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Flatness-based algebraic fault identification for a wave equation with dynamic boundary conditions

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Abstract: This paper presents a fault identification approach for a boundary controlled wave equation with dynamic boundary conditions. The faulty system is subject to an additive time-varying actuator fault and an unknown in-domain disturbance. These signals are assumed to be the solution of a finite-dimensional signal model so that polynomial and trigonometric faults as well as disturbances can be taken into account. By making use of integral transformations an algebraic expression is derived to obtain the fault from the known input and output in finite time. The kernels determining the integral transformations are obtained by solving the so-called kernel equations. This problem is traced back to the flatness-based realization of a setpoint change for an ODE-PDE casacade. From this, a condition for fault identification is derived. A simulation example demonstrates the proposed approach.

Keywords: Distributed-parameter systems, fault diagnosis, flatness, motion planning, modulating functions

1. INTRODUCTION

1.1 Background and Motivation

Several physical phenomena, e. g., vibrations of a string or the stress pulse in a bar, are described by the wave equation. It can therefore be used to model a wide range of technical systems. To ensure safe automation of these systems, efficient fault diagnosis is crucial. The fault diagnosis is divided into three tasks (see, e. g., Ding (2008)): 1) *fault detection*, the recognition if there is a fault or not, 2) *fault isolation*, separation of different faults and 3) *fault identification* determining the fault's magnitude. In particular, the latter is essential for further fault analysis and thus it is the subject of the following considerations.

Because of the spatial propagation of waves, the system under consideration belongs to the distributed parameter systems (DPS). Whereas numerous approaches are known for the fault identification of lumped-parameter systems (LPS) (see, e. g., Ding (2008); Chen and Patton (1999)), only few approaches exist for DPS so far. An observer-based fault estimation is presented, e. g., in Demetriou (2002) or Xu et al. (2019). These have the drawback, that the resulting infinite-dimensional observer must be approximated for its realization. This increases the complexity and requires further consideration of the approximation error. As an alternative, Asiri and Laleg-Kirati (2017) proposes an algebraic reconstruction of the unknown source term. However, the approach requires the distributed state to be known, which is rather restrictive. Another algebraic approach, which only requires measurements and the input is described in Fischer and Deutscher (2018) for parabolic systems.

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1.2 Contribution

In this paper, the approach from Fischer and Deutscher (2018) is extended to the fault identification problem for a wave equation with dynamic boundary conditions. The considered fault is an additive actuator fault that can be time-varying. In addition, an unknown time-varying disturbance can be present. Both, the fault and the disturbance are assumed to be describable by finite-dimensional linear signal models. Thus, the proposed approach can deal with polynomial and trigonometric signals or combinations thereof. This significantly extends the class of signals that were considered in Fischer and Deutscher (2018). In addition, the number of required measurements is reduced, as only one output is needed in the approach.

For the fault identification an algebraic input-output expression is derived. It only depends on the known input and output and allows the reconstruction of the fault, despite of disturbances in finite time. Furthermore, the implementation of the resulting expression does not require an approximation of a DPS and can therefore be easily implemented.

To derive the fault identification expression, integral transformations are applied to the DPS and to the signal models. The kernels of these transformations are the solution of the so-called kernel equations. They have the form of a coupled PDE-ODE cascade with initial and end conditions, which results in a challenging design problem. Similar as in Fischer and Deutscher (2018), the solution of the PDE can be obtained from a flatness-based setpoint change. Since the ODE contains distributed delays of the input to affect the solution, it is not directly amenable for the design. However, another transformation can be introduced in order to remove the distributed delays in the ODE facilitating the solution procedure. A fault identifiability condition is derived, which directly follows from the solvability of the kernel equations.

1.3 Organization

In the following section the considered problem is formulated. Subsequently, the fault identification is derived. Section 4 presents the solution of the kernel equations. Finally, the results of the paper are illustrated by a simulation example.

2. PROBLEM FORMULATION

Consider a wave equation with dynamic boundary conditions

$$\partial_t^2 w(z,t) = c^2 \partial_z^2 w(z,t) + g(z)d(t) \tag{1a}$$

$$\rho_0 \partial_t^2 w(0,t) = \partial_z w(0,t), \qquad t>0 \tag{1b}$$

$$\rho_1 \partial_t^2 w(1,t) = -\partial_z w(1,t) + b\left(u(t)+f(t)\right), \quad t>0 \tag{1c}$$

$$y(t) = w(0,t), \qquad t \geq 0. \tag{1d}$$

The system variable $w(z,t) \in \mathbb{R}$ in (1a) is defined on $(z,t) \in (0,1) \times \mathbb{R}^+$ and $u(t), y(t), d(t), f(t) \in \mathbb{R}$ are the input, the measurement, the unknown disturbance and the unknown fault. It is assumed that the parameters c, b, ρ_0, ρ_1 are positive and the spatial characteristic of the disturbance is $g \in C[0,1]$. Both the parameters and $g(z)$ are known. The initial conditions (ICs) of the system (1) are $w(z,0) = w_0 \in \mathbb{R}$, $\partial_t w(z,0) = w_{t,0}(z) \in \mathbb{R}$ for the distributed dynamics and $w(0,t) = w_{00} \in \mathbb{R}$, $\partial_t w(0,t) = w_{t,00} \in \mathbb{R}$, $w(1,0) = w_{10} \in \mathbb{R}$ as well as $\partial_t w(1,0) = w_{t,10} \in \mathbb{R}$ for the lumped dynamics at the boundaries.

The form of the fault $f(t)$ and the disturbance $d(t)$ are assumed to be known a priori and can be steplike, polynomial, trigonometric or a combination thereof. Thus, $f(t)$ and $d(t)$ can be represented by the solution of the known finite-dimensional *signal model*

$$\dot{v}(t) = Sv(t), \qquad t>0 \tag{2a}$$

$$f(t) = p_f^\top v(t), \qquad t \geq 0 \tag{2b}$$

$$d(t) = p_d^\top v(t), \qquad t \geq 0 \tag{2c}$$

with $v(t) \in \mathbb{R}^{n_v}$ and known

$$S = \begin{bmatrix} S_f & 0_{n_f \times n_d} \\ 0_{n_d \times n_f} & S_d \end{bmatrix} \in \mathbb{R}^{n_v \times n_v} \tag{3a}$$

$$p_f = \begin{bmatrix} \overline{p}_f \\ 0_{n_d} \end{bmatrix} \in \mathbb{R}^{n_v}, \quad p_d = \begin{bmatrix} 0_{n_f} \\ \overline{p}_d \end{bmatrix} \in \mathbb{R}^{n_v} \tag{3b}$$

where $S_f \in \mathbb{R}^{n_f \times n_f}$, $S_d \in \mathbb{R}^{n_d \times n_d}$, $\overline{p}_f \in \mathbb{R}^{n_f}$ and $\overline{p}_d \in \mathbb{R}^{n_d}$. Furthermore, the following assumption is required.

Assumption 1. The spectra $\sigma(S_f) = \{\lambda_{f,1}, \lambda_{f,2}, \ldots, \lambda_{f,n_f}\}$ and $\sigma(S_d) = \{\lambda_{d,1}, \lambda_{d,2}, \ldots, \lambda_{d,n_d}\}$ are disjoint, i. e.,

$$\sigma(S_f) \cap \sigma(S_d) = \emptyset. \tag{4}$$

Note that this assumption is linked to the fact that only one measurement is available in the presence of one fault and one disturbance. Therefore, this condition may be relaxed by considering additional measurements, which will be investigated in future work.

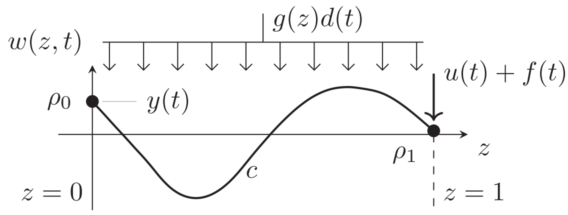

Fig. 1. Schematic representation of a vibrating string with boundary masses.

Despite the given signal model (2), the particular members in this signal class (i. e., amplitudes and phases) are determined by the IC $v(0) = v_0 \in \mathbb{R}^{n_v}$ of (2a). Since $v(0)$ is not assumed to be known, any member in the modeled system class is taken into account.

The system (1) is a model, e. g., to describe the deflection $w(z,t)$ of a vibrating string carrying loads at its ends. Then, ρ_0 and ρ_1 are the masses of the loads, c is the velocity of propagation, $u(t)$ as well as $f(t)$ are forces acting on the right load, $y(t)$ is the position of the left end and $g(z)d(t)$ is a distributed in-domain load. A representation of the corresponding string is shown in Figure 1.

This contribution considers the identification of the time-varying fault $f(t)$. The proposed fault identification is independent of the unknown disturbance $d(t)$ and only requires the known signals $u(t)$ and $y(t)$. It is shown that the fault $f(t)$ can be reconstructed exactly in finite-time, without recourse to a system approximation or an observer.

3. DERIVATION OF THE FAULT IDENTIFICATION EQUATION

The fault identification is based on an algebraic input-output expression, involving only the known signals $u(t)$ and $y(t)$ as well as the unknown $f(t)$. From this, an algebraic expression for the fault can be obtained. The input-output expression results from applying the integral transformation

$$\begin{aligned}\mathcal{M}[h](t) &= \int_0^T \int_0^1 m(z,\tau) h(z, \tau+t-T) \,\mathrm{d}z \,\mathrm{d}\tau \\ &= \langle m, h(t) \rangle_{\Omega,\mathrm{I}}, \qquad h(z,t) \in \mathbb{R}, \;\; t \geq T\end{aligned} \tag{5a}$$

to (1a) while for (2a) the transformation with

$$\begin{aligned}\mathcal{Q}[h](t) &= \int_0^T q^\top(\tau) h(\tau+t-T) \,\mathrm{d}\tau \\ &= \langle q, h(t) \rangle_{\mathrm{I}}, \qquad h(t) \in \mathbb{R}^{n_v}, \;\; t \geq T\end{aligned} \tag{5b}$$

is applied. In what follows, the kernels $m(z,\tau) \in \mathbb{R}$ and $q(\tau) \in \mathbb{R}^{n_v}$ are determined so that the input-output expression becomes independent of the unknown state $w(z,t)$ and the corresponding ICs. In the related literature (see, e. g., Preisig and Rippin (1993)), these kernels are called *modulating functions*. In (5), $\Omega = [0,1]$ denotes the spatial domain and $\mathrm{I} = [0,T]$ the detection window with $T \in \mathbb{R}^+$. The latter is a degree of freedom, but has to satisfy $T > \frac{2}{c}$ due to the delays in the system dynamics. Note that (5) evaluate their arguments on the sliding window $\mathrm{I}_t = [t-T, t]$ so that online fault detection is possible.

To determine the input-output expression, apply (5a) to (1a) yielding

$$\langle m, \partial_t^2 w(t)\rangle_{\Omega,\mathrm{I}} = \langle m, c^2\partial_z^2 w(t)\rangle_{\Omega,\mathrm{I}} + \langle m, gd(t)\rangle_{\Omega,\mathrm{I}}. \quad (6)$$

Then, using integrations by parts w. r. t. time and space, (6) results in

$$\begin{aligned}\langle \partial_\tau^2 m, w(t)\rangle_{\Omega,\mathrm{I}} = B_\Omega(t) &+ \langle c^2\partial_z^2 m, w(t)\rangle_{\Omega,\mathrm{I}} \\ &+ \langle\langle g, m\rangle_\Omega, d(t)\rangle_\mathrm{I}\end{aligned} \quad (7)$$

with

$$B_\Omega(t) = [\langle m(z), \partial_z w(z)\rangle_\mathrm{I} - \langle \partial_z m(z), w(z)\rangle_\mathrm{I}]_0^1 \quad (8)$$

when taking

$$m(z,\tau)|_{\tau\in\{0,T\}} = \partial_\tau m(z,\tau)|_{\tau\in\{0,T\}} = 0, \quad z\in\Omega, \quad (9)$$

the substitution $\partial_t w(z, \tau+t-T) = \partial_\tau w(z, \tau+t-T)$ and $\langle g, m\rangle_\Omega$ as integration w. r. t. z over Ω into account. In order to eliminate the unknown $w(z,t)$ in (7), the condition

$$\partial_\tau^2 m(z,\tau) = c^2\partial_z^2 m(z,\tau), \quad (z,\tau)\in(0,1)\times(0,T) \quad (10)$$

has to hold so that

$$0 = B_\Omega(t) + \langle\langle g, m\rangle_\Omega, d(t)\rangle_\mathrm{I}. \quad (11)$$

In B_Ω (see (8)), the BCs (1b) and (1c) must be considered, leading to

$$\begin{aligned}B_\Omega(t) = &-\langle m(1), \rho_1\partial_\tau^2 w(1,t) + b(u(t)+f(t))\rangle_\mathrm{I} \\ &- \langle \partial_z m(1), w(1,t)\rangle_\mathrm{I} - \langle m(0), \rho_0\partial_\tau^2 w(0,t)\rangle_\mathrm{I} \\ &+ \langle \partial_z m(0), w(0,t)\rangle_\mathrm{I}.\end{aligned} \quad (12)$$

The time derivatives in (12) are shifted to the kernels using integrations by parts w. r. t. time. Note, that the initial and end value terms vanish due to (9). This yields

$$\begin{aligned}B_\Omega(t) = &-\langle \rho_1\partial_\tau^2 m(1) + \partial_z m(1), w(1,t)\rangle_\mathrm{I} \\ &+ \langle \partial_z m(0) - \rho_0\partial_\tau^2 m(0), w(0,t)\rangle_\mathrm{I} \\ &+ \langle bm(1), u(t)\rangle_\mathrm{I} + \langle bm(1), f(t)\rangle_\mathrm{I}.\end{aligned} \quad (13)$$

In order to make (13) only dependent on $u(t)$, $y(t)$ and $f(t)$, substitute $w(0,t)$ by (1d) and require

$$\rho_1\partial_\tau^2 m(1,\tau) + \partial_z m(1,\tau) = 0. \quad (14a)$$

Additionally, let

$$\partial_z m(0,\tau) - \rho_0\partial_\tau^2 m(0,\tau) = n(\tau), \quad (14b)$$

in which $n(\tau)\in\mathbb{R}$ can be seen as boundary input for (10). Then,

$$B_\Omega(t) = \langle n, y(t)\rangle_\mathrm{I} + \langle bm(1), u(t)\rangle_\mathrm{I} + \langle bm(1), f(t)\rangle_\mathrm{I} \quad (15)$$

is obtained. Subsequently, by inserting (15) in (11), the *input-output expression*

$$\begin{aligned}0 = \langle n, y(t)\rangle_\mathrm{I} &+ \langle bm(1), u(t)\rangle_\mathrm{I} + \langle bm(1), f(t)\rangle_\mathrm{I} \\ &+ \langle\langle g, m\rangle_\Omega, d(t)\rangle_\mathrm{I}\end{aligned} \quad (16)$$

results.

In order to use (16) for the fault identification, it has to be solved for $f(t)$. The result, however, still depends on the unknown disturbance $d(t)$, which has to be eliminated. To this end, the signal model (2) is used. Substitute (2b) and (2c) in (16) and collect the terms dependent on $v(t)$ to get

$$\begin{aligned}\langle bm(1), f(t)\rangle_\mathrm{I} &+ \langle\langle g, m\rangle_\Omega, d(t)\rangle_\mathrm{I} \\ &= \langle p_f bm(1) - p_d\langle g, m\rangle_\Omega, v(t)\rangle_\mathrm{I}.\end{aligned} \quad (17)$$

In order to remove the dependency on $v(t)$ in (17), the integral transformation (5b) is applied to (2a) giving

$$\langle q, \mathrm{d}_t v(t)\rangle_\mathrm{I} = \langle q, Sv(t)\rangle_\mathrm{I}. \quad (18)$$

Using the substitution $\mathrm{d}_t v(\tau+t-T) = \mathrm{d}_\tau v(\tau+t-T)$, an integration by parts leads to

$$\left[q^\top(\tau)v(\tau+t-T)\right]_0^T - \langle \mathrm{d}_\tau q, v(t)\rangle_\mathrm{I} = \langle S^\top q, v(t)\rangle_\mathrm{I}. \quad (19)$$

Collecting the terms dependent on $v(t)$ and introduce

$$q(0) = 0 \quad (20)$$

to make (19) independent of $v(t-T)$, (19) becomes

$$q^\top(T)v(t) = \langle \mathrm{d}_\tau q + S^\top q, v(t)\rangle_\mathrm{I}. \quad (21)$$

In order to obtain the desired dependence of the fault $f(t)$, take (2b) into account so that, in view of (21), the choice

$$q(T) = p_f \quad (22)$$

leads to

$$f(t) = p_f^\top v(t) = \langle \mathrm{d}_\tau q + S^\top q, v(t)\rangle_\mathrm{I}. \quad (23)$$

Hence, if $q(\tau)$ satisfies

$$\dot q(\tau) + S^\top q(\tau) = -p_f bm(1,\tau) + p_d\langle g, m(\tau)\rangle_\Omega \quad (24)$$

for $\tau\in(0,T)$, then inserting (24) in (17) yields

$$\begin{aligned}\langle bm(1), f(t)\rangle_\mathrm{I} &+ \langle\langle g, m\rangle_\Omega, d(t)\rangle_\mathrm{I} \\ &= -\langle \mathrm{d}_\tau q + S^\top q, v(t)\rangle_\mathrm{I} = -f(t)\end{aligned} \quad (25)$$

in view of (23). After substitution of (25) in (16), the requested *identification equation*

$$f(t) = \langle n, y(t)\rangle_\mathrm{I} + \langle bm(1), u(t)\rangle_\mathrm{I} \quad (26)$$

is obtained.

4. SOLUTION OF THE KERNEL EQUATIONS

For (26) to hold, the integral kernels $m(z,\tau)$ and $q(\tau)$ must satisfy the conditions (9), (10), (14), (20), (22), and (24). These conditions are called the *kernel equations* and read as

$$\begin{aligned}\partial_\tau^2 m(z,\tau) &= c^2\partial_z^2 m(z,\tau) && (27a)\\ \rho_0\partial_\tau^2 m(0,\tau) &= \partial_z m(0,\tau) - n(\tau), \quad \tau\in(0,T) && (27b)\\ \rho_1\partial_\tau^2 m(1,\tau) &= -\partial_z m(1,\tau), \quad \tau\in(0,T) && (27c)\\ m(z,\tau)|_{\tau\in\{0,T\}} &= \partial_\tau m(z,\tau)|_{\tau\in\{0,T\}} = 0, \ z\in\Omega && (27d)\end{aligned}$$

for $m(z,\tau)\in\mathbb{R}$ with (27a) defined on $(z,\tau)\in(0,1)\times(0,T)$ and

$$\begin{aligned}\dot q(\tau) &= -S^\top q(\tau) - p_f bm(1,\tau) \\ &\quad + p_d\langle g, m(\tau)\rangle_\Omega, \quad \tau\in(0,T) && (28a)\\ q(0) &= 0 && (28b)\\ q(T) &= p_f && (28c)\end{aligned}$$

for $q(\tau)\in\mathbb{R}^{n_v}$. It can be seen that (27) and (28) is a cascade of a DPS and a LPS, where $n(\tau)$ is taken as the input. This degree of freedom is used to realize a transition so that the initial and end conditions (27d), (28b) as well as (28c) are satisfied. Thus, the kernels can be systematically determined, using trajectory planning methods. For this purpose, the cascade structure of the kernel equations is exploited. First, a separate solution is determined for (27) and (28), which is then joined together to form the overall solution.

4.1 Solution of the kernel equations for $m(z,\tau)$

For the wave equation (27a) with the BCs (27b) and (27c), a solution has to be determined, which satisfies the homogeneous initial and end condition (27d). Due to the attached finite-dimensional subsystem and its final state

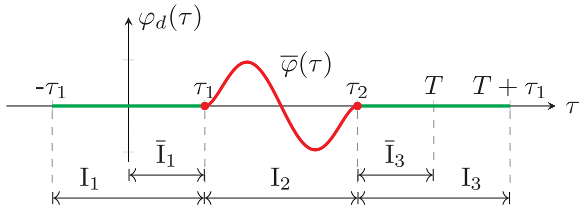

Fig. 2. The different time intervals I_i, $i = 1, 2, 3$ for the trajectory planning of $\varphi_d(\tau)$.

(28c), the trivial solution $m(z, \tau) \equiv 0$ is excluded. A solution of (27a)–(27c) can be determined by realizing a set-point change. For this, the flatness-based approach in Rudolph and Woittennek (2008) is utilized. In a first step, the *differential parameterizations*

$$m(z,\tau) = \frac{1}{2}\left(\varphi(\tau + \frac{1-z}{c}) + \varphi(\tau - \frac{1-z}{c})\right) \tag{29a}$$
$$+ \frac{c\rho_1}{2}\left(\mathrm{d}_\tau\varphi(\tau + \frac{1-z}{c}) - \mathrm{d}_\tau\varphi(\tau - \frac{1-z}{c})\right)$$

and

$$n(\tau) = \frac{1}{2c}\left(\mathrm{d}_\tau\varphi(\tau + \frac{1}{c}) - \mathrm{d}_\tau\varphi(\tau - \frac{1}{c})\right)$$
$$+ \frac{\rho_0 + \rho_1}{2}\left(\mathrm{d}_\tau^2\varphi(\tau + \frac{1}{c}) + \mathrm{d}_\tau^2\varphi(\tau - \frac{1}{c})\right)$$
$$+ \frac{c\rho_0\rho_1}{2}\left(\mathrm{d}_\tau^3\varphi(\tau + \frac{1}{c}) - \mathrm{d}_\tau^3\varphi(\tau - \frac{1}{c})\right) \tag{29b}$$

in terms of the *basic variable*

$$\varphi(\tau) = m(1, \tau) \tag{30}$$

are determined. By means of (29), the set-point change can be realized purely algebraically by introducing a suitable reference trajectory $\varphi_d(\tau)$ for the basic variable. From (29), it is easy to see that $\varphi_d \in C^2[-\tau_1, T+\tau_1]$ and $\mathrm{d}_\tau^3\varphi_d$ exists, where $\tau_1 = \frac{1}{c}$, must hold. Due to the delay τ_1 and the prediction $-\tau_1$ in $n(\tau)$ one has to consider a trajectory planning on $[-\tau_1, T+\tau_1]$. Taking (27d) and the distributed delays and predictions $\pm\frac{1-z}{c}$ in (29a) into account, $\varphi_d(\tau)$ is chosen as

$$\varphi_d(\tau) = \begin{cases} 0 & : \tau \in \mathrm{I}_1 = [-\tau_1, \tau_1) \quad (31a) \\ \overline{\varphi}(\tau) & : \tau \in \mathrm{I}_2 = [\tau_1, \tau_2] \quad (31b) \\ 0 & : \tau \in \mathrm{I}_3 = (\tau_2, T+\tau_1] \quad (31c) \end{cases}$$

where $\tau_2 = T - \tau_1$. In (31), the introduced degree of freedom $\overline{\varphi} \in C^2[\tau_1, \tau_2]$ with $\mathrm{d}_\tau^3\overline{\varphi}$ exists, must satisfy

$$\mathrm{d}_\tau^i\overline{\varphi}(\tau)\big|_{\tau\in\{\tau_1,\tau_2\}} = 0, \quad i = 0, 1, 2 \tag{32}$$

to ensure $\varphi_d \in C^2[-\tau_1, T+\tau_1]$. For a visualization of the definition of $\varphi_d(\tau)$ in (31) see Figure 2.

To fulfill (28), $\overline{\varphi}(\tau)$ has to be chosen, in order to realize the transition for $q(\tau)$ from the initial state (28b) to the final state (28c) regarding (28a). In particular, $\overline{\varphi}(\tau)$ acts as an input to (28a) through $m(1, \tau)$ and $\langle g, m(\tau)\rangle_\Omega$. To simplify the corresponding choice of $\overline{\varphi}(\tau)$, the next section introduces a transformation for (28a).

4.2 Transformation of the kernel equations for $q(\tau)$

Although (28) is an LPS, for which in general a solution can easily be determined, the coupling term $p_d\langle g, m(\tau)\rangle_\Omega$ causes the solution of $q(\tau)$ to depend on distributed delays of φ_d and its derivatives. This dependence makes a systematic determination of $\overline{\varphi}(\tau)$ very challenging. A simpler approach follows from the decoupling of (28a) from the distributed delays of φ_d by means of a transformation.

To this end, introduce the transformation

$$\tilde{q}(\tau) = q(\tau) - \int_0^1 R(z)\mu(z,\tau)\,\mathrm{d}z \tag{33}$$

with

$$\mu(z,\tau) = \begin{bmatrix} m(z,\tau) \\ \partial_\tau m(z,\tau) \end{bmatrix} \tag{34}$$

and the kernel

$$R(z) = [r_1(z)\ r_2(z)] \in \mathbb{R}^{n_v\times 2} \tag{35}$$

to be determined, where $r_1(z), r_2(z) \in \mathbb{R}^{n_v}$. This transformation maps (28) into

$$\dot{\tilde{q}}(\tau) = -S^\top\tilde{q}(\tau) + a_0\varphi_d(\tau) + a_2\ddot{\varphi}_d(\tau) \tag{36}$$

where

$$a_0 = c^2 r_2'(1) - p_f b \tag{37a}$$
$$a_2 = c^2 r_2(1)\rho_1. \tag{37b}$$

With (36), the computation of $\overline{\varphi}(\tau)$ respectively $\varphi_d(\tau)$ (see (31)) becomes much easier.

In order to determine (35), take the derivative of (33)

$$\dot{\tilde{q}}(\tau) = \dot{q}(\tau) - \int_0^1 R(z)\partial_\tau\mu(z,\tau)\,\mathrm{d}z \tag{38}$$

and write (28a) in the form

$$\dot{q}(\tau) = -S^\top q(\tau) - [p_f b\ 0_{n_v}]\,\mu(1,\tau)$$
$$- p_d\int_0^1 [g(z)\ 0]\,\mu(z,\tau)\,\mathrm{d}z. \tag{39}$$

Inserting (39) in (38) yields

$$\dot{\tilde{q}}(\tau) = -S^\top\tilde{q}(\tau) - [p_f b\ 0_{n_v}]\,\mu(1,\tau)$$
$$- \int_0^1 \left(S^\top R(z) + p_d\,[g(z)\ 0]\right)\mu(z,\tau)\,\mathrm{d}z$$
$$- \int_0^1 R(z)\partial_\tau\mu(z,\tau)\,\mathrm{d}z, \tag{40}$$

in which also (33) is utilized. In (40), the latter term is rewritten as

$$\int_0^1 R(z)\partial_\tau\mu(z,\tau)\,\mathrm{d}z \tag{41}$$
$$= \int_0^1 r_1(z)\partial_\tau m(z,\tau) + r_2(z)\partial_\tau^2 m(z,\tau)\,\mathrm{d}z$$

taking (35) into account. Subsequently, substituting (27a) in

$$\int_0^1 r_2(z)\partial_\tau^2 m(z,\tau)\,\mathrm{d}z = c^2\int_0^1 r_2(z)\partial_z^2 m(z,\tau)\,\mathrm{d}z \tag{42}$$

and applying integrations by parts yields

$$\int_0^1 r_2(z)\partial_z^2 m(z,\tau)\,\mathrm{d}z$$
$$= [r_2(z)\partial_z m(z,\tau) - r_2'(z)m(z,\tau)]_0^1$$
$$+ \int_0^1 r_2''(z)m(z,\tau)\,\mathrm{d}z. \tag{43}$$

In order to avoid a dependence on $m(0,\tau)$ and $\partial_z m(0,\tau)$ and thus a dependence on the delay $\mathrm{d}_\tau^i\varphi_d(\tau + \frac{1}{c})$, $i = 0, 1, 2, 3$, choose

$$r_2(0) = r_2'(0) = 0. \tag{44}$$

Then, insert (43) in (42) and the result in (41) as well as take (27c), (30) and (34) into account, to obtain

$$\int_0^1 R(z)\partial_\tau\mu(z,\tau)\,\mathrm{d}z = -c^2\rho_1 r_2(1)\ddot{\varphi}_d(\tau) - c^2 r_2'(1)\varphi_d(\tau) + \int_0^1 \left[c^2 r_2''(z)\ \ r_1(z)\right]\mu(z,\tau)\,\mathrm{d}z. \tag{45}$$

With (45) substituted in (40), the transformed subsystem becomes

$$\dot{\tilde{q}}(\tau) = -S^\top\tilde{q}(\tau) - p_f b\varphi_d(\tau) + r_2(1)c^2\rho_1\ddot{\varphi}_d(\tau) + r_2'(1)c^2\varphi_d(\tau) - \int_0^1 \big(S^\top R(z) + p_d\,[g(z)\ \ 0] - \left[c^2 r_2''(z)\ \ r_1(z)\right]\big)\,\mu(z,\tau)\,\mathrm{d}z. \tag{46}$$

To eliminate $\mu(z,\tau)$ in (46), let

$$S^\top R(z) + p_d\,[g(z)\ \ 0] - \left[c^2 r_2''(z)\ \ r_1(z)\right] = 0. \tag{47}$$

This yields the kernel equations

$$r_1(z) = -S^\top r_2(z) \tag{48a}$$

$$r_2''(z) = \frac{1}{c^2}\left(S^\top\right)^2 r_2(z) - \frac{1}{c^2}p_d g(z), \quad z\in(0,1) \tag{48b}$$

for (33) to be solved with the ICs (44). This is a standard initial value problem for an ODE so that the solution can be explicitly calculated.

4.3 Solution of kernel equations for $\tilde{q}(\tau)$

Since $\varphi_d(\tau)$ is piecewise defined by (31), the solution of (36) must also be considered piecewise. In view of (31), the transformed kernel equations (36) become autonomous on the domains $\bar{\mathrm{I}}_1 = (0,\tau_1)$ and $\bar{\mathrm{I}}_3 = (\tau_2, T)$, i. e.,

$$\dot{\tilde{q}}(\tau) = -S^\top\tilde{q}(\tau), \quad \tau\in\bar{\mathrm{I}}_1\cup\bar{\mathrm{I}}_3. \tag{49}$$

See Figure 2 for a visualization. In order to get an initial and end condition for $\bar{\mathrm{I}}_1$ respectively $\bar{\mathrm{I}}_3$ in (49), (28b) and (28c) have to be transformed with (33). Since $\mu(z,0)=0$ follows from (27d) and (28b) has to hold, the IC for (49) on $\bar{\mathrm{I}}_1$ results from (33) as

$$\tilde{q}(0) = 0. \tag{50}$$

Thus, $\tilde{q}(\tau)=0$ follows for $\tau\in\bar{\mathrm{I}}_1$ in view of (31a). For (49) on $\bar{\mathrm{I}}_3$ an end condition is given by (28c), which reads after the transformation with (33) as

$$\tilde{q}(T) = p_f \tag{51}$$

where $\mu(z,T)=0$ in the light of (27d). To determine $\tilde{q}(\tau)$ on $\bar{\mathrm{I}}_3$, consider the general solution

$$\tilde{q}(\tau) = \mathrm{e}^{-S^\top(\tau-\tau_2)}\tilde{q}_{\tau_2}, \quad \tau\in\bar{\mathrm{I}}_3 \tag{52}$$

of (49) on $\bar{\mathrm{I}}_3$ for the IC $\tilde{q}_2(\tau_2)=\tilde{q}_{\tau_2}\in\mathbb{R}^{n_v}$. Utilizing (51), the unknown initial value $\tilde{q}_2(\tau_2)$ in (52) is given by

$$\tilde{q}_{\tau_2} = \mathrm{e}^{S^\top(T-\tau_2)}p_f. \tag{53}$$

Then,

$$\tilde{q}(\tau) = \mathrm{e}^{S^\top(T-\tau)}p_f, \quad \tau\in\bar{\mathrm{I}}_3 \tag{54}$$

is the solution of (49) on $\bar{\mathrm{I}}_3$.

It remains to determine a solution for $\tilde{q}(\tau)$ on $\tau\in\mathrm{I}_2$ (see (31b)), which must be compatible with the solutions on $\bar{\mathrm{I}}_1$ and $\bar{\mathrm{I}}_3$. From (36) it follows that $\tilde{q}\in C(0,T)$, so that $\tilde{q}(\tau)$ on $\bar{\mathrm{I}}_3$ has to satisfy the initial and end condition

$$\tilde{q}(\tau_1) = 0 \tag{55a}$$

$$\tilde{q}(\tau_2) = \tilde{q}_{\tau_2}, \tag{55b}$$

where (55b) results from (53). In view of (31), the computation of $\tilde{q}(\tau)$ on I_2 is traced back to calculate a $\varphi_d(\tau)$, which achieves a transition

$$\tilde{q}(\tau_1) = 0 \quad\rightarrow\quad \tilde{q}(\tau_2) = \tilde{q}_{\tau_2} \tag{56}$$

for (36) under the additional constraints (32). In order to solve this problem systematically, define

$$\xi_1(\tau) = \varphi_d(\tau),\ \xi_2(\tau) = \mathrm{d}_\tau\varphi_d(\tau),\ \xi_3(\tau) = \mathrm{d}_\tau^2\varphi_d(\tau) \tag{57a}$$

as well as

$$\nu(\tau) = \mathrm{d}_\tau^3\varphi_d(\tau). \tag{57b}$$

Then, the kernel equations (36) can be represented by

$$\dot{x}(\tau) = Ax(\tau) + \bar{b}\nu(\tau) \tag{58}$$

with

$$x(\tau) = \begin{bmatrix}\tilde{q}(\tau)\\ \xi_1(\tau)\\ \xi_2(\tau)\\ \xi_3(\tau)\end{bmatrix},\quad A = \begin{bmatrix}-S^\top & a_0 & 0_{n_v} & a_2\\ 0_{n_v}^\top & 0 & 1 & 0\\ 0_{n_v}^\top & 0 & 0 & 1\\ 0_{n_v}^\top & 0 & 0 & 0\end{bmatrix},\quad \bar{b} = \begin{bmatrix}0_{n_v}\\ 0\\ 0\\ 1\end{bmatrix}.$$

Then, (56) and (32) take the form

$$x(\tau_1) = \begin{bmatrix}0_{n_v}\\ 0_3\end{bmatrix} \quad\rightarrow\quad x(\tau_2) = \begin{bmatrix}\tilde{q}_{\tau_2}\\ 0_3\end{bmatrix}. \tag{59}$$

Assume that $(A,\bar{b})$ is controllable, so that controllability Grammian

$$W(\tau_1,\tau) = \int_{\tau_1}^{\tau}\Phi(\sigma,\tau_1)\bar{b}\bar{b}^\top\Phi^\top(\sigma,\tau_1)\,\mathrm{d}\sigma, \tag{60}$$

with transition matrix $\Phi(\tau,\tau_1) = \mathrm{e}^{A(\tau-\tau_1)}$, satisfies $\det W(\tau_1,\tau)\neq 0$ for $\tau>\tau_1$ (see, e. g., (Chen, 1984, Theorem 5-7)). Then, the desired input to ensure (36) is

$$\nu(\tau) = \bar{b}^\top\Phi^\top(\tau_2,\tau)W^{-1}(\tau_1,\tau_2)x(\tau_2) \tag{61a}$$

(see, e. g., (Bernstein, 2005, Fact 12.20.4)) leading to the solution

$$x(\tau) = W(\tau_1,\tau)\Phi^\top(\tau_2,\tau)W^{-1}(\tau_1,\tau_2)x(\tau_2), \tag{61b}$$

which follows from a straightforward calculation. Based on (61), $\mathrm{d}_\tau^i\varphi_d(\tau)$, $i=0,1,2,3$, can be computed (see (57)). Then, the required $n(\tau)$ and $m(1,\tau)$ for (26) result from (29). Thus, the following theorem for the identifiability of the fault is established.

Theorem 2. Let $n(\tau)$ and $m(1,\tau)$ satisfy (27) respectively (28). Assume $f(t)$ and $d(t)$ are described by (2). If $(A,\bar{b})$ is controllable, then the fault $f(t)$ can be identified in finite-time $T>\frac{2}{c}$ by (26).

Notice, that for the setup with a single measurement, Assumption 1 is necessary for $(A,\bar{b})$ to be controllable. Furthermore, it should be noted that the kernel $q(\tau)$ must not be computed. As can be seen from (28), $q(\tau)$ determines $m(z,\tau)$, but it is not required for (26).

5. SIMULATION EXAMPLE

The proposed fault diagnosis approach is illustrated for (1) with $c,b,\rho_0,\rho_1=1$ and $g(z)=\sin(\pi z)$. The disturbance $d(t)$ has a sinusoidal form $\omega=\frac{\pi}{2}$ leading to the signal model

$$S_d = \begin{bmatrix}0 & 1\\ -\omega^2 & 0\end{bmatrix},\qquad \bar{p}_d = \begin{bmatrix}1\\ 0\end{bmatrix}. \tag{62}$$

A drifting fault is assumed, which is described by

$$S_f = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\qquad \bar{p}_f = \begin{bmatrix}1\\ 0\end{bmatrix}. \tag{63}$$

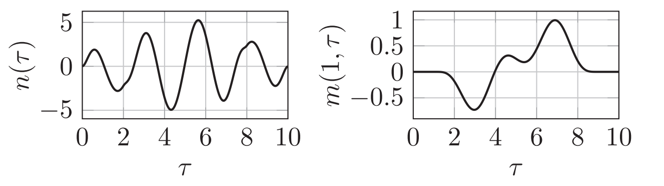

Fig. 3. Functions $n(\tau)$ and $m(1,\tau)$ resulting from (61b) with $\varphi_d \in C^4[-\tau_1, T+\tau_1]$.

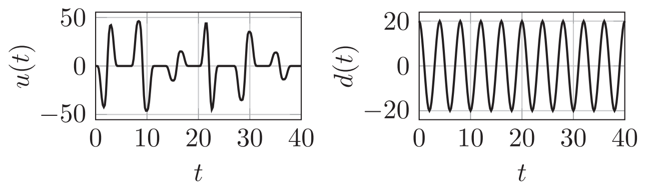

Fig. 4. Exciting signals $u(t)$ and $d(t)$ for the simulation of the wave equation.

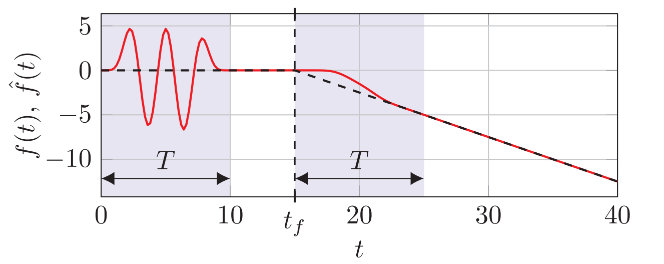

Fig. 5. Identification result $\hat{f}(t)$ (——) for the fault $f(t)$ (- - -) with the detection window marked by □.

To solve the kernel equations (27) and (28), the solution procedure presented in Section 4 is used to plan a reference trajectory on the basis of (61). To reduce numerical errors, $\varphi_d \in C^4[-\tau_1, T+\tau_1]$ is chosen. A simple approach to achieve this additional differentiability requirement, is the introduction of further auxiliary states $\xi_4(\tau) = \mathrm{d}_\tau^3\varphi_d(\tau)$, $\xi_5(\tau) = \mathrm{d}_\tau^4\varphi_d(\tau)$ and $\nu(\tau) = \mathrm{d}_\tau^5\varphi_d(\tau)$ in the light of (57). The required $\varphi_d(\tau)$ and its derivatives are obtained by the similar approach as described for (61b). Choosing $T = 10c$, the results for $n(\tau)$ and $m(1,\tau)$ are depicted in Figure 3.

The simulation of the wave equation, is based on an approximation with the finite-element method. The resulting finite-dimensional state space model has the order 100. Figure 4 shows the input $u(t)$ and the disturbance $d(t)$, which are applied for the simulation. At $t_f = 15$ the fault occurs, leading to the identification result shown in Figure 5. For $t < T$, the requirement $t \geq T$ for (5) is not fulfilled, so that the fault is not identified correctly according to (26). Hence, only an estimate $\hat{f}(t)$ is obtained in this time interval, which is marked by □ in Figure 5. Since the occurrence of the fault $f(t)$ at $t_f = 15$ means a change of the IC in the related signal model, the time T must elapse before the fault can be identified again. This interval, $t_f < t < t_f + T$ is marked by □ in Figure 5. However, if all assumptions are satisfied, the fault is identified, i. e., $\hat{f}(t) = f(t)$, for $T \leq t \leq t_f$ and $t \geq t_f + T$.

6. CONCLUDING REMARKS

This contribution presents the fault identification for a single measurement in the presence of a disturbance characterized by a signal model. In order to enable fault identification without Assumption 1, multiple outputs have to be considered. This will be dealt with in future work.

In addition also arbitrary but bounded disturbances can occur. To cope with this problem, the fault detection and estimation approach described in Fischer and Deutscher (2018), which is based on a threshold, can be directly applied to the considered setup.

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