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Robust Distributed Consensus-Based Filtering for Uncertain Systems over Sensor Networks ⋆

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Abstract: Distributed consensus-based estimation is one of the main applications of sensor networks. Most approaches are highly dependent on exact model knowledge. This limitation motivated the development of robust distributed filters that deal with model uncertainties. Many of these works, however, are not fully distributed filters or demand high communication and computational efforts. In this paper, we propose a robust distributed consensus-based filter for uncertain discrete-time linear systems. We assume norm-bounded parametric uncertainties in all matrices of both the target system and sensing models. The approach consists of adopting a purely deterministic interpretation of the robust distributed estimation problem, formulated by combining the penalty function method and the robust regularized least-squares estimation problem. The filter is presented in a fully distributed Kalman-like structure that is suitable for online applications, requiring acceptable computational and communication efforts. We evaluate the effectiveness of the proposed filter by comparing its performance with an existing robust distributed filter, as well as with a centralized strategy.

Keywords: Sensor networks, distributed estimation, consensus, robust estimation, discrete-time linear systems, least-squares problems.

1. INTRODUCTION

Distributed filtering over sensor networks has experienced an increasing research interest during the past few decades. These networks are composed of a set of interconnected nodes that have sensing, computing, and communication capabilities. Since multiple sensors observe a target system, information is shared through the network to improve estimation accuracy. Environment monitoring, intelligent transportation systems, industrial cyber-physical systems, smart grids, robotics, and healthcare are some of the applications that benefit from the flexibility and reliability of sensor networks, see, for instance, Ding et al. (2014), Ding et al. (2019), and references therein.

In a distributed architecture, each node has access to a limited set of neighboring nodes. Therefore, each node combines local information with data from its neighbors to estimate the state of the observed process. This enables a reduction in communication bandwidth, as well as improved reliability, flexibility, and scalability. The consensus protocol is an algorithm whereby, through averaging, multiple agents can reach an agreement on a certain quantity. Olfati-Saber (2005) was the pioneer in applying this strategy to distributed filtering by introducing the consensus on measurements (CM) approach. Later, in Olfati-Saber (2007), the author proposed the consensus on estimates (CE) scheme, originating the Kalman Consensus Filter (KCF). In Olfati-Saber (2009), it is shown that the KCF is a suboptimal solution to the distributed estimation problem. Nevertheless, the optimal solution (Deshmukh et al., 2017) requires the calculation of cross-covariance matrices between any two sensors, demanding excessive communication bandwidth and computational burden in exchange for minor performance improvement. A wide variety of consensus-based distributed filtering strategies can be found in the literature, see e.g. He et al. (2020) for a recent compilation.

Most works on consensus-based distributed filtering assume an exact knowledge of both system and sensor models, which is rarely valid in practice. Parametric uncertainties may arise from unmodeled dynamics, linearization, model reduction, or varying parameters. They can severely degrade the filter performance, which has, therefore, stirred considerable attention from researchers, more recently, in the distributed filtering domain. For instance, Shen et al. (2010) and Dong et al. (2014) use consensus to handle norm-bounded uncertainties and packet dropouts. They rely on linear matrix inequalities (LMIs), which are computationally intensive, and compute the filter gains all at once, requiring knowledge of the whole network. Feng et al. (2013) and Tian et al. (2016) propose distributed fusion architectures for uncertain systems with auto- and cross-correlated noises. The former is based on the involved computation of cross-covariance matrices between agents, whereas the latter applies the covariance intersection scheme to reduce computational effort. Both demand a fusion center where the local estimates of all sensors are combined. Hence, these works are not fully distributed

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strategies, as they require network-wide information or a fusion center. Rastgar and Rahmani (2018) extended the work of Deshmukh et al. (2017) to systems with stochastic uncertainties, proposing an optimal CE-based fully distributed filter. However, to achieve optimality, the computation of cross-covariance matrices is necessary, which increases computation and communication burdens.

Motivated by this discussion, in this paper, we propose a robust fully distributed consensus-based filter (RDCF) for uncertain discrete-time linear systems. We assume more general target system and sensing models, with all matrices subject to norm-bounded parametric uncertainties. The filter is obtained as the solution to a robust regularized least-squares estimation problem (Sayed, 2001; Ishihara et al., 2015), in which the penalty function method (Luenberger and Ye, 2008) is applied. It is presented in a Kalman-like structure, such that each sensor fuses local and neighboring information. Moreover, it does not rely on the computation of cross-covariance matrices, exhibiting acceptable communication and computational requirements, making it suitable for online applications.

Notation: Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^n$ the set of n-dimensional vectors with elements in $\mathbb{R}$, and $\mathbb{R}^{n\times m}$ the set of $n\times m$ real matrices. I_n is the $n\times n$ identity matrix. The superscript T denotes transposition. Let P be a real symmetric matrix, then $P\succ 0$ ($P\succeq 0$) means that P is positive (semi)definite. The **col** operator stacks its operands in a block-column matrix, whereas **diag** represents a block-diagonal matrix with its operands as diagonal elements. The weighted squared Euclidean norm of x is denoted by $\|x\|_P^2 = x^T P x$. Whenever convenient, we adopt the notation $X^T P(\bullet) = X^T P X$.

2. PROBLEM FORMULATION

Consider a sensor network with S sensors. The communication among them is represented by an undirected graph $\mathbb{G} = (\mathbb{S},\mathbb{E})$, with node set $\mathbb{S} = \{1, 2, \ldots, S\}$ and edge set $\mathbb{E}\subseteq \mathbb{S}\times\mathbb{S}$. The neighborhood of sensor i is denoted by $\mathcal{N}_i = \{j\in\mathbb{S}\,|\,(i, j)\in\mathbb{E}\}$, and has cardinality N_i. We further define the inclusive neighborhood set $\overline{\mathcal{N}}_i = \mathcal{N}_i\cup\{i\}$, with cardinality $\bar{N}_i = N_i+1$. Without loss of generality, we assume that graph $\mathbb{G}$ has a fixed topology and is connected, i.e., there is a path between any pair of nodes.

The target plant is described by the uncertain discrete-time linear system

$$x_{k+1} = (F_k+\delta F_k)x_k+(G_k+\delta G_k)u_k+(H_k+\delta H_k)w_k, \quad (1)$$

for $k = 0, 1, \ldots, N$. $x_k\in\mathbb{R}^n$ is the state vector, $u_k\in\mathbb{R}^m$ the input vector, and $w_k\in\mathbb{R}^p$ the process noise at time step k.

The uncertain sensing model of the ith sensor is given by

$$z_k^i = (C_k^i + \delta C_k^i)x_k + (D_k^i + \delta D_k^i)v_k^i, \quad \forall i\in\mathbb{S}, \quad (2)$$

where $z_k^i\in\mathbb{R}^r$ and $v_k^i\in\mathbb{R}^q$ are the measurement and noise vectors at time step k, respectively.

The nominal parameter matrices $F_k\in\mathbb{R}^{n\times n}$, $G_k\in\mathbb{R}^{n\times m}$, $H_k\in\mathbb{R}^{n\times p}$, $C_k^i\in\mathbb{R}^{r\times n}$, and $D_k^i\in\mathbb{R}^{r\times q}$ are known and the parameter uncertainty matrices $\delta F_k\in\mathbb{R}^{n\times n}$, $\delta G_k\in\mathbb{R}^{n\times m}$, $\delta H_k\in\mathbb{R}^{n\times p}$, $\delta C_k^i\in\mathbb{R}^{r\times n}$, and $\delta D_k^i\in\mathbb{R}^{r\times q}$ are modeled as

$$[\delta F_k\ \delta G_k\ \delta H_k] = M_{1,k}\,\Delta_{1,k}\,[E_{F_k}\ E_{G_k}\ E_{H_k}],$$
$$\left[\delta C_k^i\ \delta D_k^i\right] = M_{2,k}^i\,\Delta_{2,k}^i\,\left[E_{C_k}^i\ E_{D_k}^i\right],$$

with nonzero $M_{1,k}\in\mathbb{R}^{n\times s_1}$ and $M_{2,k}^i\in\mathbb{R}^{r\times s_2}$ known matrices, $E_{F_k}\in\mathbb{R}^{t_1\times n}$, $E_{G_k}\in\mathbb{R}^{t_1\times m}$, $E_{H_k}\in\mathbb{R}^{t_1\times p}$, $E_{C_k}^i\in\mathbb{R}^{t_2\times n}$, and $E_{D_k}^i\in\mathbb{R}^{t_2\times q}$ also known, and $\Delta_{1,k}\in\mathbb{R}^{s_1\times t_1}$ and $\Delta_{2,k}^i\in\mathbb{R}^{s_2\times t_2}$ arbitrary contraction matrices such that $\|\Delta_{1,k}\|\leq 1$ and $\|\Delta_{2,k}^i\|\leq 1$, for all $i\in\mathbb{S}$.

Our goal is to design a robust distributed state estimator for the uncertain target system (1). Since the system state sequence $\{x_k\}$ is not perfectly observed, the problem consists in leveraging consensus-based distributed filtering such that each sensor i can obtain an estimate $\hat{x}_{k|k}^i$ of x_k by combining its own information with data received from its neighbors, despite the presence of model uncertainties.

As discussed in Bryson and Ho (1975), Sayed (2001), and Ishihara et al. (2015), stochastic estimation problems admit a deterministic interpretation. In this context, the random variables w_k and v_k^i are seen as model fitting errors. Based on this, we associate the following one-step quadratic cost function with system (1)–(2):

$$J_k^i(x_k, w_k, v_k^j) = \sum_{j\in\overline{\mathcal{N}}_i}\|x_k - \hat{x}_{k|k-1}^j\|^2_{(P_{k|k-1}^j)^{-1}} + \|w_k\|^2_{Q_k^{-1}} + \sum_{j\in\overline{\mathcal{N}}_i}\|v_k^j\|^2_{(R_k^j)^{-1}}, \quad (3)$$

for each $k = 0, 1, \ldots, N$ and $i\in\mathbb{S}$. Matrices $P_{k|k-1}^j\succ 0$ weight the prior estimation errors $e_{k|k-1}^j = x_k - \hat{x}_{k|k-1}^j$, for $j\in\overline{\mathcal{N}}_i$, and $Q_k\succ 0$ and $R_k^j\succ 0$ weight the model fitting errors. The robust distributed consensus-based filter is then obtained by solving a min-max constrained optimization problem in which the cost function (3) is minimized under the maximum influence of the parametric uncertainties $\delta_k = \{\delta F_k, \delta G_k, \delta H_k, \delta C_k^j, \delta D_k^j\}$, for all $j\in\overline{\mathcal{N}}_i$, i.e.,

$$\begin{aligned}&\min_{x_k, x_{k+1}, w_k, v_k^j}\ \max_{\delta_k}\ J_k^i(x_k, w_k, v_k^j)\\ &\text{subject to}\begin{cases} x_{k+1} = F_{\delta,k}x_k + G_{\delta,k}u_k + H_{\delta,k}w_k\\ z_k^j = C_{\delta,k}^j x_k + D_{\delta,k}^j v_k^j, \quad \forall j\in\overline{\mathcal{N}}_i,\end{cases}\end{aligned} \quad (4)$$

with $F_{\delta,k} = F_k + \delta F_k$, $G_{\delta,k} = G_k + \delta G_k$, $H_{\delta,k} = H_k + \delta H_k$, $C_{\delta,k}^j = C_k^j + \delta C_k^j$, and $D_{\delta,k}^j = D_k^j + \delta D_k^j$.

For ease of notation, consider the following definitions:

$$\hat{\boldsymbol{x}}_{k|k-1}^i = \mathbf{col}\left(\hat{x}_{k|k-1}^i, \hat{x}_{k|k-1}^{i_1}, \ldots, \hat{x}_{k|k-1}^{i_{N_i}}\right),$$
$$\boldsymbol{e}_{k|k-1}^i = \mathbf{col}\left(x_k - \hat{x}_{k|k-1}^i, x_k - \hat{x}_{k|k-1}^{i_1}, \ldots, x_k - \hat{x}_{k|k-1}^{i_{N_i}}\right),$$
$$\boldsymbol{v}_k^i = \mathbf{col}\left(v_k^i, v_k^{i_1}, \ldots, v_k^{i_{N_i}}\right),\ \boldsymbol{z}_k^i = \mathbf{col}\left(z_k^i, z_k^{i_1}, \ldots, z_k^{i_{N_i}}\right),$$
$$\mathfrak{P}_{k|k-1}^i = \mathbf{diag}\left(P_{k|k-1}^i, P_{k|k-1}^{i_1}, \ldots, P_{k|k-1}^{i_{N_i}}\right),$$
$$\mathfrak{R}_k^i = \mathbf{diag}\left(R_k^i, R_k^{i_1}, \ldots, R_k^{i_{N_i}}\right),\ \mathfrak{I} = \mathbf{col}\left(I_n, I_n, \ldots, I_n\right),$$
$$\mathfrak{C}_{\delta,k}^i = \mathfrak{C}_k^i + \delta\mathfrak{C}_k^i = \mathbf{col}\left(C_{\delta,k}^i, C_{\delta,k}^{i_1}, \ldots, C_{\delta,k}^{i_{N_i}}\right),$$
$$\mathfrak{D}_{\delta,k}^i = \mathfrak{D}_k^i + \delta\mathfrak{D}_k^i = \mathbf{diag}\left(D_{\delta,k}^i, D_{\delta,k}^{i_1}, \ldots, D_{\delta,k}^{i_{N_i}}\right),$$
$$\boldsymbol{E}_{\mathfrak{C}_k}^i = \mathbf{col}\left(E_{C_k}^i, E_{C_k}^{i_1}, \ldots, E_{C_k}^{i_{N_i}}\right),$$
$$\boldsymbol{E}_{\mathfrak{D}_k}^i = \mathbf{diag}\left(E_{D_k}^i, E_{D_k}^{i_1}, \ldots, E_{D_k}^{i_{N_i}}\right),$$

$$\mathcal{M}_{2,k}^{i} = \mathbf{diag}\left(M_{2,k}^{i}, M_{2,k}^{i_1}, \ldots, M_{2,k}^{i_{N_i}}\right),$$
$$\mathbf{\Delta}_{2,k}^{i} = \mathbf{diag}\left(\Delta_{2,k}^{i}, \Delta_{2,k}^{i_1}, \ldots, \Delta_{2,k}^{i_{N_i}}\right),$$

in which the superscript i_j, $j \in \mathcal{N}_i$, denotes the jth neighbor of sensor i.

The constrained problem (4) can be transformed into an equivalent unconstrained problem by application of the penalty function method (Luenberger and Ye, 2008). The constraints are weighted by a penalty parameter $\mu > 0$ and included in the cost function (3). Violations of the constraints are thus penalized by μ. Therefore, for each $\mu > 0$, problem (4) can be rewritten as follows.

$$\min_{x_{k,\mu}} \max_{\delta_k} \mathcal{J}_{k,\mu}^{i} \tag{5}$$

with cost function given in (6). According to Luenberger and Ye (2008), the optimal solution to the original constrained problem (4) is obtained when $\mu \to +\infty$ in (5)–(6).

In the next section, we present the strategy that will be used to solve the proposed optimization problem.

3. ROBUST REGULARIZED LEAST-SQUARES ESTIMATION PROBLEM

Consider the robust regularized least-squares estimation problem

$$\min_{x} \max_{\delta A, \delta b} \|x\|_{\mathcal{Q}}^2 + \|(A+\delta A)x - (b+\delta b)\|_{\mathcal{W}}^2, \tag{7}$$

where x is an unknown vector we wish to estimate, A is a nominal matrix, b is a measurement vector, $\mathcal{Q} \succ 0$ and $\mathcal{W} \succ 0$ are weighting matrices, and δA and δb are parametric uncertainties modeled as

$$[\delta A \;\; \delta b] = M\Delta\,[E_A \;\; E_b], \quad \|\Delta\| \le 1,$$

with known matrices M, E_A, and E_b.

Sayed (2001) proposed a unique solution to problem (7), which was later presented in alternative convenient symmetric matrix arrangement framework by Ishihara et al. (2015), as the following lemma states.

Lemma 1. (Ishihara et al., 2015) Consider the robust regularized least-squares estimation problem (7). The optimal solution x^* and the respective weighting matrix $\mathcal{P}$ for an estimation error $e = x - x^*$ are given by

$$[x^* \;\; \mathcal{P}] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}^{-1} & 0 & 0 & I \\ 0 & \widehat{\mathcal{W}}^{-1} & 0 & A \\ 0 & 0 & \widehat{\lambda}^{-1} I & E_A \\ I & A^T & E_A^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ b & 0 \\ E_b & 0 \\ 0 & -I \end{bmatrix}, \tag{8}$$

with

$$\widehat{\mathcal{W}} = (\mathcal{W}^{-1} - \widehat{\lambda}^{-1} M M^T)^{-1},$$
$$\widehat{\lambda} = (1+\alpha)\,\|M^T \mathcal{W} M\|, \text{ for some } \alpha > 0.$$

Remark 1. In case $\mathcal{Q} \succeq 0$ in (7), such that $\mathcal{Q} = \begin{bmatrix} \bar{\mathcal{Q}} & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{\mathcal{Q}} \succ 0$, it can be shown that solution (8) can be alternatively rewritten as

$$[x^* \;\; \mathcal{P}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \bar{\mathcal{Q}}^{-1} & 0 & 0 & I & 0 \\ 0 & \widehat{\mathcal{W}}^{-1} & 0 & A_1 & A_2 \\ 0 & 0 & \widehat{\lambda}^{-1} I & E_{A_1} & E_{A_2} \\ I & A_1^T & E_{A_1}^T & 0 & 0 \\ 0 & A_2^T & E_{A_2}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ E_b & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix}, \tag{9}$$

where $A = [A_1 \;\; A_2]$ and $E_A = [E_{A_1} \;\; E_{A_2}]$.

4. ROBUST DISTRIBUTED CONSENSUS-BASED FILTER

We are now ready to propose a solution to the unconstrained min-max optimization problem (5)–(6). Notice that it is a special case of the robust regularized least-squares estimation problem (7), considering the following identifications:

$$x \leftarrow \begin{bmatrix} \boldsymbol{e}_{k|k-1}^{i} \\ w_k \\ \boldsymbol{v}_k^{i} \\ x_k \\ x_{k+1} \end{bmatrix}, \; \mathcal{Q} \leftarrow \begin{bmatrix} (\boldsymbol{\mathcal{P}}_{k|k-1}^{i})^{-1} & 0 & 0 & 0 & 0 \\ 0 & Q_k^{-1} & 0 & 0 & 0 \\ 0 & 0 & (\boldsymbol{\mathcal{R}}_k^{i})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A + \delta A \leftarrow \begin{bmatrix} I_{\bar{n}} & 0 & 0 & -\boldsymbol{\mathcal{I}} & 0 \\ 0 & H_{\delta,k} & 0 & F_{\delta,k} & -I_n \\ 0 & 0 & \boldsymbol{\mathcal{D}}_{\delta,k}^{i} & \boldsymbol{\mathcal{C}}_{\delta,k}^{i} & 0 \end{bmatrix}, \; \Delta \leftarrow \begin{bmatrix} \Delta_{1,k} & 0 \\ 0 & \mathbf{\Delta}_{2,k}^{i} \end{bmatrix},$$

$$b + \delta b \leftarrow \begin{bmatrix} -\hat{\boldsymbol{x}}_{k|k-1}^{i} \\ -G_{\delta,k} u_k \\ \boldsymbol{z}_k^{i} \end{bmatrix}, \; M \leftarrow \begin{bmatrix} 0 & 0 \\ M_{1,k} & 0 \\ 0 & \mathcal{M}_{2,k}^{i} \end{bmatrix}, \; \mathcal{W} \leftarrow \mu I,$$

$$E_A \leftarrow \begin{bmatrix} 0 & E_{H_k} & 0 & E_{F_k} & 0 \\ 0 & 0 & \boldsymbol{E}_{\mathcal{D}_k}^{i} & \boldsymbol{E}_{\mathcal{C}_k}^{i} & 0 \end{bmatrix}, \; E_b \leftarrow \begin{bmatrix} -E_{G_k} u_k \\ 0 \end{bmatrix}.$$

Notice that we have $\mathcal{Q} \succeq 0$, therefore, as stated in Remark 1, the alternative solution (9) can be used in this case. In order to do so, consider the following additional identifications:

$$\bar{\mathcal{Q}} \leftarrow \begin{bmatrix} (\boldsymbol{\mathcal{P}}_{k|k-1}^{i})^{-1} & 0 & 0 \\ 0 & Q_k^{-1} & 0 \\ 0 & 0 & (\boldsymbol{\mathcal{R}}_k^{i})^{-1} \end{bmatrix}, \; A_1 \leftarrow \begin{bmatrix} I_{\bar{n}} & 0 & 0 \\ 0 & H_k & 0 \\ 0 & 0 & \boldsymbol{\mathcal{D}}_k^{i} \end{bmatrix},$$

$$A_2 \leftarrow \begin{bmatrix} -\boldsymbol{\mathcal{I}} & 0 \\ F_k & -I_n \\ \boldsymbol{\mathcal{C}}_k^{i} & 0 \end{bmatrix}, \; E_{A_1} \leftarrow \begin{bmatrix} 0 & E_{H_k} & 0 \\ 0 & 0 & \boldsymbol{E}_{\mathcal{D}_k}^{i} \end{bmatrix}, \; E_{A_2} \leftarrow \begin{bmatrix} E_{F_k} & 0 \\ \boldsymbol{E}_{\mathcal{C}_k}^{i} & 0 \end{bmatrix}.$$

Lemma 2. Consider the optimization problem (5)–(6) with fixed $\mu > 0$. The posterior and prior state estimates, $\hat{x}_{k|k}^{i}$ and $\hat{x}_{k+1|k}^{i}$, respectively, obtained by the ith sensor of the network, as well as their respective error weighting matrices, $P_{k|k}^{i}$ and $P_{k+1|k}^{i}$, are recursively given by

$$\mathcal{J}_{k,\mu}^{i} = \begin{bmatrix} \boldsymbol{e}_{k|k-1}^{i} \\ w_k \\ \boldsymbol{v}_k^{i} \\ x_k \\ x_{k+1} \end{bmatrix}^T \begin{bmatrix} (\boldsymbol{\mathcal{P}}_{k|k-1}^{i})^{-1} & 0 & 0 & 0 & 0 \\ 0 & Q_k^{-1} & 0 & 0 & 0 \\ 0 & 0 & (\boldsymbol{\mathcal{R}}_k^{i})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{k|k-1}^{i} \\ w_k \\ \boldsymbol{v}_k^{i} \\ x_k \\ x_{k+1} \end{bmatrix} + \left(\begin{bmatrix} I_{\bar{n}} & 0 & 0 & -\boldsymbol{\mathcal{I}} & 0 \\ 0 & H_{\delta,k} & 0 & F_{\delta,k} & -I_n \\ 0 & 0 & \boldsymbol{\mathcal{D}}_{\delta,k}^{i} & \boldsymbol{\mathcal{C}}_{\delta,k}^{i} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{k|k-1}^{i} \\ w_k \\ \boldsymbol{v}_k^{i} \\ x_k \\ x_{k+1} \end{bmatrix} - \begin{bmatrix} -\hat{\boldsymbol{x}}_{k|k-1}^{i} \\ -G_{\delta,k} u_k \\ \boldsymbol{z}_k^{i} \end{bmatrix} \right)^T \mu I \left(\bullet \right) \tag{6}$$

with fixed $\mu > 0$ and $\bar{n} = \bar{N}_i n$.

$$\left[\begin{bmatrix}\hat{x}^i_{k|k}\\ \hat{x}^i_{k+1|k}\end{bmatrix}\ \begin{bmatrix}P^i_{k|k} & *\\ * & P^i_{k+1|k}\end{bmatrix}\right] = \begin{bmatrix}0\\0\\0\\I\end{bmatrix}^T \begin{bmatrix}\mathscr{P}^i_k & 0 & I & 0\\ 0 & \Lambda^i_{\mu,k} & \mathscr{D}^i_k & \mathscr{C}^i_k\\ I & (\mathscr{D}^i_k)^T & 0 & 0\\ 0 & (\mathscr{C}^i_k)^T & 0 & 0\end{bmatrix}^{-1} \begin{bmatrix}0 & 0\\ \psi^i_k & 0\\ 0 & 0\\ 0 & -I\end{bmatrix}, \quad (10)$$

for $k = 0, 1, \ldots, N$, with

$$\mathscr{P}^i_k = \begin{bmatrix}\boldsymbol{\mathcal{P}}^i_{k|k-1} & 0 & 0\\ 0 & Q_k & 0\\ 0 & 0 & \boldsymbol{\mathcal{R}}^i_k\end{bmatrix},\ \Lambda^i_{\mu,k} = \begin{bmatrix}\mu^{-1}I_{\bar{n}} & 0 & 0\\ 0 & \Sigma^i_{1,\mu,k} & 0\\ 0 & 0 & \Sigma^i_{2,\mu,k}\end{bmatrix},$$

$$\Sigma^i_{1,\mu,k} = \begin{bmatrix}\mu^{-1}I_n - (\widehat{\lambda}^i_k)^{-1}M_{1,k}M^T_{1,k} & 0\\ 0 & (\widehat{\lambda}^i_k)^{-1}I_{t_1}\end{bmatrix},$$

$$\Sigma^i_{2,\mu,k} = \begin{bmatrix}\mu^{-1}I_{\bar{r}} - (\widehat{\lambda}^i_k)^{-1}\boldsymbol{\mathcal{M}}^i_{2,k}(\boldsymbol{\mathcal{M}}^i_{2,k})^T & 0\\ 0 & (\widehat{\lambda}^i_k)^{-1}I_{\bar{t}_2}\end{bmatrix},$$

$$\mathscr{D}^i_k = \begin{bmatrix}I_{\bar{n}} & 0 & 0\\ 0 & \widehat{H}_k & 0\\ 0 & 0 & \widehat{\boldsymbol{\mathcal{D}}}^i_k\end{bmatrix},\ \mathscr{C}^i_k = \begin{bmatrix}-\boldsymbol{\mathcal{I}} & 0\\ \widehat{F}_k & -\widehat{I}\\ \widehat{\boldsymbol{\mathcal{C}}}^i_k & 0\end{bmatrix},\ \psi^i_k \leftarrow \begin{bmatrix}-\hat{\boldsymbol{x}}^i_{k|k-1}\\ -\widehat{G}_k u_k\\ \hat{\boldsymbol{z}}^i_k\end{bmatrix},$$

$$\widehat{F}_k = \begin{bmatrix}F_k\\ E_{F_k}\end{bmatrix},\ \widehat{G}_k = \begin{bmatrix}G_k\\ E_{G_k}\end{bmatrix},\ \widehat{H}_k = \begin{bmatrix}H_k\\ E_{H_k}\end{bmatrix},\ \widehat{I} = \begin{bmatrix}I_n\\ 0\end{bmatrix},$$

$$\widehat{\boldsymbol{\mathcal{C}}}^i_k = \begin{bmatrix}\boldsymbol{\mathcal{C}}^i_k\\ \boldsymbol{E}^i_{\mathcal{C}_k}\end{bmatrix},\ \widehat{\boldsymbol{\mathcal{D}}}^i_k = \begin{bmatrix}\boldsymbol{\mathcal{D}}^i_k\\ \boldsymbol{E}^i_{\mathcal{D}_k}\end{bmatrix},\ \hat{\boldsymbol{z}}^i_k = \begin{bmatrix}\boldsymbol{z}^i_k\\ 0\end{bmatrix},$$

$$\widehat{\lambda}^i_k = (1+\alpha)\left\|\mu\,\mathbf{diag}\left(M^T_{1,k}M_{1,k},\ (\boldsymbol{\mathcal{M}}^i_{2,k})^T\boldsymbol{\mathcal{M}}^i_{2,k}\right)\right\|,$$

for some $\alpha > 0$, $\bar{n} = \bar{N}_i n$, $\bar{r} = \bar{N}_i r$, and $\bar{t}_2 = \bar{N}_i t_2$.

Proof. If follows directly from applying Lemma 1 to problem (5)–(6), considering the alternative solution in Remark 1. Note that the entries of the solution marked with $*$ are byproducts with no particular meaning.

Remark 2. The optimal solution to the original constrained problem (4) is thus obtained by making $\mu \to +\infty$ in the solution (10) of the equivalent unconstrained problem (5)–(6). As a consequence, $\widehat{\lambda}^i_k \to +\infty$, such that $\Lambda^i_{\mu,k} \to 0$. In this case, notice that the matrix block $\left[\mathscr{D}^i_k\ \mathscr{C}^i_k\right]$ must have full row rank in order to ensure the invertibility of the central matrix in (10).

The main result of this paper is the following theorem, in which the solution in Lemma 2 is reduced to an equivalent distributed Kalman-like recursive structure.

Theorem 1. Consider the optimization problem (5)–(6) with fixed $\mu > 0$. The RDCF recursive equations to compute the posterior and prior state estimates, $\hat{x}^i_{k|k}$ and $\hat{x}^i_{k+1|k}$, respectively, obtained by the ith sensor of the network, as well as their respective error weighting matrices, $P^i_{k|k}$ and $P^i_{k+1|k}$, for $k = 0, 1, \ldots, N$, are

- **Correction step:**

$$(P^i_{k|k})^{-1} = \sum_{j\in\overline{\mathbb{N}}_i}(\bar{P}^j_{k|k-1})^{-1} + \sum_{j\in\overline{\mathbb{N}}_i}(\widehat{C}^j_k)^T(\bar{R}^j_k)^{-1}\widehat{C}^j_k + \widehat{F}^T_k\Upsilon^i_k\widehat{F}_k, \quad (11)$$

$$(P^i_{k|k})^{-1}\hat{x}^i_{k|k} = \sum_{j\in\overline{\mathbb{N}}_i}(\bar{P}^j_{k|k-1})^{-1}\hat{x}^j_{k|k-1} + \sum_{j\in\overline{\mathbb{N}}_i}(\widehat{C}^j_k)^T(\bar{R}^j_k)^{-1}\hat{z}^j_k - \widehat{F}^T_k\Upsilon^i_k\widehat{G}_k u_k, \quad (12)$$

- **Prediction step:**

$$P^i_{k+1|k} = \left[\widehat{I}^T(\bar{Q}^i_k)^{-1}\widehat{I}\right]^{-1} + \Gamma^i_k\widehat{F}_k P^i_{k|k}\widehat{F}^T_k(\Gamma^i_k)^T, \quad (13)$$

$$\hat{x}^i_{k+1|k} = \Gamma^i_k\left(\widehat{F}_k\hat{x}^i_{k|k} + \widehat{G}_k u_k\right), \quad (14)$$

with

$$\widehat{F}_k = \begin{bmatrix}F_k\\ E_{F_k}\end{bmatrix},\ \widehat{G}_k = \begin{bmatrix}G_k\\ E_{G_k}\end{bmatrix},\ \widehat{H}_k = \begin{bmatrix}H_k\\ E_{H_k}\end{bmatrix},\ \hat{z}^j_k = \begin{bmatrix}z^j_k\\ 0\end{bmatrix},$$

$$\widehat{C}^j_k = \begin{bmatrix}C^j_k\\ E^j_{C_k}\end{bmatrix},\ \widehat{D}^j_k = \begin{bmatrix}D^j_k\\ E^j_{D_k}\end{bmatrix},\ \bar{P}^j_{k|k-1} = \mu^{-1}I_n + P^j_{k|k-1},$$

$$\bar{Q}^i_k = \Sigma^i_{1,k} + \widehat{H}_k Q_k\widehat{H}^T_k,\quad \Gamma^i_k = \left[\widehat{I}^T(\bar{Q}^i_k)^{-1}\widehat{I}\right]^{-1}\widehat{I}^T(\bar{Q}^i_k)^{-1},$$

$$\bar{R}^j_k = \sigma^j_{2,k} + \widehat{D}^j_k R^j_k(\widehat{D}^j_k)^T,\quad \Upsilon^i_k = (\bar{Q}^i_k)^{-1} - (\bar{Q}^i_k)^{-1}\widehat{I}\,\Gamma^i_k,$$

$$\Sigma^i_{1,k} = \begin{bmatrix}\mu^{-1}I_n - (\widehat{\lambda}^i_k)^{-1}M_{1,k}M^T_{1,k} & 0\\ 0 & (\widehat{\lambda}^i_k)^{-1}I_{t_1}\end{bmatrix},\ \widehat{I} = \begin{bmatrix}I_n\\ 0\end{bmatrix},$$

$$\sigma^j_{2,k} = \begin{bmatrix}\mu^{-1}I_r - (\widehat{\lambda}^i_k)^{-1}M^j_{2,k}(M^j_{2,k})^T & 0\\ 0 & (\widehat{\lambda}^i_k)^{-1}I_{t_2}\end{bmatrix},$$

$$\widehat{\lambda}^i_k = (1+\alpha)\left\|\mu\,\mathbf{diag}\left(M^T_{1,k}M_{1,k},\ (\boldsymbol{\mathcal{M}}^i_{2,k})^T\boldsymbol{\mathcal{M}}^i_{2,k}\right)\right\|, \quad (15)$$

for some $\alpha > 0$, and $\forall j \in \overline{\mathbb{N}}_i$.

Proof. It follows from solving the system of simultaneous equations represented by the matrix arrangement (10) and performing some algebraic manipulations.

Remark 3. In order to improve estimation accuracy, additional consensus steps can be carried out by the sensors when computing the summation terms in (11) and (12).

Remark 4. The invertibility of $P^i_{k|k}$, $\forall i \in \mathbb{S}$ and $k = 0, 1, \ldots, N$ is guaranteed by requiring that $P^i_{0|-1} \succ 0$ and $(\widehat{C}^i_k)^T(\bar{R}^i_k)^{-1}\widehat{C}^i_k \succ 0$, $\forall i \in \mathbb{S}$. Thus, matrices $\widehat{C}^i_k$ should have full column rank. Furthermore, it can be shown that $\widehat{F}^T_k\Upsilon^i_k\widehat{F}_k \succeq 0$.

Remark 5. Regarding optimality of the solution, as mentioned in Remark 2, it is required that $\mu \to +\infty$. In this reduced form, it is easier to verify that, for this to be possible, matrices $\widehat{H}_k$ and $\widehat{D}^i_k$ should have full row rank, $\forall i \in \mathbb{S}$. This, in turn, means that $n + t_1 \le p$ and $r + t_2 \le q$. When this is not the case, the penalty parameter μ acts as a measure of robustness of the proposed filter.

Algorithm 1 summarizes the robust distributed consensus-based filtering procedure for each sensor i of the network.

5. ILLUSTRATIVE EXAMPLES

In this section, we assess the performance of the proposed RDCF with two examples. In both, we compare our results with an existing robust distributed filtering strategy (Rastgar and Rahmani, 2018) and with a robust centralized scheme, in which a fusion center has access to information from all sensors of the network.

Algorithm 1 Robust Distributed Consensus-Based Filter

Model: Assume a target plant with model (1) and a network of sensors with model (2).
Initialization: Set $\hat{x}^i_{0|-1}$, $P^i_{0|-1} \succ 0$, $\mu > 0$, and $\alpha > 0$.
for $k = 0, 1, \ldots, N$ **do**:
1: Obtain measurement z^i_k;
2: Compute $\bar{P}^i_{k|k-1}$, as in (15);
3: Broadcast message

$$m^i_k = \{(\bar{P}^i_{k|k-1})^{-1}\hat{x}^i_{k|k-1},\ z^i_k,\ (\bar{P}^i_{k|k-1})^{-1},\ \widehat{C}^i_k,\ \widehat{D}^i_k R^i_k (\widehat{D}^i_k)^T,\ M^i_{2,k}\};$$

4: Receive messages m^j_k from neighbors $j \in \mathcal{N}_i$;
5: Compute $\widehat{\lambda}^i_k$, $\Sigma^i_{1,k}$, $\bar{Q}^i_k$, Γ^i_k, and Υ^i_k using (15);
6: Compute $\sigma^j_{2,k}$ and $\bar{R}^j_k$, $\forall j \in \overline{\mathcal{N}}_i$, according to (15);
7: Obtain posterior error weighting matrix $P^i_{k|k}$ and state estimate $x^i_{k|k}$ using (11) and (12), respectively;
8: Obtain prior error weighting matrix $P^i_{k+1|k}$ and state estimate $x^i_{k+1|k}$ using (13) and (14), respectively.
end for

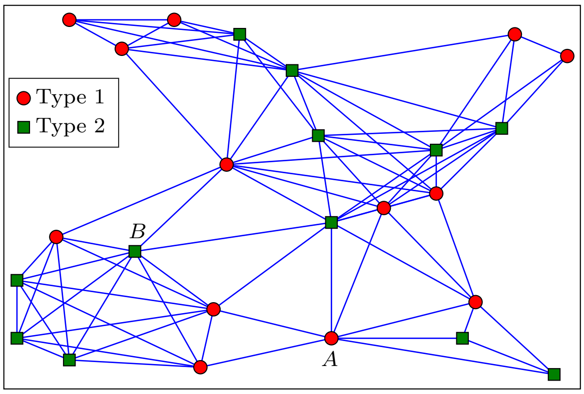

Fig. 1. Sensor network with 25 nodes and 81 edges.

Example 1. (Adapted from Rastgar and Rahmani (2018)) Consider a target plant model as in (1), with matrices:

$$F_k = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix},\ G_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ H_k = \begin{bmatrix} -6 \\ 1 \end{bmatrix},\ M_{1,k} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$E_{F_k} = [0.01\sin(6k)\ \ 0.2]\,,\ E_{G_k} = 0,\ E_{H_k} = 0.3.$$

No input signal u_k is present and w_k is a zero-mean white Gaussian noise signal with variance $Q_k = 0.00125$. The initial state is $x_0 = [1\ 0]^T$.

A set of $S = 25$ sensors arranged in a random geometric undirected network, depicted in Fig. 1, observe the target system. The sensing model is described in (2), with v^i_k as zero-mean white Gaussian noise signals with variances R^i_k. Two distinct types of sensors are considered. Sensors with odd number, i.e., $i = 1, 3, \ldots, 25$, are of the first type, having matrices

$$C^i_k = [-100\ \ 50]\,,\ D^i_k = 1,\ M^i_{2,k} = 1,$$

$$E^i_{C_k} = [0.2\ \ 0.5]\,,\ E^i_{D_k} = 0.2,\ R^i_k = 0.9.$$

Sensors with even number, i.e., $i = 2, 4, \ldots, 24$, are of the second type, with matrices

$$C^i_k = [-100\ \ 53]\,,\ D^i_k = 1,\ M^i_{2,k} = 1,$$

$$E^i_{C_k} = [0.3\ \ 0.5]\,,\ E^i_{D_k} = 0.3,\ R^i_k = 0.9.$$

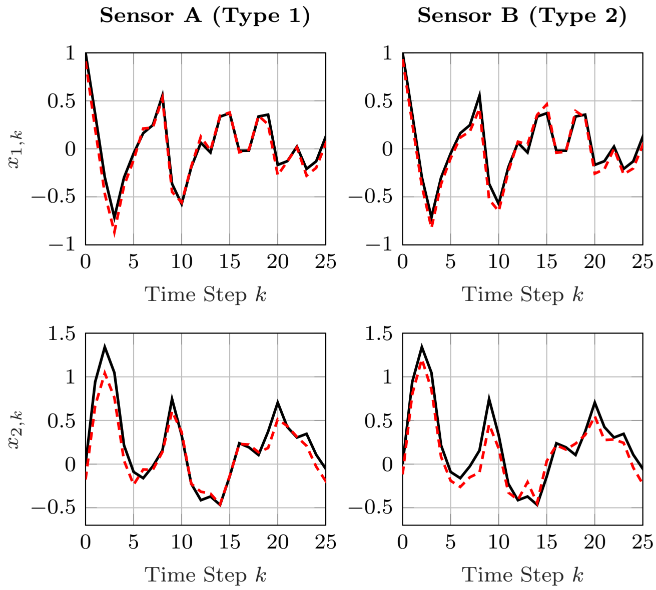

Fig. 2. Actual (solid lines) and estimated (dashed lines) state of the target system by the two types of sensors in the network.

We apply Algorithm 1 with the following initialization data for all sensors:

$$P^i_{0|-1} = I_2,\ \mu = 10^6,\ \alpha = 0.5,$$

and initial prior state estimates $x^i_{0|-1}$ randomly selected from a normal distribution with mean x_0 and variance $P^i_{0|-1}$, for all $i \in \mathbb{S}$. Figure 2 shows the evolution of the target state along with the estimation performed by sensors A (type 1) and B (type 2), which are identified in Fig. 1. At each time step, $\Delta_{1,k}$ and $\Delta^i_{2,k}$ are real numbers picked from a zero-mean Gaussian distribution with variance $\xi = 0.1$. The results show that the proposed RDCF can successfully track the state of the target, despite the norm-bounded parametric uncertainties, present in all matrices of both the process and sensor models.

We further evaluate the RDCF by comparing its performance with that of the optimal robust distributed CE-based filter (ORDCF) proposed in Rastgar and Rahmani (2018) and with a robust centralized filter (RCF), based on the filter presented in Ishihara et al. (2015), in which all sensor data is fused at once. The simulation consists of performing $L = 1000$ Monte Carlo experiments, each with time horizon $N = 100$. At each time step k, the root mean squared estimation error (RMSE) averaged along all experiments and sensors in the network is computed as

$$\mathrm{RMSE}_k = \left(\frac{1}{SL} \sum_{i=1}^{S} \sum_{\ell=1}^{L} \| x_k - \hat{x}^i_{k|k,\ell} \|^2 \right)^{\frac{1}{2}}.$$

The results for the three filters are shown in Fig. 3. As expected, the centralized strategy presents the smallest estimation error. The proposed RDCF, even in a suboptimal setting, since $\mu < +\infty$ (see Remark 5), outperforms the ORDCF. Moreover, the communication burden and computational complexity of the latter are significantly higher, as it depends on the calculation of cross-covariance matrices between every pair of sensors.

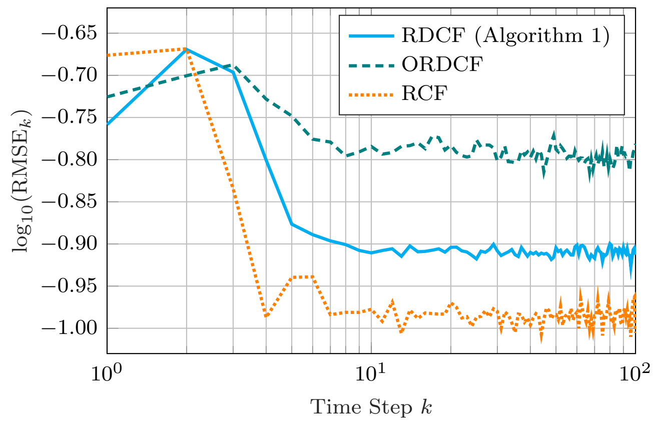

Fig. 3. RMSE of the three filters for Example 1.

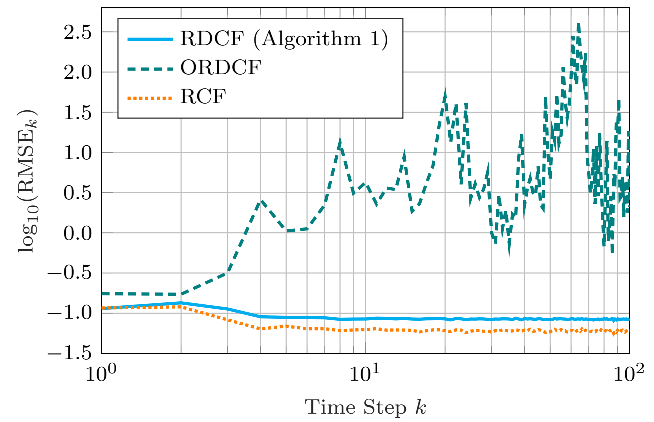

Fig. 4. RMSE of the three filters for Example 2.

Example 2. In this second example, we consider the same data from Example 1, except for the following changes in the sensing model matrices:

$$C_k^i = [-100 \;\; 10]\,,\; i = 1, 3, \ldots, 25.$$
$$C_k^i = [-95 \;\; 16]\,,\; i = 2, 4, \ldots, 24.$$

As in Example 1, a set of $L = 1000$ Monte Carlo experiments are performed. The resulting evolution of the RMSE for the three filters over time is depicted in Fig. 4. As before, the RCF exhibits the best performance, followed by the RDCF. Nevertheless, the ORDCF presents performance degradation.

6. CONCLUSION

In this paper, we proposed a new robust distributed consensus-based filter that is suitable for sensor networks applications. Norm-bounded parametric uncertainties are assumed in every matrix of both the target plant and sensing models. The filter results from the solution of a purely deterministic robust regularized least-squares estimation problem, along with the application of the penalty function method. The relation between optimality and robustness is encapsulated in the penalty parameter μ, which can be tuned in order to obtain satisfactory tracking performance.

The effectiveness of the proposed filter was verified by means of two numerical examples. A comparison with the optimal robust distributed filter of Rastgar and Rahmani (2018) and with a centralized strategy was carried out. The results show that, in both examples, our filter presents a performance close to the centralized scheme and outperforms the other distributed strategy. Moreover, it does not require the calculation of cross-covariance matrices, saving significant communication and computational efforts.

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