6) that

$$\ln\|\Phi_{c_*}(kT,0)\| = \sum_{\ell=1}^{k}\ln\|\Phi_{c_*}(\ell T,(\ell-1)T)\| = -k\mu(a,b,n).$$

Then, standard arguments yield

$$\begin{aligned} R(2a,2b,2T,n) &\leq R(c_*) \\ &\leq -\lim_{\ell\to+\infty}\frac{\ln\|\Phi_{c_*}(2\ell T,0)\|}{2\ell T} = \frac{\mu(a,b,n)}{T}, \end{aligned}$$

concluding the proof.

The next result allows to reduce the proof of Theorem 1 to the analysis of (OCP) in dimension 2.

Lemma 5. The map $(a,b,n)\mapsto\mu(a,b,n)$ is non-decreasing with respect to a, and non-increasing with respect to b and n, respectively.

Proof. The statements regarding a and b are trivial. The statement regarding n, follows by first considering $\mu(a,b,n)$ as the minimal value for problem (Conv). Then, we observe that any admissible trajectory ω of (5) in dimension n, associated with some $S\in\mathrm{Sym}_n^0(a,b,T)$, yields an admissible trajectory $\tilde\omega$ in dimension $q>n$ associated with $\tilde S\in\mathrm{Sym}_q^0(a,b,T)$ such that $\bar J(S,\omega(0)) = \bar J(\tilde S,\tilde\omega(0))$. Indeed, it suffices to let

$$\tilde S = \begin{pmatrix} S & 0 \\ 0 & \frac{a}{T}\mathrm{Id}_{q-n}\end{pmatrix},$$

and to observe that the associated trajectory of (7), with initial condition $(\omega(0),0)$, is simply $t\mapsto(\omega(t),0)$.

The proof of Theorem 1 thus reduces to the following.

Proposition 6. There exists a constant $C_0>0$ such that, for all $0<a\leq b$ and $T>0$, it holds

$$\mu(a,b,2) \leq \frac{C_0 a}{1+b^2}.$$

3. UPPER BOUND FOR THE MINIMAL VALUE OF THE 2D PROBLEM

In this section we present an argument of proof for Proposition 6. Due to its complexity, we are obliged to defer part of the proofs to Chitour et al. (2019).

We observe that, thanks to Lemma 5, it is enough to consider a fixed and b arbitrarily large. In that case, notice that $\mu(a,b,2)$ must necessarily tend to zero as b tends to infinity. Indeed, if this were not the case, one could easily contradict the result of Barabanov et al. (2005), see Remark 1.0.1. The following result yields a simple upper bound for μ.

Proposition 7. It holds

$$\mu(a,b,n) \leq a.$$

Proof. Let $\{e_1,\dots,e_n\}$ be the canonical basis of $\mathbb{R}^n$. Let the control $c:\mathbb{R}\to\mathbb{R}^n$ be defined for $t\in[0,T]$ by

$$c(t) = \sqrt{an}\,e_j \quad \text{if } t\in\left[\frac{(j-1)T}{n},\frac{jT}{n}\right), \quad j\in\{1,\dots,n\},$$

and then extended by periodicity for $t\in\mathbb{R}$. Namely, $c(t) = c(t-\lfloor t/T\rfloor T)$. Then $c\in\mathcal{C}_n(a,b,T)$. Indeed, for any $t\geq 0$, we have

$$\int_t^{t+T} c(s)c^\top(s)\,ds = \int_0^T c(s)c^\top(s)\,ds = a\,\mathrm{Id}_n.$$

As a consequence, the restriction to $[0,T]$ of c is an admissible control for (OCP). Finally, letting $\omega_0=e_1$, we have $\omega(t)=\omega_0$ for $t\in[0,T]$, and thus

$$\mu(a,b,n) \leq J(c,\omega_0) = \int_0^{T/n}\|c(s)\|^2\,ds = a.$$

This completes the proof.

The optimal control problem (Conv) can then be recast as follows: Minimize $\bar J(S,\omega_0)$ with respect to $S\in\mathrm{Sym}_2^{+,0}(a,b,T)$ and $\omega_0\in\mathbb{S}^1$ along trajectories of

$$\begin{aligned} \dot\omega &= -S\omega + \left(\omega^\top S\omega\right)\omega, \\ \dot Q &= S, \end{aligned}$$

starting at any $(\omega_0,0)$ so that $Q(T)\in\mathrm{Sym}_2(a,b)$.

Thanks to Proposition 3, this optimal control problem admits a minimiser. The following proposition, proved in Appendix A, gives necessary optimality conditions.

Proposition 8. Let S be an optimal control for problem (OCP) associated with the trajectory (ω,Q). Then,

(1) $S=cc^\top$ for some $c\in\mathcal{C}_2^0(a,b,T)$;
(2) Up to a time reparametrization and a rotation, we can assume $\|c\|=1$, $T=a+b$, and $Q(T)=\mathrm{diag}(a,b)$;
(3) The adjoint state $(p,P_Q)\in T_\omega^*\mathbb{S}^1\times T_Q^*\mathrm{Sym}_2$ satisfy

$$\dot p = Sp - \left(\omega^\top S\omega\right)p - \dot\omega, \quad \dot P_Q = 0, \tag{10}$$

with $p(0)=p(T)=0$ and $P_Q=\mathrm{diag}(\alpha,-d)$ for $\alpha\in(0,1]$ and $d\geq 0$.
(4) It holds $M\leq 0$ and $MS\equiv SM\equiv 0$, where

$$M(t) = P_Q - (\omega p^\top + p\omega^\top + \omega\omega^\top). \tag{11}$$

Identifying vectors of $\mathbb{R}^2$ with complex numbers, we set

$$\omega = e^{i\theta/2}, \quad \omega^\perp = ie^{i\theta/2}, \quad p = \eta\omega^\perp,$$

where $\theta\in\mathbb{S}^1$ and $\eta\in\mathbb{R}$. Let $S=cc^\top$ be a control satisfying the necessary optimality conditions of Proposition 8. Then, since $Mc\equiv 0$ and $\mathrm{Tr}\,M\equiv\alpha-d-1$, we have $M=(\alpha-d-1)cc^\top$. Comparing the time derivative of this expression with the one of (11), letting $c=e^{i\phi/2}$, by (8) and (10), we get

$$\begin{aligned}\dot\theta &= \sin(\theta-\phi),\\ \dot\eta &= -\frac{\sin(\theta-\phi)}{2}+\eta\cos(\theta-\phi), \qquad (12)\\ \dot\phi &= \frac{2\eta}{1-\alpha+d}.\end{aligned}$$

Observe that, up to rotating ω and c, we can always assume $\phi_0 := \phi(0) \in (0,\pi]$. Moreover, $M(0)c(0) = 0$ yields

$$\cos\theta_0 = 1 - \frac{2d(1-\alpha)}{\alpha+d} \qquad (13)$$

$$\cos\phi_0 = \frac{2d(d+1)}{\alpha(1-\alpha)+d(d+1)} - 1. \qquad (14)$$

A crucial point is that differentiating $\det M \equiv 0$ yields $\eta(2\dot\eta - (\alpha+d)\sin\phi) \equiv 0$. This and (12), allow then to show that the angle ϕ of the control behaves following an inverted pendulum equation. This fact is stated in the following lemma, proved in Chitour et al. (2019), where we also obtain explicit formulae linking (α, d) with (a, b).

Lemma 9. It holds

$$\ddot\phi = \frac{1}{2\nu^2}\sin\phi, \quad \text{where} \quad \nu = \sqrt{\frac{1-\alpha+d}{2(\alpha+d)}}.$$

Moreover, there exists $\kappa \in \mathbb{N}^*$ such that $\eta(t) = 0$ if and only if $t = \ell T/\kappa$ for $\ell \in \{0,\dots,\kappa\}$

$$a = \nu\kappa K_+(\phi_0) \quad b = \nu\kappa K_-(\phi_0),$$

where we let

$$K_\pm(\gamma) = \int_\gamma^\pi \frac{1\pm\cos\phi}{\sqrt{\cos\gamma-\cos\phi}}\,d\phi.$$

Since $T = a + b$, one can check that

$$J(c,\omega(0)) = \int_0^{a+b}\left(\cos\frac{\theta-\phi}{2}\right)^2 dt. \qquad (15)$$

Putting together the above result and (12), it is possible to estimate this quantity. However, it turns out that the result depends on the integer κ. The next lemma, proven in detail in Chitour et al. (2019), shows that for any (α, d) it is possible to construct a control c for which $\kappa = 1$.

Lemma 10. Let $(\alpha, d) \in (0,1)\times[0,+\infty)$. Then, there exist $0 < a \le b$ and a control $c = e^{i\phi/2} \in \mathcal{C}_2^0(a, b, a+b)$ satisfying the necessary conditions of Proposition 8, where (θ, η) is given by (12) with $\eta(0) = 0$ and $\theta(0)$ satisfying (13). Moreover, $\eta(t) \ne 0$ for $t \in (0, a+b)$. In particular, ϕ_0 satisfies (14) and

$$a = \nu K_+(\phi_0) \quad \text{and} \quad b = \nu K_-(\phi_0).$$

Sketch of the proof. Let (α, d) be fixed. Then, in order for (θ, η) to satisfy $\det M \equiv 0$, it has to hold that

$$\cos\theta - 2\eta\sin\theta = \frac{2\eta^2+2\alpha d+\alpha-d}{\alpha+d}.$$

Letting γ be the unique angle in $[0, 2\pi)$ such that $\sin\gamma = (1+4\eta^2)^{-1/2}$ and $\cos\gamma = 2\eta(1+4\eta^2)^{-1/2}$, the above can be recast as

$$\sin(\gamma-\theta) = F(\eta^2), \qquad (16)$$

where we let

$$F(\xi) = \frac{2\xi+2\alpha d+\alpha-d}{(\alpha+d)\sqrt{1+4\xi}}.$$

This allows to write $\theta = f(\eta)$ via the implicit function theorem. Moreover, considering (θ, η) as functions of t, differentiating (16) w.r.t. this variable, and replacing (12), we get that the control ϕ must satisfy

$$\sin(\theta-\phi)\left(\frac{4\eta}{1+4\eta^2}\cos(\gamma-\theta)+F'(\eta^2)\right) + \cos(\theta-\phi)\left(2\eta F'(\eta^2)-\cos(\gamma-\theta)\right) = 0.$$

This allows to recover $(\theta-\phi) \mod \pi$, and thus to reconstruct (ϕ, θ, η) by (12), with $\phi(0) = \phi_0$ satisfying (14).

One proves that the remaining optimality conditions of Proposition 8 are satisfied on a certain interval $[0,\tau]$, i.e., $Mc \equiv 0$. Finally, it turns out that letting $a = \nu K_+(\phi_0)$ and $b = \nu K_-(\phi_0)$, it holds $\tau = a+b$, completing the proof.

The final essential ingredient in order to prove Proposition 6 is the following consequence of the above, whose proof reduces to a careful asymptotic study of $K_\pm$ and an application of the Inverse Function Theorem.

Proposition 11. For every $0 < a \le b$, with $b \gg 1$, there exists a control $c \in \mathcal{C}_2^0(a, b, a+b)$ and an initial condition $\omega_0 \in \mathbb{S}^1$ satisfying the optimality conditions of Proposition 8 and such that $\eta(t) \ne 0$ for $t \in (0, a+b)$. In addition, it holds

(1) The initial condition ϕ_0 is the largest solution in $(0,\pi)$ of
$$\frac{K_+(\phi_0)}{K_-(\phi_0)} = \frac{a}{b};$$
(2) There exist two universal constants $c_0, c_1 > 0$ such that
$$c_0\frac{a}{b} \le \left(\cos\frac{\phi_0}{2}\right)^2 \le c_1\frac{a}{b}$$
(3) As $b \to +\infty$ it holds
$$\alpha \sim_{b\to+\infty} \frac{K_-(\phi_0)^2}{2b^2} \quad \text{and} \quad d = \frac{1+\cos\phi_0}{2}\alpha.$$

We now describe the argument for Proposition 6: Taking into account the fact that $\bar J(S, \omega(0)) < a$, it is enough to establish the result for sequences $(a_\ell, b_\ell)_{\ell\in\mathbb{N}}$ such that b_ℓ tends to infinity as ℓ tends to infinity. Moreover, since we need to upper bound $\mu(a, b, 2)$, it is enough to find a control $c \in \mathcal{C}_2^0(a, b, a+b)$ and an initial condition ω_0 whose cost $J(c, \omega_0)$ is indeed smaller than $C_0 a/(1+b^2)$ for some universal constant C_0. We claim that such quantities are provided by Proposition 11. The computation of the corresponding cost can be done from (15), via a precise study of the dynamics of $\theta-\phi$, and on the asymptotic estimates in Proposition 11. We collect the resulting estimate in the following.

Lemma 12. Assume that b is large enough. Let c, ω_0 be the control and the initial condition provided by Proposition 11. Then, there exist two positive constants c_2, c_3, independent of (a, b), such that

$$c_2\frac{a}{b^2} \le J(c,\omega_0) \le c_3\frac{a}{b^2}.$$

4. CONCLUSION

In this paper, we have confirmed the sharpness of the lower bounds obtained in previous works for the worst rate of exponential decay of degenerate gradient flows in $\mathbb{R}^n$ subject to the persistent excitation condition (PE). More precisely, these lower bounds are of the type $\frac{a}{(1+nb^2)T}$, and we provided upper bounds of the type $\frac{a}{(1+b)^2 T}$. Since our analysis is based on a reduction to the 2-dimensional case, it cannot detect an eventual dependence on the dimension n of the upper bounds. However, we conjecture the factor n in the denominator of the known upper bounds to be the optimal one. In order to obtain it, we are currently extending our analysis to treat the case of a general dimension n.

Finally, as shown in Chitour et al. (2019), these results also allow to address L_2-gain issues related to the above dynamics, and in particular to an open problem posed in Rantzer (1999); see also Efimov and Fradkov (2015) and Efimov et al. (2019) for recent results on this subject.

Appendix A. NECESSARY CONDITIONS FOR OPTIMALITY IN THE 2D CASE

In this section, we prove Proposition 8 by applying the Pontryagin Maximum Principle, cf. Clarke (1990), to the optimal control problem (Conv).

The state space of the system is $\mathbb{S}^1 \times \mathrm{Sym}_2$. The covector associated to a state (ω, Q) is $(p, P_Q) \in T^*_\omega \mathbb{S}^1 \times T^*_Q \mathrm{Sym}_2$. Letting $\lambda = (\omega, Q, p, P_Q)$, the Hamiltonian turns out to be

$$H(\lambda, \lambda_0, S) = \frac{\mathrm{Tr}\,(SM(\omega, p, P_Q))}{2},$$

where $\lambda_0 \in \{0, 1\}$, and $M \in \mathrm{Sym}_2$ is defined by

$$M(\omega, p, P_Q) := P_Q - \big(\omega p^\top + p\omega^\top + \lambda_0 \omega\omega^\top\big). \qquad \text{(A.1)}$$

At any (ω, Q) we make the identifications $T^*_Q \mathrm{Sym}_2 \simeq \mathrm{Sym}_2$ and $T^*_\omega \mathbb{S}^1 \simeq \omega^\perp$.

To an optimal constant trace control S must then correspond λ, with $(p, P_Q) \not\equiv 0$, and $\lambda_0 \in \{0, 1\}$. These satisfy the hamiltonian equations, that read:

$$\dot p = \frac{\partial H}{\partial \omega} = Sp - \big(\omega^\top S\omega\big)p - \lambda_0 \dot\omega, \quad \dot P_Q = \frac{\partial H}{\partial Q} = 0.$$

Moreover, the transversality conditions $p(t) \perp T^*_{\omega(t)}\mathbb{S}^1$, for $t \in \{0, T\}$, yield $p(0) = p(T) = 0$, while the maximisation condition and the linearity of the Hamiltonian, yield $H(\lambda(t), \lambda_0, S(t)) \equiv 0$, the fact that $M \le 0$, and $MS \equiv SM \equiv 0$.

This immediately implies that $\lambda_0 = 1$. Indeed, if $\lambda_0 = 0$ we have $p \equiv 0$. The maximisation condition then implies that $S(t)P_Q \equiv 0$ which yields $P_Q = 0$, since $S \in \mathrm{Sym}_2^{+,0}(a, b)$, which contradicts $(p, P_Q) \not\equiv 0$. This fact and the previous observations complete the proof of item 4 and (10).

Let us prove that P_Q and $Q(T)$ commute. This follows by considering a parameterized version of (Conv), and more precisely by introducing a *constant* state variable $U \in \mathrm{O}(2)$. Then, (9) becomes $\dot Q(t) = USU^\top$ and one adds to the dynamics the equation $\dot U = 0$. If p_U denotes the co-state associated with U, then one has, after computations, that $\dot p_U = U[S, U^\top P_Q U]$ with transversality conditions $p_U(0) = p_U(T) = 0$, since there is no constraint on the choice of U. By integrating the above differential equation between $t = 0$ and $t = T$ and since both P_Q and U are constant, we derive that $[Q(T), P_Q]U = 0$, which yields the claim.

We now claim that P_Q has one positive and one non-positive eigenvalue. Firstly, since $M(0) = P_Q - \omega_0\omega_0^\top \le 0$, we observe that the restriction of the quadratic form defined by P_Q to $(\mathbb{R}\omega_0)^\perp$ is also negative semi-definite. This implies that P_Q has at least 1 non-positive eigenvalues. To complete the claim, let us assume that both eigenvalues of P_Q are non-positive. Then, for every $t_1 \le t_2$ in $[0, T]$, we have $\mathrm{Tr}\,\big(P_Q\big(Q(t_2) - Q(t_1)\big)\big) \le 0$. Notice first that, for every $t \in [0, T]$, one has $\int_0^t p^\top Sp\, dt \le \max_{s \in [0,t]} \|p(s)\|^2 \,\mathrm{Tr}\, Q$ by Cauchy-Schwarz's inequality. Let $\bar t \in [0, T]$ such that $\|p(\bar t)\| = \max_{s\in[0,\bar t]} \|p(s)\|$ and $\mathrm{Tr}\, Q(\bar t) \le 1/2$. Multiplying (10) by $p^\top$ and integrating it over $[0, \bar t]$, one gets

$$2\int_0^{\bar t} p^\top S\omega\, dt \ge \frac{\|p(\bar t)\|^2}{2}\Big(1 - \mathrm{Tr}\, Q(\bar t)\Big). \qquad \text{(A.2)}$$

Recall that $MS = 0$ where M is defined in (A.1). Integrate this expression over $[0, \bar t]$ and use (A.2) to get $\int_0^{\bar t} \omega^\top S\omega\, dt \le \mathrm{Tr}\,\big(P_Q Q(\bar t)\big) \le 0$. This implies that $S\omega \equiv 0$ and $p \equiv 0$ on $[0, \bar t]$. Note also that $\omega \equiv \omega(0)$ on that interval. Let then $T_0 \le T$ be the largest time in $[0, T]$ such that $S\omega \equiv 0$, $p \equiv 0$ on $[0, T_0]$ and $\mathrm{Tr}\, Q(T_0) = 1$. Necessarily, $T_0 < T$ otherwise $Q(T)\omega_0 = 0$, which is impossible. Redoing the above reasoning starting from T_0, we can extend the interval on which both $S\omega$ and p are zero beyond T_0, hence contradicting the definition of T_0.

As a consequence of the above facts, up to an orthogonal transformation simultaneously diagonalizing P_Q and $Q(T)$, we have $P_Q = \mathrm{diag}(\alpha, -d)$ with $\alpha > 0, d \ge 0$ and necessarily $Q(T) = \mathrm{diag}(a, b)$ by the transversality condition on P_Q (i.e., $-P_Q$ belongs to the normal cone of $\mathrm{Sym}_2(a, b)$ at $Q(T)$). Finally, the fact that $M(0) \le 0$, and thus $P_Q \le \omega(0)\omega(0)^\top$, yields that $\alpha \le 1$. This completes the proof of item 3.

We now prove item 1, i.e. that $S = cc^\top$ for some $c \in \mathcal{C}^0_n(a, b, T)$. To this effect, observe that S is never zero, since it has constant trace and its integral on $[0, T]$ equals $Q(T) = \mathrm{diag}(a, b)$. Moreover, the same is true for M, since $\mathrm{Tr}\, M \equiv \alpha - d - 1 < 0$, and $MS \equiv 0$ on $[0, T]$. These facts imply that both M and S have constant rank equal to one. Since S is symmetric and positive semidefinite, the conclusion follows.

Finally, we consider the time reparametrisation $\sigma(t) = (\mathrm{Tr}\, S)t$ and replace $S(t)$ by $\tilde S(\sigma) = S(\sigma)/\,\mathrm{Tr}\, S$. A simple computation shows that

$$\int_0^{\sigma(T)} \tilde S(\sigma)\, d\sigma = \int_0^T \tilde S(t)\, dt = Q(T),$$

which immediately implies $\tilde S \in \mathrm{Sym}_2^{+,0}(a, b)$ and

$$a + b = \mathrm{Tr}\, Q(T) = \int_0^{\sigma(T)} \mathrm{Tr}\, \tilde S(t)\, dt = \sigma(T).$$

This proves item 2, and hence the proof of Proposition 8.

Appendix B. PROOF OF THEOREM 2

Consider three sequences $(a_\ell)_{\ell\ge1}$, $(b_\ell)_{\ell\ge1}$ and $(\tau_\ell)_{\ell\ge1}$ not verifying (2). In this section we construct a signal c that satisfies (PEG) w.r.t. to these three sequences, and such that trajectories of system (DGF) do not converge to the origin.

For every $\ell \in \mathbb{N}$, we apply Proposition 3 to deduce that there exists $c_\ell \in \mathcal{C}^0_n(a_\ell, b_\ell, \tau_{\ell+1} - \tau_\ell)$ and $\omega_\ell \in \mathbb{S}^{n-1}$ such that $J(c_\ell, \omega_\ell) = \mu(a_\ell, b_\ell, n)$. Choose a sequence $(U_\ell)_{\ell \geq 0}$ in $\mathrm{O}(n)$ such that, if C_ℓ is the function defined on $[0, \tau_\ell]$ as the concatenation of the $U_j c_j$, $j \in \{0, \ldots, \ell-1\}$, and if $y_j \in \mathbb{R}^n$, $j \in \{0, \ldots, \ell-1\}$, is defined by $y_0 := \omega_0$ and $y_{j+1} := \Phi_{C_\ell}(\tau_{j+1}, \tau_j) y_j$, then one has

$$\frac{\|y_{\ell+1}\|}{\|y_j\|} = \|\Phi_{c_j}(T, 0)\omega_0\|, \quad \text{for } j \in \{0, \ldots, \ell-1\}.$$

By summing up these relations, and using (6), one obtains

$$\ln \|y_{\ell+1}\| = -\sum_{j=0}^{\ell-1} \mu(a_j, b_j, n).$$

Finally, by Lemma 5 and Proposition 6, we get

$$\lim_{\ell \to +\infty} \|y_\ell\| \geq \exp\left(-C_0 \sum_{\ell=0}^{\infty} \frac{a_\ell}{(1+b_\ell)^2}\right) > 0,$$

proving the statement.

REFERENCES

Aeyels, D. and Sepulchre, R. (1994). On the convergence of a time-variant linear-differential equation arising in identification. *KYBERNETIKA*, 30(6), 715–723.

Anderson, B. (1977). Exponential stability of linear equations arising in adaptive identification. *IEEE Transactions on Automatic Control*, 22(1), 83–88. doi: 10.1109/TAC.1977.1101406.

Andersson, S. and Krishnaprasad, P.S. (2002). Degenerate gradient flows: a comparison study of convergence rate estimates. In *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, volume 4, 4712–4717 vol.4. doi:10.1109/CDC.2002.1185122.

Barabanov, N., Ortega, R., and Astolfi, A. (2005). Is normalization necessary for stable model reference adaptive control? *IEEE Transactions on Automatic Control*, 50(9), 1384–1390. doi:10.1109/TAC.2005.854625.

Barabanov, N. and Ortega, R. (2017). On global asymptotic stability of $\dot{x} = -\phi(t)\phi^\top(t)x$ with ϕ not persistently exciting. *Systems and Control Letters*, 109, 24–29. doi:10.1016/j.sysconle.2017.09.005.

Brockett, R. (2000). The rate of descent for degenerate gradient flows. In *Proceedings of the 2000 MTNS, Perpignan, France.*

Chitour, Y., Mason, P., and Prandi, D. (2019). Worst exponential decay rate for degenerate gradient flows subject to persistence of excitation. URL `dprn.github.io/docs/papers/PE.pdf`.

Clarke, F.H. (1990). *Optimization and nonsmooth analysis*, volume 5 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition. doi: 10.1137/1.9781611971309.

Efimov, D., Barabanov, N., and Ortega, R. (2018). Robust stability under relaxed persistent excitation conditions. In *CDC 2018 - 57th IEEE Conference on Decision and Control*. Fontainebleau (FL), United States.

Efimov, D., Barabanov, N., and Ortega, R. (2019). Robustness of linear time-varying systems with relaxed excitation. *International Journal of Adaptive Control and Signal Processing*, 0(0), 1–16.

Efimov, D. and Fradkov, A. (2015). Design of impulsive adaptive observers for improvement of persistency of excitation. *International Journal of Adaptive Control and Signal Processing*, 29(6), 765–782. doi:10.1002/acs.2506.

Praly, L. (2017). Convergence of the gradient algorithm for linear regression models in the continuous and discrete time cases. Research report, PSL Research University ; Mines ParisTech.

Rantzer, A. (1999). To estimate the L_2-gain of two dynamic systems. In *Open problems in mathematical systems and control theory*, Comm. Control Engrg. Ser., 177–179. Springer, London.

Sondhi, M.M. and Mitra, D. (1976). New results on the performance of a well-known class of adaptive filters. *Proceedings of the IEEE*, 64(11), 1583–1597. doi: 10.1109/PROC.1976.10378.