Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Gain-scheduled $\mathscr{H}_\infty$ controller synthesis for LPV systems subject to multiplicative noise ⋆

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Abstract: This paper proposes LMI conditions to design parameter-dependent (i.e. gain-scheduled) state-feedback controllers that ensure closed-loop stability with guaranteed $\mathscr{H}_\infty$ performance for both continuous and discrete-time LPV systems with state multiplicative noise. The state-space matrices and the multiplicative noise matrix are considered polytopic and independent. The time-varying parameters can be considered time-invariant, arbitrarily fast or with bounded rates of variation. The advantages of the proposed technique are illustrated by numerical examples borrowed from the literature.

Keywords: LPV systems, multiplicative noise, gain-scheduled, LMIs, $\mathscr{H}_\infty$ norm.

1. INTRODUCTION

Demands for improved performance in physical systems have imposed the use of increasingly sophisticated analysis and synthesis tools. In this line, the trade-off between accuracy and numerical complexity of the algorithms rises as an important issue to be considered. In order to obtain faithful and precise modeling of phenomena that occur in real systems, representations based on linear models have increasingly incorporated information about the unidentified part or frequently unconsidered, such as the existence of noise, nonlinearities, delays, and specious dynamics. The inclusion of parametric uncertainties as affine and polytopic representations, with time-invariant or time-varying parameters (with bounded or arbitrarily fast rates of variations) is another aspect that is widely explored in system modeling Åström and Wittenmark (1984), Åström and Wittenmark (1995), Barmish (1994), Boyd and Barratt (1991), Boyd et al. (1994), Khalil (1996), De Caigny et al. (2010), Agulhari et al. (2013), Takaku et al. (2014).

Currently, the design of controllers to guarantee stable closed-loop systems and the estimation of the state variables from the output measurements (associated to performance criteria that characterize the behavior of the state process over time) are among the main purposes of control theory. Concerning linear systems, an important class comprises models where the state variables are affected by multiplicative noises with stochastic properties, characterizing what is known as *bilinear stochastic dynamics*, Kumar and Varaiya (1986), with applications in chemistry, biology, economy, etc. Costa and Kubrusly (1996).

The $\mathscr{H}_\infty$ norm is an import performance index with several applications in terms of allowable disturbance and robustness of uncertain linear (with time-invariant or time-varying parameters) and nonlinear systems, De Caigny et al. (2010). Using the concept of quadratic stability, it is possible to guarantee, through LMI feasibility tests, stability of linear systems affected by time-varying parameters with arbitrary rates of variation, leading to numerical tests with low numerical complexity and allowing to deal with control problems using the concept of guaranteed costs (see Geromel et al. (1991), Geromel et al. (2007), Boyd et al. (1994)). The gains-scheduled strategy can reduce conservatism of the solutions, as well to provide better performance for the closed-loop system when compared to the performance obtained with robust gain strategies, at the price of measuring (or estimating the time-varying parameters to update the controller gain, Leith and Leithead (2000), Rugh and Shamma (2000). This control technique is particularly interesting for treating linear parameter varying systems (LPVs) Hoffmannet et al. (2015).

This paper investigates LPV systems subject to multiplicative noise in the states and proposes gain-scheduled state-feedback synthesis conditions using the $\mathscr{H}_\infty$ norm as performance criterion. In this line, the aim is to design controllers that generate closed-loop stable systems associated to an $\mathscr{H}_\infty$ guaranteed cost. The approach is general in the sense of dealing with continuous and discrete-time LPV systems and considering arbitrary and bounded rates of variation for the time-varying parameters. Numerical examples illustrate the applicability of the proposed approach and the advantages when compared with the existing methods.

2. GENERAL NOTATION

Throughout the paper the symbol $^\mathsf{T}$ stands for the transpose of a matrix or a vector. The symbol $\star$ denotes transposed blocks in a symmetric matrix. The identity matrix and the zero matrix are denoted by I and 0, respectively, diag$(\cdot)$ denotes a

⋆ Research supported by PNPD/CAPES and Department of Electrical Engineering of Federal University of Technology of Paraná – UTFPR, São Paulo Research Foundation (FAPESP) (grant 2017/18785-5).

diagonal matrix of blocks, and $\text{He}\{X\} = X + X^\intercal$ is used for any square matrix X. $\mathbb{N}$ is the set of natural number, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ represents the set of matrices of dimension $n \times m$ with real elements. Inside this paper, the point above of any matrix $\dot{X}(t)$ denotes the derivative with respect to time t. The notation $P = P^\intercal > 0$ is used when $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. The expected value is denoted by $E[\cdot]$. The Euclidean norm is represented as follows $\|\cdot\|_{\mathbb{R}^n}$, such that $\|x(\cdot)\|^2_{\mathbb{R}^n} = \langle x(\cdot), x(\cdot)\rangle$ with inner product $\langle , \rangle$.Consider $L^2(\Omega,\mathbb{R}^n)$ as the space of quadratically integrable functions and $(\Omega,\mathscr{F},\mathbf{P})$ the space of the probabilities, such that, $\mathscr{F}$ represents a σ-algebra, $\mathbf{P}$ represents probability law on $\mathscr{F}$. $\{\mathscr{F}_t\}_{t\geq 0}$ denotes a family of σ-algebras, such that, $\mathscr{F}_t \subseteq \mathscr{F}_s$ for all $0 < t \leq s$. For the continuous-time systems $\tilde{L}^2([0,\infty+),\mathbb{R}^n)$ represents the space of unanticipated stochastic processes with respect to $(\mathscr{F}_\tau)_{\tau\in[0,\infty+)}$ satisfying in $\tilde{L}^2$ (abbreviated form of $\tilde{L}^2([0,\infty+),\mathbb{R}^n)$)

$$\|f(\cdot)\|^2_{\tilde{L}^2} = \int_0^\infty E\left[\|f(t)\|^2_{\mathbb{R}^n}\right] dt < \infty.$$

In discrete-time, $[0,\infty+)$ is replaced by $\mathbb{N}$ that represents the set of natural numbers, in this case the norm in $\tilde{\ell}^2(\mathbb{N},\mathbb{R}^n)$ is defined as

$$\|f_k(\cdot)\|^2_{\tilde{\ell}^2} = \sum_{k=0}^{\infty} E\left[\|f_k\|^2_{\mathbb{R}^n}\right] < \infty.$$

Denote δ_{ij} as the Kronecker delta for all $k, j \geq 0$. In this paper, we consider exponential stability, Kozin (1969).

Throughout the text $t \in \mathbb{R}_+$ is used for continuous-time and $k \in \mathbb{N}$ is used for discrete-time samples.

3. CONTINUOUS-TIME SYSTEMS

Consider the following continuous-time LPV system

$$\begin{cases} dx(t) = (A(\alpha(t)) + D(\beta(t))\nu(t))x(t)dt \\ \qquad\qquad + B_1(\alpha(t))w(t) + B_2(\alpha(t))u(t), \\ z(t) = C_1(\alpha(t))x(t) + D_{11}(\alpha(t))w(t) \\ \qquad\qquad + D_{12}(\alpha(t))u(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the vector of states, $w(t) \in \mathbb{R}^p$ is the external disturbance, $u(t) \in \mathbb{R}^\ell$ is the control input, $z(t) \in \mathbb{R}^r$ is the output to be controlled and $\nu(t)$ is a multiplicative noise affecting the states. Matrices $A(\alpha(t))$, $B_1(\alpha(t))$, $B_2(\alpha(t))$, $C_1(\alpha(t))$, $D_{11}(\alpha(t))$, $D_{12}(\alpha(t))$ and $D(\beta(t))$ depend affinely on bounded time-varying parameters

$$(\alpha(t),\beta(t)) = (\alpha_1(t),\ldots,\alpha_{N_1}(t),\beta_1(t),\ldots,\beta_{N_2}(t))$$

for all $t \in \mathbb{R}^+$, such that $\alpha(t) \in \Lambda_{N_1}$ and $\beta(t) \in \Lambda_{N_2}$ where Λ_N is the unit simplex given by

$$\Lambda_N = \left\{\zeta \in \mathbb{R}^N : \sum_{i=1}^{N} \zeta_i = 1,\ \zeta_i \geq 0, i = 1,\ldots,N\right\}.$$

As $\alpha(t) \in \Lambda_{N_1}$, system (1) is known in the literature as a *polytopic* LPV system. Matrix $D(\beta(t))$ is also polytopic LPV and weights the multiplicative noise. Additionally, it is assumed that $\alpha(t)$ and $\beta(t)$ are continuously differentiable, with bounded rates of variation given in the form $|\dot{\alpha}_i(t)| \leq b_i$, $i = 1,\ldots,N_1$, $|\dot{\beta}_i(t)| \leq d_j$, $j = 1,\ldots,N_2$, such that, the vectors $\dot{\alpha}(t) \in \Omega_{M_1}$ and $\dot{\beta}(t) \in \Omega_{M_2}$ where Ω_M is a bounded polyhedron (polytope) defined as

$$\Omega_M = \left\{\delta \in \mathbb{R}^N : \delta = \sum_{i=1}^{M} \zeta_i h^i,\ \sum_{j=1}^{M} h^i_j = 0, \right. \\ \left. j = 1,\ldots,M, \zeta \in \Lambda_M\right\}. \tag{2}$$

The vertices of the polytope Ω_M, given by the vectors h^i, $i = 1,\ldots,M$ (in principle the value of M is unknown) can be obtained from the bounds b_i (or d_j) and from

$$\dot{\alpha}_1(t) + \cdots + \dot{\alpha}_{N_1}(t) = 0, \text{ or, } \dot{\beta}_1(t) + \cdots + \dot{\beta}_{N_2}(t) = 0.$$

These restrictions are nothing but linear constraints that can be put in the form $Ax \leq b$, where vertex enumeration algorithms Avis and Fukuda (1992) can be employed to generate the vectors h^i in a systematic way using, for instance, the Multi-Parametric Toolbox (MPT) toolbox Herceg et al. (2013).

The scalar $\nu(t) \in \mathbb{R}$ represents a Gaussian process with zero expected value satisfying

$$E[d\nu(t)] = 0, \quad E[d\nu(t)^2] = dt.$$

The goal to design a gain-scheduled control law, such that, $u(t) = K(\alpha(t),\beta(t))x(t)$ stabilizes system (1) and optimizes the following cost functional

$$J_{SE} := E\left[\int_0^\infty (\|z(t)\|^2 - \gamma^2\|w(t)\|^2)dt\right], \tag{3}$$

that is, to determine an upper bound $\gamma > 0$ for the $\mathscr{H}_\infty$ norm from noise input $w(t)$ to output $z(t)$. In the specialized literature there are several works that propose a solution to this problem for linear systems affected by time-invariant uncertainty, for instance see (Gershon et al., 2015, Th. 1). Next theorem presents an extension of this result to cope with LPV systems, including the treatment of $D(\beta(t))$ as time-varying.

Theorem 1. *The system* (1) *in open-loop is exponentially stable in the mean quadratic sense and for some scalar* $\gamma > 0$, $J_{SE} < 0$ *hold for all* $w(t) \in \tilde{L}^2([0,\infty+),\mathbb{R}^p)$ *not null, if there exists a symmetric matrix* $P(\alpha(t),\beta(t)) > 0$*, satisfying the following parameter-dependent LMIs*

$$\begin{bmatrix} \Xi_1 & \Xi_2 & P(\alpha(t),\beta(t))B_1(\alpha(t)) & D(\beta(t))^\intercal P(\alpha(t),\beta(t)) \\ \star & -\gamma^2 I_r & D_{11}(\alpha(t)) & 0_{r\times n} \\ \star & \star & -I_p & 0_{p\times n} \\ \star & \star & \star & -P(\alpha(t),\beta(t)) \end{bmatrix} < 0, \tag{4}$$

where $\Xi_1 = He\{A(\alpha(t))^\intercal P(\alpha(t),\beta(t))\} + \dot{P}(\alpha(t),\beta(t))$ *and* $\Xi_2 = C_1(\alpha(t))^\intercal$*, for all* $(\alpha(t),\beta(t)) \in \Lambda_{N_1} \times \Lambda_{N_2}$ *and for all* $(\dot{\alpha}(t),\dot{\beta}(t)) \in \Omega_{M_1} \times \Omega_{M_2}$.

A proof for Theorem 1 in the case of time-invariant parameters can be found in (Gershon et al., 2015, Th. 1). The time-varying case requires only the inclusion of the term $\dot{P}(\alpha(t),\beta(t))$ and that inequality (4) be tested for all $(\alpha(t),\beta(t)) \in \Lambda_{N_1} \times \Lambda_{N_2}$ and for all $(\dot{\alpha}(t),\dot{\beta}(t)) \in \Omega_{M_1} \times \Omega_{M_2}$.

Note that Theorem 1 presents the most general case, where the Lyapunov matrix also depends on $\beta(t)$. In what follows we present the first result of the paper, which is a synthesis condition expressed in terms of parameter-dependent LMIs combined with a scalar search. The main advantage is the fact that the Lyapunov matrix is not used to construct the control gain. In this case different structures for the controller can be chosen, for instance, depending only on $\alpha(t)$, only on $\beta(t)$ or parameter-independent (robust).

Theorem 2. *For given scalars ξ and $\gamma > 0$, if there exist a positive defined symmetric matrix $W(\alpha(t),\beta(t)) \in \mathbb{R}^{n\times n}$, and matrices $X(\alpha(t),\beta(t)) \in \mathbb{R}^{n\times n}$, $Z(\alpha(t),\beta(t)) \in \mathbb{R}^{\ell\times n}$, $Y_1(\alpha(t),\beta(t))$, $Y_2(\alpha(t),\beta(t))$, $Y_4(\alpha(t),\beta(t))$, $Y_6(\alpha(t),\beta(t)) \in \mathbb{R}^{n\times n}$ which satisfy the following parameter-dependent LMI condition*

$$\begin{bmatrix} \Sigma_{11} & \star & \star & \star & \star & \star \\ \Sigma_{21} & \Sigma_{22} & \star & \star & \star & \star \\ \Sigma_{31} & \Sigma_{32} & -\gamma^2 I_r & \star & \star & \star \\ \Sigma_{41} & \Sigma_{42} & 0_{n\times r} & \Sigma_{44} & \star & \star \\ B_1(\alpha(t))^\intercal & 0_{p\times n} & D_{11}(\alpha(t))^\intercal & 0_{p\times n} & -I_p & \star \\ \Sigma_{61} & \Sigma_{62} & 0_{n\times r} & \Sigma_{64} & 0_{n\times p} & \Sigma_{66} \end{bmatrix} < 0, \quad (5)$$

for all $(\alpha(t),\beta(t)) \in \Lambda_{N_1}\times\Lambda_{N_2}$ and for all $(\dot\alpha(t),\dot\beta(t)) \in \Omega_{M_1}\times\Omega_{M_2}$, where

$$\Sigma_{11} = He\{A(\alpha(t))X(\alpha(t),\beta(t)) + B_2(\alpha(t))Z(\alpha(t),\beta(t))\} - \dot W(\alpha(t),\beta(t)),$$

$$\Sigma_{21} = \xi\left(A(\alpha(t))X(\alpha(t),\beta(t)) + B_2(\alpha(t))Z(\alpha(t),\beta(t))\right)^\intercal + W(\alpha(t),\beta(t)) - X(\alpha(t),\beta(t)),$$

$$\Sigma_{22} = -\xi He\{X(\alpha(t),\beta(t))\},$$

$$\Sigma_{31} = C_1(\alpha(t))X(\alpha(t),\beta(t)) + D_{12}(\alpha(t))Z(\alpha(t),\beta(t)),$$

$$\Sigma_{32} = \xi\left(C_1(\alpha(t))X(\alpha(t),\beta(t)) + D_{12}(\alpha(t))Z(\alpha(t),\beta(t))\right),$$

$$\Sigma_{41} = W(\alpha(t),\beta(t)) - Y_1(\alpha(t),\beta(t)),$$

$$\Sigma_{42} = -Y_2(\alpha(t),\beta(t)),\ \ \Sigma_{44} = -He\{Y_4(\alpha(t),\beta(t))\},$$

$$\Sigma_{6j} = D(\beta(t))Y_j(\alpha(t),\beta(t)), \quad j = 1,2,$$

$$\Sigma_{64} = -Y_6(\alpha(t),\beta(t))^\intercal + D(\beta(t))Y_4(\alpha(t),\beta(t)),$$

$$\Sigma_{66} = -W(\alpha(t),\beta(t)) + He\{D(\beta(t))Y_6(\alpha(t),\beta(t))\},$$

then $K(\alpha(t),\beta(t)) = Z(\alpha(t),\beta(t))X(\alpha(t),\beta(t))^{-1}$ is a gain-scheduled state-feedback gain that stabilizes system (1) and ensures that $J_{SE} < 0$ for all $w \in L^2$ not null.

Proof. Using the change of variables $K(\alpha(t),\beta(t))X(\alpha(t),\beta(t)) = Z(\alpha(t),\beta(t))$, notice that (5) can be rewritten as

$$\mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}^\intercal\mathscr{X}^\intercal < 0 \quad (6)$$

such that [1]

$$\mathscr{Q} = \begin{bmatrix} -\dot W & W & 0 & W & B_1 & 0 \\ \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -\gamma^2 I_r & 0 & D_{11} & 0 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & -I_p & 0 \\ \star & \star & \star & \star & \star & -W \end{bmatrix}, \quad (7)$$

$$\mathscr{X} = \begin{bmatrix} X & \xi X & 0 & 0 & 0 & 0 \\ Y_1 & Y_2 & 0 & Y_4 & 0 & Y_6 \end{bmatrix}^\intercal, \quad (8)$$

$$\mathscr{B} = \begin{bmatrix} A_{cl}^\intercal & -I_n & C_{cl}^\intercal & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n & 0 & D^\intercal \end{bmatrix}. \quad (9)$$

The system (1) in closed-loop is represented by

$$\begin{cases} dx(t) = A_{cl}x(t)dt + D(\beta(t))\nu(t)x(t)dt + B_1(\alpha(t))w(t), \\ z(t) = C_{cl}x(t) + D_{11}(\alpha(t))w(t). \end{cases} \quad (10)$$

where $A_{cl} = A(\alpha(t)) + B_2(\alpha(t))K(\alpha(t),\beta(t))$ and $C_{cl} = C_1(\alpha(t)) + D_{12}(\alpha(t))K(\alpha(t),\beta(t))$.

[1] The dependence on $\alpha(t)$ and $\beta(t)$ was omitted to save space.

Therefore, multiplying (6) on the right by

$$B^\perp = \begin{bmatrix} I_n & 0 & 0 & 0 \\ A_{cl}^\intercal & C_{cl}^\intercal & 0 & 0 \\ 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & D^\intercal \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}$$

and on the left by $B^{\perp\intercal}$, one gets

$$\begin{bmatrix} A_{cl}W + WA_{cl}^\intercal - \dot W & WC_{cl}^\intercal & B_1 & WD^\intercal \\ \star & -\gamma^2 I_r & D_{11} & 0 \\ \star & \star & -I_p & 0 \\ \star & \star & \star & -W \end{bmatrix} < 0. \quad (11)$$

Finally, pre and post multiplying (11) by $\operatorname{diag}(W^{-1}, I_r, I_p, W^{-1})$ we have (4) with $A = A_{cl}$, $C_1 = C_{cl}$ and $P = W^{-1}$ (note that $-W^{-1}\dot W W^{-1} = \dot P$). $\square$

A procedure to check the conditions of Theorem 2 in terms of a finite-dimensional test (not depending on time-varying parameters and t) is discussed in Section 5.

Remark 1. *It is possible to design a robust K gain (parameter-independent) by simply fixing $X(\alpha(t),\beta(t)) = X \in \mathbb{R}^{n\times n}$, $Z(\alpha(t),\beta(t)) = Z \in \mathbb{R}^{\ell\times n}$ such that $K = ZX^{-1}$. Note that the structure of these matrices is a choice of the designer (see Section 5 for more details). It is worth of mentioning that robust gains tend to provide more conservative results than gain-scheduled gains.*

4. DISCRETE-TIME SYSTEMS

Consider the following linear system subject to multiplicative noise ν_k in the state variable,

$$\begin{cases} x_{k+1} = (A(\alpha(k)) + D(\beta(k))\nu_k)x_k \\ \qquad\qquad + B_1(\alpha(k))w_k + B_2(\alpha(k))u_k, \\ z_k = C_1(\alpha(k))x_k + D_{11}(\alpha(k))w_k + D_{12}(\alpha(k))u_k, \end{cases} \quad (12)$$

where $x_k \in \mathbb{R}^n$ is the vector of states, $w_k \in \mathbb{R}^p$ is the exogenous input vector, $u_k \in \mathbb{R}^\ell$ is the control input vector and $z_k \in \mathbb{R}^r$ is the output signal to be controlled. Matrices $A(\alpha(k))$, $B_1(\alpha(k))$, $B_2(\alpha(k))$, $C_1(\alpha(k))$, $D_{11}(\alpha(k))$, $D_{12}(\alpha(k))$ and $D(\beta(k))$ are structured as in the continuous-time case (polytopic LPV). The time-varying parameters $\alpha(k) \in \Lambda_{N_1}$ and $\beta(k) \in \Lambda_{N_2}$ are such that

$$|\alpha_i(k+1) - \alpha_i(k)| \le b_i \le 1,\ b_i \in \mathbb{R}^+,\ i = 1,\dots,N_1$$
$$|\beta_j(k+1) - \beta_j(k)| \le c_j \le 1,\ c_j \in \mathbb{R}^+,\ j = 1,\dots,N_2$$

for all $k \ge 0$. If $b_i = 0$ (or $c_j = 0$), the parameters are time-invariant, whereas $b_i = 1$ (or $c_j = 1$) indicates that the parameter is arbitrarily fast (maximum allowed variation rate). Bounded rates of variation are represented by $0 < b_i < 1$ (or $0 < c_j < 1$). Taking into account the linear constraints that define the bounds and the variation rates of the parameters, the space where $(\alpha(k),\alpha(k+1))$ or $(\beta(k),\beta(k+1))$ lies also defines a polytope, given in the form

$$\Xi_M = \Big\{\delta \in \mathbb{R}^{2N} : \delta = \sum_{i=1}^{M}\zeta_i h^i,\ \sum_{j=1}^{N} h^i_j = \sum_{j=N+1}^{2N} h^i_j = 1,\ i = 1,\dots,M,\ \zeta \in \Lambda_M\Big\}.$$

Differently from the continuous-time case, the parameters $\alpha(k)$ and $\alpha(k+1)$ (or $\beta(k)$ and $\beta(k+1)$) must be modeled jointly, giving rise to a single polytope whose vertices have dimension $2N_1$ (or $2N_2$), Bertolin et al. (2019).

The scalar variable v_k, which represents the multiplicative noise, has null expected value and satisfies

$$E[v_k v_j] = \delta_{kj},\ \forall k,j \geq 0.$$

As in the continuous-time case, the purpose is to design a state-feedback gain-scheduled control gain optimizing the cost functional

$$J_{SEd} := E_v\left[\|z_k\|_{\ell^2}^2 - \gamma^2\|w_k\|_{\ell^2}^2\right],$$

that is, to compute a guaranteed cost $\gamma > 0$ for the $\mathscr{H}_\infty$ norm from the noise input w_k to the output z_k of system (12). Next theorem presents an extension of the bounded real lemma for discrete-time systems affected by multiplicative noise Gershon et al. (2005) to deal with time-varying parameters.

Theorem 3. *For a given scalar $\gamma > 0$ system (12) in open-loop exponentially stable in the mean quadratic sense condition $J_{SE} < 0$ holds for all $w_k \in \tilde{\ell}^2$ not null, if there exist a symmetric matrix $P(\alpha(t),\beta(t)) > 0$, satisfying the following LMI*

$$\begin{bmatrix} P(\alpha(k+1),\beta(k+1)) & \star & \star \\ 0_{p\times n} & \gamma^2 I_p & \star \\ A(\alpha(k))P(\alpha(k+1),\beta(k+1)) & B_1(\alpha(k)) & P(\alpha(k),\beta(k)) \\ D(\beta(k))P(\alpha(k+1),\beta(k+1)) & 0_{n\times p} & 0_{n\times n} \\ C_1(\alpha(k))P(\alpha(k+1),\beta(k+1)) & D_{11}(\alpha(k)) & 0_{r\times n} \end{bmatrix}$$

$$\begin{bmatrix} \star & \star \\ \star & \star \\ \star & \star \\ P(\alpha(k),\beta(k)) & \star \\ 0_{r\times n} & I_r \end{bmatrix} > 0. \quad (13)$$

for all $(\alpha(k),\alpha(k+1)) \in \Xi_{M_1}$ and $(\beta(k),\beta(k+1)) \in \Xi_{M_2}$.

The following theorem presents the second result of this paper.

Theorem 4. *For given scalars $\gamma > 0$ and $\xi \neq 0$, if there exist a positive definite symmetric matrix $P(\alpha(k),\beta(k)) \in \mathbb{R}^{n\times n}$, matrices $X(\alpha(k),\beta(k)) \in \mathbb{R}^{n\times n}$, $Z(\alpha(k),\beta(k)) \in \mathbb{R}^{\ell\times n}$, $Y_2(\alpha(k),\beta(k))$, $Y_6(\alpha(k),\beta(k)) \in \mathbb{R}^{n\times n}$, satisfying the following parameter-dependent LMI condition*

$$\begin{bmatrix} \gamma^2 I_p & \star & \star \\ B_1(\alpha(k)) & \Phi_{22} & \star \\ 0_{n\times p} & \Phi_{32} & P(\alpha(k),\beta(k)) \\ D_{11}(\alpha(k)) & \Phi_{42} & 0_{r\times n} \\ 0_{n\times p} & \Phi_{52} & \xi Y_2(\alpha(k),\beta(k))D(\beta(k))^\intercal \\ 0_{n\times p} & -Y_2(\alpha(k),\beta(k))^\intercal & Y_6(\alpha(k),\beta(k))D(\beta(k))^\intercal \end{bmatrix}$$

$$\begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ I_r & \star & \star \\ \Phi_{54} & \Phi_{55} & \star \\ 0_{n\times r} & \Phi_{65} & \Phi_{66} \end{bmatrix} > 0 \quad (14)$$

for all $(\alpha(k),\alpha(k+1)) \in \Xi_{M_1}$ and $(\beta(k),\beta(k+1)) \in \Xi_{M_2}$ where

$$\Phi_{22} = P(\alpha(k),\beta(k)) + He\{A(\alpha(k))X(\alpha(k),\beta(k))^\intercal + B_2(\alpha(k))Z(\alpha(k),\beta(k))\},$$

$$\Phi_{32} = D(\beta(k))Y_2(\alpha(k),\beta(k))^\intercal,$$

$$\Phi_{42} = C_1(\alpha(k))X(\alpha(k),\beta(k))^\intercal + D_{12}(\alpha(k))Z(\alpha(k),\beta(k)),$$

$$\Phi_{52} = -X(\alpha(k),\beta(k))^\intercal + \xi\left(X(\alpha(k),\beta(k))A(\alpha(k))^\intercal + Z(\alpha(k),\beta(k))^\intercal B_2(\alpha(k))^\intercal\right),$$

$$\Phi_{54} = \xi\left(X(\alpha(k),\beta(k))C_1(\alpha(k))^\intercal + Z(\alpha(k),\beta(k))^\intercal D_{12}(\alpha(k))^\intercal\right),$$

$$\Phi_{55} = -P(\alpha(k+1),\beta(k+1)) - \xi He\{X(\alpha(k),\beta(k))\},$$

$$\Phi_{65} = -P(\alpha(k+1),\beta(k+1)) - \xi Y_2(\alpha(k),\beta(k))^\intercal,$$

$$\Phi_{66} = -P(\alpha(k+1),\beta(k+1)) - He\{Y_6(\alpha,\beta)\},$$

then $K(\alpha(k),\beta(k)) = Z(\alpha(k),\beta(k))X(\alpha(k),\beta(k))^{-\intercal}$ is a gain-scheduled gain that stabilizes system (12) and ensures that $J_{SEd} < 0$ for all $w \in \ell^2$ not null.

Proof. Observe that (14) can be rewritten as

$$\mathscr{Q} + \mathscr{V}^\intercal\mathscr{X}^\intercal\mathscr{U} + \mathscr{U}^\intercal\mathscr{X}\mathscr{V} < 0 \quad (15)$$

where [2]

$$\mathscr{Q} = \begin{bmatrix} \gamma^2 I_p & \star & \star & \star & \star & \star \\ B_1 & P & \star & \star & \star & \star \\ 0 & 0 & P & \star & \star & \star \\ D_{11} & 0 & 0 & I_r & \star & \star \\ 0 & 0 & 0 & 0 & -P^+ & \star \\ 0 & 0 & 0 & 0 & -P^+ & -P^+ \end{bmatrix},\quad \mathscr{V}^\intercal = \begin{bmatrix} 0 & 0 \\ I & Y_2 \\ 0 & 0 \\ 0 & 0 \\ \xi I & \xi Y_2 \\ 0 & Y_6 \end{bmatrix},$$

$$\mathscr{U} = \begin{bmatrix} 0 & A_{cl}^\intercal & 0 & C_1^\intercal & -I & 0 \\ 0 & 0 & D^\intercal & 0 & 0 & -I \end{bmatrix},\quad \mathscr{X}^\intercal = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}$$

and $A_{cl}(\alpha)$ and $C_{cl}(\alpha)$ as in (10). Considering the following basis for the null spaces of $\mathscr{V}$ and $\mathscr{U}$, respectively,

$$\mathscr{V}^\perp = \begin{bmatrix} 0 & I & 0 & 0 \\ \xi I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad \mathscr{U}^\perp = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & A_{cl}^\intercal & 0 & C_1^\intercal \\ 0 & 0 & D^\intercal & 0 \end{bmatrix},$$

and multiplying (15) on the right by $\mathscr{U}^\perp$ and on the left by its transpose, one obtains

$$\begin{bmatrix} \gamma^2 I_p & \star & \star & \star \\ B_1 & P - A_{cl}P^+A_{cl}^\intercal & \star & \star \\ 0 & -DP^+A_{cl}^\intercal & P - DP^+D^\intercal & \star \\ D_{11} & -C_1P^+A_{cl}^\intercal & -C_1P^+D^\intercal & I_r - C_1P^+C_1^\intercal \end{bmatrix} > 0. \quad (16)$$

Applying a Schur complement in (16) yields (13).Finally, pre-multiplying (15) by $\mathscr{V}^\perp$ and post-multiplying by its transpose, one obtains

$$\begin{bmatrix} \xi^2 P(\alpha(k),\beta(k)) - P(\alpha(k+1),\beta(k+1)) & \star & \star & \star \\ \xi B_1^\intercal & \gamma^2 I_p & \star & \star \\ 0 & 0 & P & \star \\ 0 & D_{11} & 0 & I_r \end{bmatrix} > 0 \quad (17)$$

proving that ξ cannot be zero. □

Regarding the scalar parameter ξ in Theorem 4, note that as the variation rates of both $\alpha(k)$ and $\beta(k)$ tend to zero, one has that $\xi^2 P(\alpha(k),\beta(k)) - P(\alpha(k+1),\beta(k+1)) \to (\xi^2 - 1)P(\alpha(k),\beta(k))$. In this case it is necessary that $|\xi| > 1$ to have feasible solutions. This particular case was investigated in Morais et al. (2017).

5. PROGRAMMING THE SYNTHESIS CONDITIONS

The proposed synthesis conditions were presented in a high level of abstraction (parameter-dependent LMIs) for two rea-

[2] The dependence on $\alpha(k)$, $\alpha(k+1)$, $\beta(k)$ and $\beta(k+1)$ was omitted to save space.

sons. The first is a clearer presentation, avoiding the trick notation to represent polynomials. The second is that nowadays to solve parameter-dependent LMIs by polynomial approximations is a well established technique with software support. As a matter of fact, the proposed conditions can be programmed in high level using ROLMIP (Robust LMI Parser) Agulhari et al. (2019). The only task of the user is to pick the polynomial degrees associated to each optimization variable. The notation $g = (g_\alpha, g_\beta)$ is used to indicate the polynomial degrees associated to the variable in terms of α and β (time-dependence can be dropped since α and β are constrained to the set Λ_N for all $t \geq 0$. The same applies to $\dot{\alpha}$ and $\dot{\beta}$, and to the discrete-time case). Note that the degrees chosen for the variables $X(\alpha,\beta)$ and $Z(\alpha,\beta)$ define the type of the controller. If any degree is nonzero, then the controller is gain-scheduled, otherwise the controller is robust (parameter-independent). Moreover, the computation of $\dot{P}(\alpha(t),\alpha(t))$ and a systematic treatment for $(\alpha(k),\alpha(k+1))$ is also provided by the software once the bounds for the time-derivatives (continuous-time case) or for the variation rates (discrete-time case) are informed. The scripts used in the numerical examples presented in the next section were programmed using ROLMIP, YALMIP Löfberg (2004) and solved using SeDuMi Sturm (1999).

6. NUMERICAL EXPERIMENTS

Comparisons with the methods from Gershon et al. (2015) and Gershon and Shaked (2015) are also presented when the parameters are time-invariant. All the results presented in this section, including the figures, are based only on the bounds for the $\mathscr{H}_\infty$ norm provided by the synthesis conditions.

Example 1 Consider the spring mass system proposed in Trofino et al. (2005), with the matrices given below,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1(t) & k_1(t) & -0.2 & 0.2 \\ \frac{k_1(t)}{2} & -\frac{k_1(t)+9.6}{2} & 0.1 & -0.15 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, k_1(t) \in [1.6 \;\; 2.4], \; |\dot{k}_1(t)| < b_i.$$

The matrix associated to the multiplicative noise is given by $D = diag(d(i), \; -d(i), \; 0, \; 0)$, $i = 1,2$, with $d = [0.1 \quad 0.45]$. A polytopic model for the dynamic matrix $A(\alpha)$ can be easily obtained by considering the extreme values of $k_1(t)$.

The aim is to evaluate the effectiveness of the conditions of Theorem 2 in terms of the scalar parameter ξ when the variation rates grow. More precisely, considering $|\alpha_i(t)| = |\beta_i(t)| \leq b$, the conditions of Theorem 2 are tested with $b \in \{0.01, \; 0.1, \; 1{,}5, \; 10\}$. In the first part of this experiment, $(g_\alpha, g_\beta) = (1{,}0)$ is adopted and the gain is robust (variables $X(\alpha,\beta)$ and $Z(\alpha,\beta)$ are fixed with $(g_\alpha, g_\beta) = (0{,}0)$). The guaranteed costs are depicted in Figure 1 considering $\xi \in [0.01, \; 8.65]$ and, as can be seen, a search on ξ can provide stabilizing gains associated with better guaranteed costs. Considering all parameters time-invariant ($b = 0$), it is possible to compare the proposed approach with the method in (Gershon et al., 2015, Cor. 2), that provides $\gamma = 0.6756$. In this situation Theorem 2 provides $\gamma = 0.6546$ with $\xi = 1.389$. In terms of numerical complexity, Theorem 2 demands $V = 169$ scalar variables and $L = 160$ LMI rows. On the other hand, the method in Gershon requires $V = 57$ and $L = 53$. To conclude, the problem

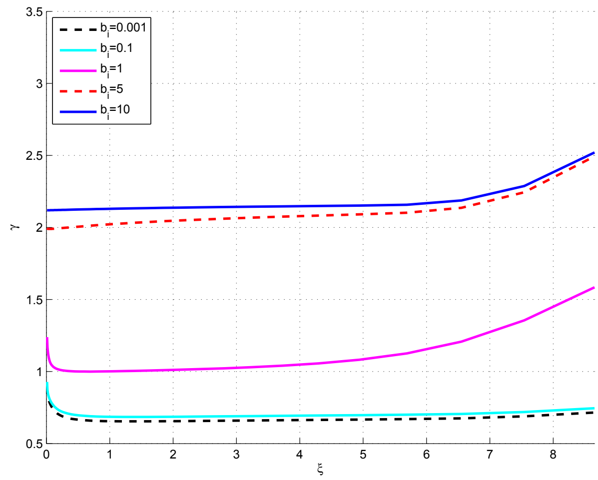

Figure 1. $\mathscr{H}_\infty$ guaranteed costs (γ) computed by Theorem 2 as a function of the scalar parameter ξ considering the variation rate $b_i \in \{0.001, \, 0.1, \, 1, \, 5, \, 10\}$ in Example 1.

of gain-scheduled is investigated. Adopting $(g_\alpha, g_\beta) = (1{,}0)$, the conditions of Theorem 2, tested with $\xi = 1.389$, provide $\gamma = 0.6157$ for $b = 0.01$ and $\gamma = 2.1188$ for $b = 10$. Note that to obtain the better performance provided by the scheduled controller the parameter α needs to be measured or estimated in real-time.

Example 2 The purpose of this example is to design state feedback gains for the discrete system (11) with matrices given by

$$A_{\{1\}} = \begin{bmatrix} 0.4 & 0.7 \\ 0.5 & 0.3 \end{bmatrix}, A_{\{2\}} = \begin{bmatrix} 0.1 & 0.6 \\ 0.6 & 0.5 \end{bmatrix}, B_{2\{1,2\}} = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 1 \end{array}\right],$$

$$D_{\{1\}} = \begin{bmatrix} 0.21 & 0.42 \\ 0.105 & 0.21 \end{bmatrix}, D_{\{2\}} = \begin{bmatrix} -0.01 & -0.02 \\ -0.01 & -0.01 \end{bmatrix},$$

$B_{1\{1\}} = [2.5 \;\, 1]^\intercal$, $B_{1\{2\}} = 2B_{1\{1\}}$, $C_{1\{1\}} = [1 \;\, 0]$, $C_{1\{2\}} = [0 \;\, 1]$, $D_{12\{1\}} = 0.1$, $D_{12\{2\}} = 0.2$. The synthesis conditions of Theorem 4 are tested using $\xi \in [-10{,}10]$ and considering the variation rates bounded in the form $|\alpha_i(k+1) - \alpha_i(k)| \leq b$, $b \in \{0.01{,}0.1{,}0.5{,}1\}$ and β time-invariant. Regarding the degree of the optimization variables, $(g_\alpha, g_\beta) = (1{,}0)$ is chosen for all variables except $X(\alpha,\beta)$ and $Z(\alpha,\beta)$, that are chosen degree zero (robust gain). The results in terms of the guaranteed costs are shown in Figure 2. As can be seen, feasible results were obtained when $|\xi| \geq 1.5$. Moreover, negative values of ξ provided better results. If all parameters are time-invariant, it is possible to apply the method from (Gershon and Shaked, 2015, Lema. 2), which provides $\gamma = 4.4872$. In this case Theorem 4 provided $\gamma = 4.4778$ with $\xi = -2.069$. In terms of numerical complexity, Theorem 4 demands $V = 29$ scalar variables and $L = 44$ LMI rows, whereas the method in Gershon requires $V = 13$ and $L = 20$. Considering $X(\alpha,\beta)$ and $Z(\alpha,\beta)$ with degrees $(g_\alpha, g_\beta) = (1{,}0)$, that is, a gain-scheduled controller, Theorem 4 using $\xi = -2.069$ provides $\gamma = 3.9822$ for $b = 0.01$ and $\gamma = 7.4383$ for $b = 10$.

7. CONCLUSIONS

This paper proposed synthesis conditions for the design of $\mathscr{H}_\infty$ state-feedback controllers for continuous- and discrete-time LPV systems affected by multiplicative noise. The merits

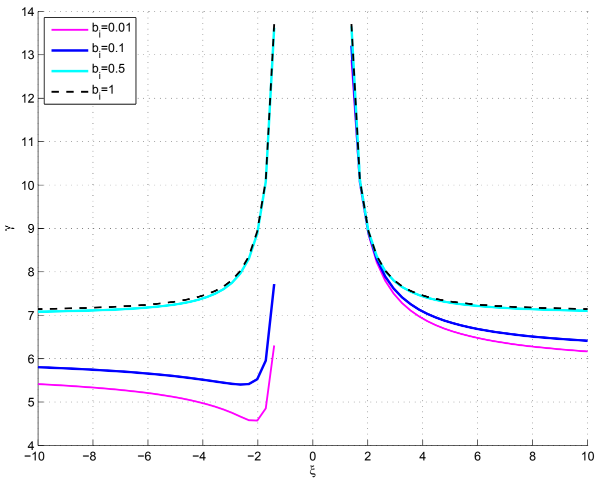

Figure 2. $\mathscr{H}_\infty$ guaranteed costs (γ) computed by Theorem 4 as a function of the scalar parameter with limited variation rate $b_i \in \{0.01, 0.1, 0.5, 1\}$ in Example 2. Note that when $|\xi| < 1.5$ no feasible solutions were obtained.

of the approach are an independent treament of the uncertainty affecting the system matrices and the multiplicative noise matrix (a slack variable approach is used to address this issue) and a general approach to cope with bounded rates of variation for continuous- and discrete-time systems, which in the LPV case require distinct treatments. Numerical experiments including comparisons with methods from the literature illustrated the results.

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