Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Environmental Feedback incorporated on a Collective Decision Making Model

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Abstract: We study a collective decision making model where each player needs to commit to one of two options. The fractions of committed individuals are the states of this evolutionary model. As element of novelty we incorporate environmental feedback to our model, which translates to system parameters that are now depending on the state of the system. In the first scenario of environmental feedback, we show how we reach a stable unique equilibrium that only depends on the factor of spontaneous commitment. In a second and third scenario, we show that under a suitable form of environmental feedback, we obtain limit cycles in the behavior. All our findings are covered by simulations.

Keywords: Collective decision making, evolutionary dynamics, environmental feedback, limit cycles.

1. INTRODUCTION

In recent years, there has been an increase in literature on environmental feedback (Weitz et al., 2016). Here, systems are studied where certain parameters are depending on the actual state of the system: the environment. The state serves thus as a feedback term to the system, and by considering the closed-loop system, a rise to interesting phenomena is gained.

Another big stream of literature focuses on evolutionary dynamics of game-theoretic models (Pais et al., 2013). One of these models is the bio-inspired honeybee model, where a colony of bees need to chose a new home, and the bees have to choose between two options. This example of collective decision making has been studied quite extensively (Stella and Bauso, 2018b), (Stella and Bauso, 2018a).

Another stream of literature within game-theory focuses on limit cycles (Hommes and Ochea, 2012), (Toni, 2017), (Gilpin, 1975). The idea here, is to consider a game-theoretic model where the attractive asymptotic steady-state behavior is not given by a unique equilibrium point, but rather, where there is convergence to a limit cycle. Interestingly, in this case, the optimal strategy is not a unique fixed strategy, but players are continuously cycling through their action profiles. As examples we can think of the rock-paper-scissors game (Toupo and Strogatz, 2015), or games with evolutionary dynamics (Pais et al., 2012), (Hofbauer and Sigmund, 2003), (Foster and Young, 1990).

The main contributions of this paper are as follows. As starting point we consider a collective decision-making model, but as element of novelty we let the system parameters be state-varying. The system parameters are now no longer fixed constants, but are adapting and act as environmental feedback. For different choices of parameters, we show that we can obtain multiple different dynamical systems. In the first model we develop, the role of one particular parameter, called the spontaneous commitment factor, is highlighted. We show that an increase in this factor yields an increase of committed individuals. By another choice of environmental feedback, we show how we can transform our collective decision-making model to a dynamical system that shows limit-cyclic behavior. The contribution of this paper is relevant in that it can serve as stepping stone to study the way in which environmental feedback can be incorporated in game-theoretic models and can lead to asymptotic behavior such as limit cycles.

Our paper is organized as follows. In Section 2 we study the bio-inspired evolutionary model for collective decision making. In Sections 3, 4, and 5 we integrate the previous model with environmental feedback, thus by making the system parameters state-depending. Extensive numerical experiments are conducted in Section 6, validating our theoretical results. In Section 7 conclusions are drawn and future research directions are given.

2. COLLECTIVE DECISION MAKING MODEL

In (Stella and Bauso, 2018b), (Stella and Bauso, 2018a), a collective decision making model has been studied in

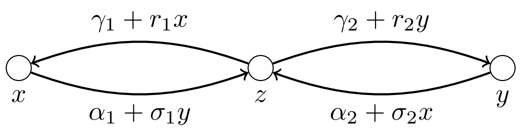

Fig. 1. The evolutionary dynamics for the a-symmetric case.

great detail. It describes a bio-inspired evolutionary model as it stems from the study of how a swarm of honeybees decide on a new home. The idea behind the model is to consider a large population of players, and every player needs to choose between two options. As applications one can think of honeybees that can choose to live in two nests, or consumers that can buy a similar product from two manufacturers, or voters that need to choose between left and right politics.

The average fractions of the crowd committed to each option form the states of this dynamical model. Here, x denotes the fraction of the population that is committed to option 1, y denotes the fraction of players committed to option 2, and z denotes the fraction of the uncommitted players. Note that by construction we have that $x + y + z = 1$.

These fractions x, y and z can change over time due to numerous phenomena. First of all, uncommitted players can spontaneously decide to commit to option 1 or 2, and this is quantified by a factor α_1 and α_2, respectively. Uncommitted players are also attracted to an option by means of imitation, quantified by r_1 and r_2, and the strength of this imitation factor is proportional to the number of individuals to an option. Thus, more players committed to option 1 (2) leads to a stronger imitation factor $r_1 x$ ($r_2 y$).

Committed individuals to option 1 or 2, i.e., players that are located in the x and y fractions, can also become uncommitted. This can happen through either spontaneously uncommitment, quantified by a rate α_1 and α_2, or by means of cross-inhibitory signals σ_1 and σ_2. The idea here, is that players from option 1 (2), can send cross-inhibitory signals to the opposing team 2 (1), to lure them to become uncommitted. The strength of these signals are proportional to the magnitude of the fraction of committed individuals. As practical example, in politics we can think of these cross-inhibitory signals as smear-campaigns.

All in all, these phenomena are captured in Figure 1. The mathematical model describing this evolutionary model is thus given by

$$\begin{aligned}\dot{x} &= (\gamma_1 + r_1 x)z - x(\alpha_1 + \sigma_1 y),\\ \dot{y} &= (\gamma_2 + r_2 y)z - y(\alpha_2 + \sigma_2 x),\\ \dot{z} &= -(\gamma_1 + r_1 x)z + x(\alpha_1 + \sigma_1 y) - (\gamma_2 + r_2 y)z +\\ &\quad y(\alpha_2 + \sigma_2 x).\end{aligned}$$

Since $x + y + z = 1$, the variable z is actually redundant. Thus, an equivalent representation is given by

$$\begin{aligned}\dot{x} &= (\gamma_1 + r_1 x)(1 - x - y) - x(\alpha_1 + \sigma_1 y),\\ \dot{y} &= (\gamma_2 + r_2 y)(1 - x - y) - y(\alpha_2 + \sigma_2 x).\end{aligned} \tag{1}$$

It should be remarked that this is the asymmetric case, by which we mean that the system parameters $\alpha_i, \gamma_i, r_i, \sigma_i$ are different for the two options. If the strengths of these parameters are the same, thus $\gamma_1 = \gamma_2, r_1 = r_2, \alpha_1 = \alpha_2, \sigma_1 = \sigma_2$, we refer to this scenario as the symmetric case, and the model simplifies to

$$\begin{aligned}\dot{x} &= (\gamma + rx)(1 - x - y) - x(\alpha + \sigma y),\\ \dot{y} &= (\gamma + ry)(1 - x - y) - y(\alpha + \sigma x).\end{aligned} \tag{2}$$

3. ENVIRONMENTAL FEEDBACK

In the next sections, we will incorporate environmental feedback in the system. This means that the system parameters depend on the current state of the system. By doing so, we can transform the nature of the behavior and we will show how this will lead to interesting behavior. In this section, for the imitation factors r_1 and r_2, we let them be proportional to the fractions of committed individuals. There is a clear intuition behind this, because the more individuals are committed to a certain option, the higher the imitation factor should become. In particular, we let

$$r_1 = x, \; r_2 = y.$$

For the cross-inhibitory signals σ_1 and σ_2, the following choice is made

$$\sigma_1 = y - x, \quad \sigma_2 = x - y.$$

We note that $\sigma_1 = -\sigma_2$. Furthermore, depending on the magnitude of x and y, the cross-inhibitory signal σ_i can change sign. The intuition behind this is as follows. Consider $\sigma_1 = y - x$. If there are more players committed to option 2 than to option 1, so $y > x$, we have that $\sigma_1 > 0$. The cross-inhibitory signals can thus be interpreted as a positive effect to make committed players to option 1 uncommitted. On the other hand, if $y < x$, we have that $\sigma_1 < 0$, and the interpretation is that the fraction of players committed to option 2 is too small to send a cross-inhibitory signal. Due to the negativity of the parameter σ_1, this weak cross-inhibitory signal now in fact works as benefactor for the growth of the fraction of committed individuals to option 1. Having these environmental feedback laws in mind, the dynamics of (1) becomes

$$\begin{aligned}\dot{x} &= (\gamma_1 + x^2)(1 - x - y) - x(\alpha_1 + y^2 - xy),\\ \dot{y} &= (\gamma_2 + y^2)(1 - x - y) - y(\alpha_2 + x^2 - xy),\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\dot{x} &= \gamma_1 - \gamma_1 x - \gamma_1 y + x^2 - \alpha_1 x - x(x^2 + y^2),\\ \dot{y} &= \gamma_2 - \gamma_2 x - \gamma_2 y + y^2 - \alpha_2 y - y(x^2 + y^2).\end{aligned} \tag{3}$$

We consider the case where we have

$$\alpha_1 = x, \quad \alpha_2 = y.$$

In this case, the dynamics reduce to

$$\begin{aligned}\dot{x} &= \gamma_1 - \gamma_1 x - \gamma_1 y - x(x^2 + y^2),\\ \dot{y} &= \gamma_2 - \gamma_2 x - \gamma_2 y - y(x^2 + y^2).\end{aligned} \tag{4}$$

We remark that, due to the effect of the constant parameters γ_1 and γ_2, assuming that at least one parameter is nonzero, the origin $(x, y) = (0, 0)$ is not an equilibrium. In the remainder of this paper, we will study the nonzero equilibrium of the above system and we do that for two different variants of the model in (4). In the first case, we consider a symmetric model in which $\gamma_1 = \gamma_2$. In the second case we consider the asymmetric model, where $\gamma_1 \neq \gamma_2$.

3.1 Symmetric case

Consider the dynamics of (4), but now in the case where the parameter for spontaneous commitment is the same for each option, so $\gamma_1 = \gamma_2 = \gamma$. The model is now described by

$$\begin{aligned} \dot{x} &= \gamma - \gamma x - \gamma y - x(x^2 + y^2), \\ \dot{y} &= \gamma - \gamma x - \gamma y - y(x^2 + y^2). \end{aligned} \tag{5}$$

In order to find an equilibrium $(x, y) = (x^*, y^*)$, we must solve $\dot{x} = \dot{y} = 0$. Subtracting these two equations one obtains

$$(-x + y)(x^2 + y^2) = 0.$$

We note that $x^2 + y^2$ can only be zero for $x = y = 0$, since otherwise we would have complex values for the fractions of committed players. Hence, we must have that $-x + y = 0$ so in fact we have a symmetric equilibrium $x = y$. Substituting this back in the equation for $\dot{x} = 0$, we see that this equilibrium value satisfies

$$\gamma - 2\gamma x - 2x^3 = 0.$$

Theorem 1. For the symmetric case $\gamma_1 = \gamma_2$ whose dynamics are described by (5), we reach a unique symmetric equilibrium $(x, y) = (x^*, y^*)$ whose value is given by

$$x^* = y^* = w - \tfrac{\gamma}{3w}, \tag{6}$$

with w given by

$$w = \left(\frac{\gamma}{4} + \sqrt{\frac{\gamma^2}{16} + \frac{\gamma^3}{27}} \right)^{\frac{1}{3}}. \tag{7}$$

Proof. We have already established that we reach a symmetric equilibrium $(x, y) = (x^*, y^*)$, where the equilibrium value satisfies the following equality

$$x^3 + \gamma x - \tfrac{\gamma}{2} = 0.$$

This is a cubic equation in x, which can be solved by using a Vieta's substitution. Let us introduce

$$x = w - \tfrac{\gamma}{3w}.$$

Then, the cubic equation $x^3 + \gamma x - \frac{\gamma}{2} = 0$ reads

$$w^3 - \tfrac{\gamma^3}{27w^3} - \tfrac{\gamma}{2} = 0.$$

Multiplying both sides by w^3, we end up with a quadratic solution in w^3, which is easily solved. This quadratic equation has two solutions, but the one with a minus sign in front of the square root cannot be a solution, as this value is negative and we have that $x^* \in [0, 1]$. Thus, we find the value in equation (7), and this completes the proof.

In the simulation depicted in Figure 2, the equilibrium value x^* is plotted as a function of γ. We see that as γ increases, the value x^* converges to a half, which means that half of the players are committed to option 1 and the other half is committed to option 2. The fraction of uncommitted players goes to zero.

3.2 Asymmetric case

We now consider the asymmetric case, where the factors of commitment are not necessarily the same. The model is described by the differential equations in (4). We can establish the following result.

Theorem 2. For the asymmetric model, whose dynamics are described by (4), we reach a unique equilibrium $(x, y) = (x^*, y^*)$ and this equilibrium is

$$x^* = w - \frac{\frac{\gamma_1+\gamma_2}{1+\left(\frac{\gamma_2}{\gamma_1}\right)^2}}{3w}, \tag{8}$$

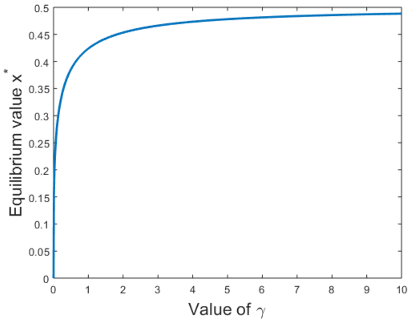

Fig. 2. The equilibrium value x^* as function of the system parameter γ.

where w is given by

$$w = \left(\frac{\gamma_1}{2\left(1 + \left(\frac{\gamma_2}{\gamma_1}\right)^2\right)} + \sqrt{\frac{\gamma_1^2}{4\left(1 + \left(\frac{\gamma_2}{\gamma_1}\right)^2\right)^2} + \frac{(\gamma_1 + \gamma_2)^3}{27\left(1 + \left(\frac{\gamma_2}{\gamma_1}\right)^2\right)^3}} \right)^{\frac{1}{3}}, \tag{9}$$

and

$$y^* = \frac{\gamma_2}{\gamma_1} x^*. \tag{10}$$

Proof. In order to find the equilibria of (4), we set $\dot{x} = 0$ and $\dot{y} = 0$ and solve for x and y. Subtracting γ_1 times the second equation from γ_2 times the first equation, we obtain

$$\gamma_2 \dot{x} - \gamma_1 \dot{y} = (-\gamma_2 x + \gamma_1 y)(x^2 + y^2) = 0.$$

From this it follows immediately that $y = \frac{\gamma_2}{\gamma_1} x$, or that $x = y = 0$ since imaginary solutions are not part of the feasible set. Substituting $y = \frac{\gamma_2}{\gamma_1} x$ into the first equation for $\dot{x} = 0$, we obtain a cubic equation in x

$$\gamma_1 - (\gamma_1 + \gamma_2)x - x^3 \left(1 + \left(\tfrac{\gamma_2}{\gamma_1}\right)^2\right) = 0.$$

This is again a cubic polynomial in x and we can find the roots analytically using a Vieta's substitution. Following the same steps as in the proof of Theorem 1, we find that the equilibrium $(x, y) = (x^*, y^*)$ is given by

$$x^* = w - \frac{\frac{\gamma_1+\gamma_2}{1+\left(\frac{\gamma_2}{\gamma_1}\right)^2}}{3w},$$

where w is given by (9). This completes the proof.

We remark that, whenever $\gamma_1 > \gamma_2$, since $y^* = \frac{\gamma_2}{\gamma_1} x^*$, we have that $x^* > y^*$. On the other hand, if $\gamma_2 > \gamma_1$, we will find that $y^* > x^*$. As second observation, we note that if $\gamma_2 = \gamma_1$, indeed $x^* = y^*$, and we reach the same symmetric equilibrium as expected.

To show how the value of the equilibrium (x^*, y^*) changes for different values of γ_1 and γ_2, we made a plot in Figure 3. The diagonal $\gamma_1 = \gamma_2$ is represented in a solid red line, and we note that these curves are indeed the same for both x^* and y^*. Furthermore, this curve follows the shape as displayed in Figure 2.

3.3 Stability

In the previous subsections we showed that we have a nonzero equilibrium for the systems (5) and (4). The

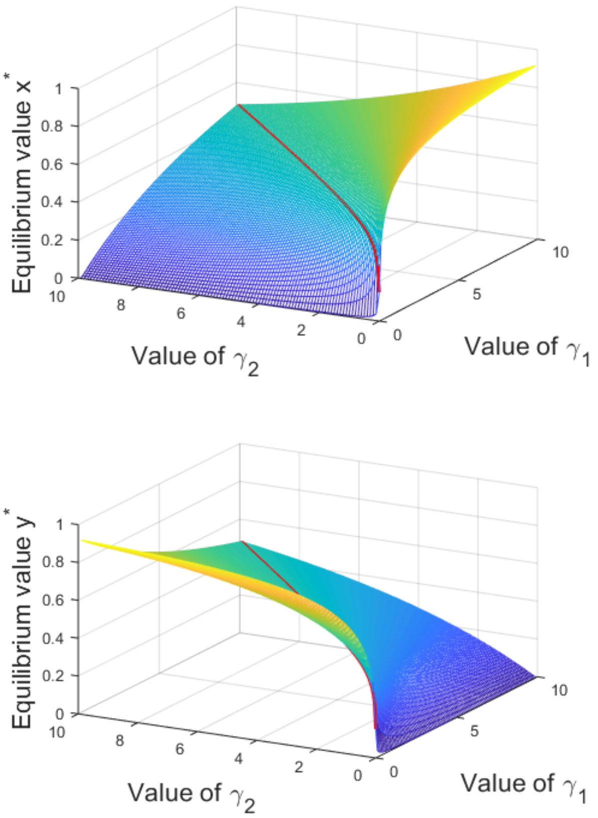

Fig. 3. Equilibrium value x^* (top) and equilibrium value y^* (bottom) for different values of γ_1, γ_2.

next theorem shows that this unique equilibrium is locally stable.

Theorem 3. Consider either the symmetric model (5) or the asymmetric model (4). In either case, the unique equilibrium that we reach, given by Theorem 1 or 2, is locally stable.

Proof. We prove the theorem for the asymmetric case, since the symmetric case follows then immediately by imposing $\gamma_1 = \gamma_2$. For the equilibrium value (x^*, y^*), the linearized system of (4) has the following Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix}\Bigg|_{(x,y)=(x^*,y^*)} = \begin{bmatrix} -\gamma_1 - 3x^2 - y^2 & -\gamma_1 - 2xy \\ -\gamma_2 - 2xy & -\gamma_2 - x^2 - 3y^2 \end{bmatrix}\Bigg|_{(x,y)=(x^*,y^*)}.$$

From Theorem 2, we furthermore know that $y^* = \frac{\gamma_2}{\gamma_1} x^*$. We use this to prove that the matrix is diagonally dominant. For diagonally dominance, we require

$$\gamma_1 + 3(x^*)^2 + (y^*)^2 \geq \gamma_1 + 2x^* y^*$$
$$3 + \left(\tfrac{\gamma_2}{\gamma_1}\right)^2 \geq 2\tfrac{\gamma_2}{\gamma_1},$$

the above inequality is always satisfied. This follows immediately from the fact that the quadratic polynomial $a^2 - 2a + 3$ has negative discriminant. Similarly, one can show that $\gamma_2 + (x^*)^2 + 3(y^*)^2 \geq \gamma_2 + 2x^* y^*$ holds as well. Hence, the Jacobian J is diagonally dominant. Now, by the Gershgorin's Circle Theorem, the eigenvalues of J are contained in disks with centre $-\gamma_1 - 3x^* - y^*$ and $-\gamma_2 - x^* - 3y^*$, with radius $\gamma_1 + 2x^* y^*$ and $\gamma_2 + 2x^* y^*$, respectively. Since the matrix is diagonally dominant, these disks do not cross the imaginary axis, so the eigenvalues of J have negative real part. This completes the proof.

4. LIMIT CYCLES

In this section, we will show how we can derive limit cycles in the behavior of our collective decision making model (1) by incorporating environmental feedback laws. The idea here again, is that the system parameters $\gamma_i, r_i, \alpha_i, \sigma_i$ can be thought of as control inputs, and the choice serves as input strategy to the system. In the end, we will show how we can arrive at the following dynamical system, by using a particular choice of our system parameters:

$$\begin{aligned} \dot{x} &= (y - d) + (x - e)\left(a - b(x - e)^2 - c(y - d)^2\right), \\ \dot{y} &= -(x - e) + (y - d)\left(a - b(x - e)^2 - c(y - d)^2\right). \end{aligned} \tag{11}$$

This model contains limit cycles around the point $(x, y) = (e, d)$, and the values for a, b, c influence the width and height of the limit cycle. These parameters a, b, c can be thought of as a measure of stiffness or damping, while d, e simply denote the translation.

We first recall the existence of limit cycles in the system described by (11), since this is just a translated system of

$$\begin{aligned} \dot{x} &= y + x(a - bx^2 - cy^2), \\ \dot{y} &= -x + y(a - bx^2 - cy^2). \end{aligned} \tag{12}$$

Lemma 4. The system (12) contains a limit cycle.

Proof. The proof follows by direct application of the Poincaré-Bendixson Theorem, see e.g. (Khalil, 2003) or (Hirsch et al., 2012).

We now prove that by a suitable choice of parameters $\gamma_i, r_i, \alpha_i, \sigma_i$, we can transform our collective decision making model (1) into (11).

Theorem 5. The collective decision model (1) can be transformed into (11) by choosing

$$\begin{aligned}
\gamma_1 &= cey^2 + y + be^3 + cd^2 e, \\
r_1 &= 3bex + 2cdy + a, \\
\alpha_1 &= -ax - 3bex^2 + bx^2 - be^3 - cd^2 e + 3be^2 + cd^2 + \\
&\quad \tfrac{d}{x} + \tfrac{ae}{x}, \\
\sigma_1 &= -1 - a - cey - 2cdy + cy - 2cdx - 3bex - \tfrac{be^3}{x} - \\
&\quad \tfrac{cd^2 e}{x} + \tfrac{2cde}{x} - \tfrac{y}{x} - \tfrac{cey^2}{x}, \\
\gamma_2 &= bdx^2 + e + bde^2 + cd^3, \\
r_2 &= 3cdy + 2bex + a, \\
\alpha_2 &= -e - bde^2 - cd^3 + be^2 + 3cd^2 - ay - 3cdy^2 + \\
&\quad cy^2 + \tfrac{ad}{y}, \\
\sigma_2 &= -a - 2bey - 3cdy - 2bex - bdx + bx - \tfrac{bdx^2}{y} - \tfrac{e}{y} - \\
&\quad \tfrac{bde^2}{y} - \tfrac{cd^3}{y} + \tfrac{1}{y} + \tfrac{2bde}{y}.
\end{aligned}$$

Proof. This follows from direct substitution. For the first dynamics equation we obtain:

$$\begin{aligned}\dot{x} &= (\gamma_1 + r_1 x)(1-x-y) - x(\alpha_1 + \sigma_1 y)\\ &= cey^2 + y + be^3 + cd^2e + 3bex^2 + 2cdxy + ax - bx^3 - \\ &\quad 3be^2x - cd^2x - d - ae - cxy^2 - 2cdey.\end{aligned}$$

We now observe that, if we substitute $x = e$, the above expression reduces to $y - d$. Hence, if we neglect the terms $y - d$ in the above expression, the remainder can be factorized with a factor $(x - e)$. By long division one can obtain:

$$\begin{aligned}&cey^2 + y + be^3 + cd^2e + 3bex^2 + 2cdxy + ax - bx^3 - \\ &\quad 3be^2x - cd^2x - d - ae - cxy^2 - 2cdey\\ &= (x-e)\left(-cy^2 - be^2 - cd^2 + 2cdy + a - bx^2 + 2bex\right).\end{aligned}$$

Thus, the dynamics of $\dot{x}$ reduce further to

$$\dot{x} = (y-d) + (x-e)\left(a - b(x-e)^2 - c(y-d)^2\right).$$

A similar argument holds for the equation of $\dot{y}$, and this completes the proof.

Remark 1. There is an important remark we need to make. Although there is some freedom in choosing the values a, b, c, d, e, it is important to stress that there are constraints. First of all, the values d, e account for the translation and obviously we cannot choose d, e outside the set of feasible states $\{(x, y) \in [0, 1]^2 \mid 0 \leq x + y \leq 1\}$. Following up on this, as a, b, c are used to create the shape and size of the limit cycle, we cannot choose them such that the limit cycle becomes too large, crossing the set of feasible states. The reason why it can happen that for some values we exit the feasible set, is due to the way we chose the system parameters $\gamma_i, r_i, \alpha_i, \sigma_i$. For particular choices of a, b, c, d, e, it can happen that $\alpha_1, \sigma_1, \alpha_2, \sigma_2$ become negative for some x, y. In that case, the physical interpretation of our model (1) is lost.

5. LIMIT CYCLES: VAN DER POL OSCILLATORS

Next up, we show that by another choice of environmental feedback, we can obtain the following type of dynamical systems.

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= (a - bx^2)y - cx,\end{aligned} \tag{13}$$

where a, b, c are positive constants that we still need to specify, that can be thought of as stiffness or damping coefficients. A special case of this system is known as the Van der Pol oscillator. An equivalent representation of (13) is

$$\ddot{x} + (bx^2 - a)\dot{x} + cx = 0,$$

which is a so-called Liénard equation, and we can think of it as a model for a mass-spring system, where $bx^2 - a$ is a damping force and cx is a spring constant. Interestingly, this system permits limit cycles which we will prove in the next lemma.

Lemma 6. The system (13) contains a limit cycle around the origin.

Proof. This is again a classical result and it follows immediately by application of the Levinson-Smith Theorem.

We will show in the next theorem how it is possible to transform our collective decision making model (1) into the following dynamical system

$$\begin{aligned}\dot{x} &= y - d,\\ \dot{y} &= \left(a - b(x-e)^2\right)(y-d) - c(x-e).\end{aligned} \tag{14}$$

Note that this is simply a translated version of (13). Thus, by the previous lemma, the above system contains an attractive limit cycle around the point $(x, y) = (e, d)$.

Theorem 7. Under the following choice of environmental feedback, we can transform (1) into (14), where the values a, b, c, d, e are yet to be assigned

$$\begin{aligned}&\gamma_1 = y, \quad r_1 = 0, \quad \alpha_1 = \tfrac{d}{x}, \quad \sigma_1 = -1 - \tfrac{y}{x},\\ &\gamma_2 = bdx^2 + bde^2 + ce, \quad r_2 = 2bex + a,\\ &\alpha_2 = -bde^2 - ce + be^2 - ay + \tfrac{ad}{y},\\ &\sigma_2 = -a - 2bey + bx - 2bex - bdx - \tfrac{bdx^2}{y} - \\ &\qquad \tfrac{bde^2}{y} - \tfrac{ce}{y} + \tfrac{2bde}{y} + \tfrac{c}{y}.\end{aligned}$$

Proof. The proof follows from direct substitution. We first analyze the dynamics in $\dot{x}$

$$\begin{aligned}\dot{x} &= (\gamma_1 + r_1 x)(1-x-y) - x(\alpha_1 + \sigma_1 y)\\ &= (y)(1-x-y) - x\left(\tfrac{d}{x} - y - \tfrac{y^2}{x}\right) = y - d,\end{aligned}$$

as desired. Next up, consider the dynamics in $\dot{y}$

$$\begin{aligned}\dot{y} &= (\gamma_2 + r_2 y)(1-x-y) - y(\alpha_2 + \sigma_2 x)\\ &= bdx^2 + bde^2 + ce + 2bexy + ay - be^2y - ad - bx^2y - \\ &\quad 2bdex - cx.\end{aligned}$$

We note that if we substitute $y = d$ in the last expression, this reduces to $-cx + ce$. Thus, the above expression can be factored with $(y - d)$ and remainder $-cx + ce$. By long division, we obtain that

$$\begin{aligned}&bdx^2 + bde^2 + 2bexy + ay - be^2y - ad - bx^2y - 2bdex\\ &= (y-d)(-bx^2 - be^2 + 2bex + a),\end{aligned}$$

Thus, the dynamics of $\dot{y}$ reduce to

$$\begin{aligned}\dot{y} &= ce - cx + (y-d)(-bx^2 - be^2 + 2bex + a)\\ &= \left(a - b(x-e)^2\right)(y-d) - c(x-e).\end{aligned}$$

This completes the proof.

Of course a similar remark as to Theorem 5 is in order. Although we did not specify the values a, b, c, d, e yet, they are not complete free, as we would like to have the limit cycle to be contained in the set of feasible states $\{(x, y) \in [0, 1]^2 \mid 0 \leq x + y \leq 1\}$. The values d, e simply account for a translation, while the values a, b, c have influence on the shape and magnitude of the limit cycle.

6. NUMERICAL EXPERIMENTS

In this section, we will run various numerical simulations to validate our results. For the system (4) developed in Section 3, we have run the dynamics with $\gamma_1 = 1$ and $\gamma_2 = 0.1$. The results are shown in Figure 4. Since $\gamma_1 = 10\gamma_2$, we will find that $x^* = 10y^*$. The time-evolution of the fraction of committed players to option 1 and option 2 is depicted in red and blue, respectively. We reach a stable equilibrium (x^*, y^*) which value is given by the result in Theorem 2, and indeed we see that $x^* = \frac{\gamma_2}{\gamma_1}y^*$.

Another simulation is done to show the existence of limit cycles in the phase plane (x, y), validating our results in Section 4. For the system given by (11), we take $d = e = \frac{1}{3}$ and $a = 0.4, b = 3, c = 15$. This means that we took the following environmental feedback laws

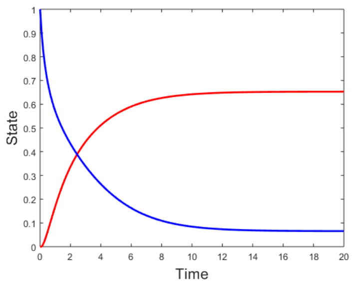

Fig. 4. Time-evolution of the fraction of players committed to option 1, x, in red, and of players committed to option 2, y, in blue.

$$\gamma_1 = 5y^2 + y + \tfrac{2}{3}, \qquad \gamma_2 = x^2 + 1,$$
$$r_1 = 3x + 10y + \tfrac{2}{5}, \qquad r_2 = 15y + 2x + \tfrac{2}{5},$$
$$\alpha_1 = -\tfrac{2}{5}x + 2 + \tfrac{7}{15x}, \qquad \alpha_2 = \tfrac{13}{3} - \tfrac{2}{5}y + \tfrac{2}{15y},$$
$$\sigma_1 = -\tfrac{7}{5} - 13x + \tfrac{8}{3x} - \tfrac{y}{x} - 5\tfrac{y^2}{x}, \qquad \sigma_2 = -\tfrac{2}{5} - 17y - \tfrac{x^2}{y} + \tfrac{2}{3y}.$$

For three different initial conditions, we plotted the trajectory in the xy-plane, and the result is shown in Figure 5 (left). We see that every trajectory converges to the same limit cycle, and this limit cycle is an elliptic shape around the point $(e, d) = (\frac{1}{3}, \frac{1}{3})$.

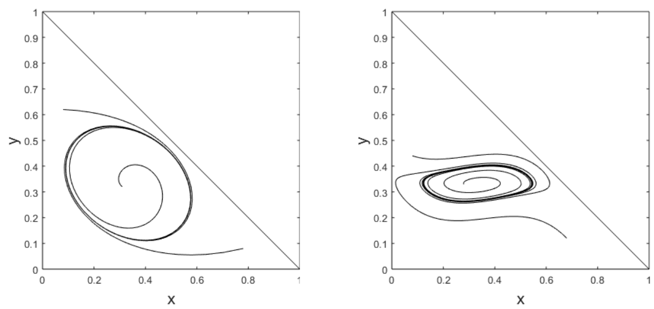

Fig. 5. Left: Time-evolution in the phase-plane for system (11). Right: Time-evolution in the phase-plane for system (14).

Finally, an experiment on the limit cycles obtained in Section 5 is conducted. For system (14) we took the values $a = 0.1, b = 10, c = 0.1$, and we translated to $d = \frac{1}{3}, e = \frac{1}{3}$. The environmental feedbacks were in this case

$$\gamma_1 = y, \qquad \gamma_2 = \tfrac{10}{3}x^2 + \tfrac{109}{270},$$
$$r_1 = 0, \qquad r_2 = \tfrac{20}{3}x + \tfrac{1}{10},$$
$$\alpha_1 = \tfrac{1}{3x}, \qquad \alpha_2 = -\tfrac{79}{270} - \tfrac{1}{10}y + \tfrac{1}{30y},$$
$$\sigma_1 = -1 - \tfrac{y}{x}, \qquad \sigma_2 = -\tfrac{1}{10} - \tfrac{20}{3}y - \tfrac{10}{3y}x^2 + \tfrac{518}{270y}.$$

Again, for three different initial conditions, we plotted the trajectories in the xy-plane and the results are shown in Figure 5 (right). We see that every trajectory approached the limit cycle.

7. CONCLUSION

In this paper we combined a collective decision making model with environmental feedback. We started with a description of a system that models the average commitment to two distinct options. We showed that by making the system parameters state-dependent, this gives rise to interesting phenomena. In the first scenario, where the environmental feedback laws were practical in nature, we reduced the dynamics to a system that was only depending on one parameter, the spontaneously commitment factor. For this case, we showed that there is a unique equilibrium, which can be computed a priori. Furthermore we showed that this unique equilibrium is locally stable.

In order to obtain limit cycles in our dynamics, the environmental feedback laws are more complicated, but it is nevertheless possible. Interestingly, for such scenarios, state trajectories do not converge to a single equilibrium, but rather, they end up in cyclic dynamics. Practically, this means that the average commitment fractions are also changing continuously.

For future research we would like to investigate under which conditions the environmental feedback laws cannot change sign, such that they always remain positive. Otherwise, and this is the case currently, it can happen that for a bad choice of feedback the system trajectories leave the set of feasible states.

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