A Robust Method for Dual Faceted Linearization

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Abstract: The dynamics of nonlinear systems become linear systems when lifted to higher or infinite dimensional spaces. We call such linear system representations and approximations, 'lifting linear' representations. The lifting linear representations are linear system representations that are closer to the original systems than Taylor series approximations. Once we have such a linear system representation, we can apply linear control theory to the nonlinear systems. In Model Predictive Control (MPC), the computation time is reduced because the nonlinear optimization problem becomes a convex quadratic optimization problem. In this paper, we propose a method to make Dual Faceted Linearization (DFL) robust for uncertainties of the plants. It will be shown that the proposed method can yield a lifting linearization leading to better control results for MPC by numerical examples.

Keywords: Lifting Linearization, Model Predictive Control(MPC), Nonlinear System Control, Robustification, Optimal Control

1. INTRODUCTION

It is now well known that dynamics of nonlinear systems can be transformed to linear systems in higher or infinite dimensional spaces. We call such linear system representations and approximations, 'lifting linear' representations. As described in Koopman (1931), Koopman theory shows that general nonlinear systems can be represented as linear systems in infinite dimensional spaces. From Asada and Sotiropoulos (2019), Dual Faceted Linearization (DFL) method offers a consistent method for selecting the augmented states based on the system model. There are other methods such as Carleman linearization for Steeb and Wilhelm (1980) and machine learning for Lusch et al. (2018). These linearization methods give very different linear models compared to Taylor series linear approximations, and yield behaviors that are valid in larger state spaces. Once such linear representations are obtained, linear control theory, e.g., analysis of oscillation modes, design of observer and predictor, Model Predictive Control (MPC) can be applied. See Korda and Mezić (2018), Surana (2016) and Arbabi et al. (2018). In addition, a fast Stochastic MPC algorithms can be constructed in Oyama et al. (2016).

When we apply MPC for nonlinear systems and higher order terms of states and input are considered in the stage cost of MPC, we are required to solve nonlinear optimization problems online. This leads to significant problems due to computational time and convergence of the optimal solution. When the target systems are linear and conventional cost functions can be applied with linear constraints, the optimization problems become convex quadratic optimization problems, which are expected to have much shorter computational times and yield solutions with improved optimality.

In papers Igarashi et al. (2020), we verified the computational speed and optimality of MPC for a nonlinear system using DFL. From the numerical simulations, it was confirmed that the computational time can be shortened much and the optimality is maintained. And we studied that computational time of MPC dramatically improves and performances of MPC for nonlinear systems are improved by taking account of higher order terms of the states.

DFL is a lifting linearization method that can be used when the system model is known. The augmented state variables have a clear physical meaning and the augmented state space should not be too large in dimension. However, this method cannot be used if the system model is unknown.

In this paper, we propose a method to make DFL robust for uncertainties of the plants. It will be shown that the proposed method can yield a lifting linearization with better control performance for MPC. First we summarize the concept of lifting linearizations and then propose a robust method for DFL. Finally, the efficiencies of the proposed method are shown through numerical simulations.

2. LIFTING LINEARIZATION

In this section, we simply explain the lifting linearization by an illustrative nonlinear system. In addition, Koopman method and DFL are described as lifting linearization methods related to the proposed method.

2.1 Illustrative example

Let us consider a nonlinear system given as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \rho x_1 \\ \mu(x_2 + x_1^2) \end{bmatrix}, \tag{1}$$

where x_1, x_2 are state variables and ρ, μ are constant parameters. For this system, we introduce an augmented state x_3^* as $x_3^* = x_1^2$ and the dimension of the system is increased. Then the time derivative of x_3^* is given as

$$\dot{x}_3^* = (\dot{x_1^2}) = 2x_1\dot{x}_1 = 2\rho x_1^2 = 2\rho x_3^* \tag{2}$$

and the state space representation of the augmented system with x_1, x_2, x_3^* can be given as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3^* \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \mu & \mu \\ 0 & 0 & 2\rho \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3^* \end{bmatrix}. \tag{3}$$

The obtained system is a Linear Time Invariant (LTI) system, but the state variables are not independent. In general, however, we can not obtain such complete LTI system as in eq. (3) and some approximation must be made, ignoring some nonlinear terms when the dimensions of the augmented system are finite.

2.2 Koopman theory

In Koopman theory, Koopman operators are introduced and linear system representations are obtained. Let us consider the following discrete time nonlinear system:

$$x(k+1) = f(x(k)), \tag{4}$$

where $x(k)$ is a state vector at time instance k. For this system we introduce observation functions (or basis functions) $g(x(k))$. We define a space of observation functions $\mathcal{F}$ which contains the original state variable x and their corresponding observation functions $g(x(k))$. In general, $\mathcal{F}$ becomes an infinite dimensional space. For this observation space, we define a Koopman operator $\mathcal{K} : \mathcal{F} \to \mathcal{F}$ as

$$(\mathcal{K}g)(x(k)) := g(f(x(k))). \tag{5}$$

It is shown that the Koopman operator $\mathcal{K}$ is linear operator. (For the details, please refer Koopman (1931).)

Time advance of the observation function can be represented using the Koopman operator $\mathcal{K}$ as

$$\begin{aligned} g(x(k)) = \ & g(f(x(k-1))) = (\mathcal{K}g)(x(k-1)) \\ & = (\mathcal{K}^2 g)(x(k-2)) = \cdots = (\mathcal{K}^k g)(x(0)). \end{aligned} \tag{6}$$

The problem with the above representation is how to determine the observation function and the approximation of Koopman operator $\mathcal{K}$. For these problems, Extended Dynamic Mode Decomposition (EDMD) (Korda and Mezić (2018), Arbabi et al. (2018) and Williams et al. (2015)) and methods based on machine learning (Yeung et al. (2019) and Takeishi et al. (2017)) have been proposed in the literatures. These methods can approximate a linear expression even with respect to the inputs.

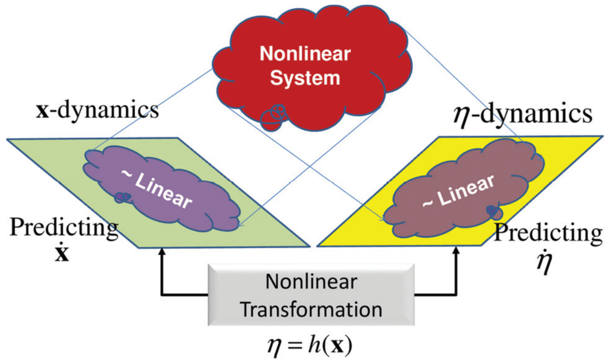

Fig. 1. Conceptual diagram of DFL (Asada and Sotiropoulos (2019)) .

2.3 Dual Faceted Linearization

Dual Faceted Linearization (DFL) is a lifting linearization method which possesses the following two features:

1) Usage of natural linearity in physical modeling
When we consider lumped parameter systems, elements, e.g., mass, spring, damper or capacitor, inductor, resistor, are connected linearly. For example, it can be formulated that the summation of forces for each of the mechanical elements is zero using Newton's law and d'Alembert's principle as

$$F_1(x) + F_2(x) + \cdots + F_N(x) = 0, \tag{7}$$

which is a linear relationship. In such systems, nonlinearity arises from the property of each element. When the nonlinear quantities are defined as auxiliary variables and they are used with the new variables, the system behaviors can be represented as linear relationships.

2) Usage of dual variables for one nonlinear element
Let us consider two sets of equation of motion using two different variables for one dynamical system. When one expression can be transformed to another one by a linear transformation, there is no advantage gained by using both expressions. However, if two variables are related by a nonlinear relationship, they may correspond to one physical system but show different behaviors in different coordinate systems. As shown in Fig. 1, each representation is (approximately) linear of the original system and when both are combined, more accurate behavior of the original system can be obtained.

Let us consider the following nonlinear system:

$$\dot{x} = f(x, u), \tag{8}$$

where x is a state vector and u is an input vector. We introduce an auxiliary vector $\eta(x)$ for nonlinear elements and we assume that system (8) can be represented as

$$\dot{x} = A_x x + A_\eta \eta + B_x u, \tag{9}$$

where A_x, A_η, B_x are constant system matrices. Even when the input is not linear in the original system, a linear expression can be obtained by approximation. Next we approximate the dynamics of $\eta(x)$ with a linear dynamic relationship in the form

$$\dot{\eta} = H_x x + H_\eta \eta + H_u u, \tag{10}$$

where H_x, H_η, H_u are constant system matrices. When we define the following matrix:

$$H := [H_x, H_\eta, H_u] \tag{11}$$

and $\xi = [x^T, \eta^T, u^T]^T$, then we consider the following criterion function with observation data of $\dot{\eta}$ as $\dot{\eta}_{ob}$:

$$H = \arg\min_H E[\|\dot{\eta} - \dot{\eta}_{ob}\|^2] \tag{12}$$

to determine the parameter matrices where $E[\cdot]$ stands for expectation operator and H is calculated as

$$H = E[\dot{\eta}_{ob}\xi^T](E[\xi\xi^T])^{-1}. \tag{13}$$

3. MODEL PREDICTIVE CONTROL

3.1 Model Predictive Control

In MPC, future state sequences are predicted in a receding horizon, and a control input sequence is determined to optimize a criterion function based on a system model, and this process is repeated. Let us consider a nonlinear system given by

$$\dot{x}(t) = f(x(t), u(t)), \tag{14}$$

where $x(t)$ is a state vector and $u(t)$ is an input vector. In MPC, we consider and minimize the following criterion function J defined within $[t, t+T]$:

$$\min J = \varphi(x(t+T)) + \int_t^{t+T} L(x(\tau), u(\tau))d\tau, \tag{15}$$

$$s.t. \quad \dot{x}(t) = f(x(t), u(t)), \tag{16}$$

$$x_0(t) = x(t), \tag{17}$$

$$C(x(t), u(t)) \leq 0, \tag{18}$$

where $\varphi(x(T))$ is a terminal cost, $L(x(\tau), u(\tau))$ is a stage cost, $C(x(t), u(t))$ is a constraint vector. When an optimal control sequence $u(\tau)$ $(t \leq \tau \leq t+T)$ is determined, the first control input $u(t)$ is injected. In nonlinear MPC, the system dynamics are usually discretized by Euler approximation to suppress the computation in nonlinear dynamics analysis. When the receding horizon discretized by N interval and the discrete time interval is defined as $d\tau = T/N$, the corresponding discrete time optimization problem is defined as

$$\min J = \varphi(x_N(t)) + \sum_{i=0}^{N-1} L(x_i(t), u_i(t))d\tau, \tag{19}$$

$$s.t. \quad x_{i+1}(t) = x_i(t) + f(x_i(t), u_i(t))d\tau, \tag{20}$$

$$x_0(t) = x(t), \tag{21}$$

$$C(x_i(t), u_i(t)) \leq 0. \tag{22}$$

The optimal control input is determined by solving a static nonlinear optimization problem.

3.2 Model predictive control using lifting linearization

In nonlinear MPC, it is difficult to apply a static nonlinear optimization method to determine the optimal control input sequence due to large computation burden, and some approximation and continuation method such as C/GMRES method in Ohtsuka (2004) is used. In this paper, we consider the use of lifting linearizations to reduce the computation time.

When the augmented state vector represented as x^* and the linear system obtained by the lifting linearization for the original system (14) is given as

$$\dot{x}^* = Ax^* + Bu, \tag{23}$$

where A, B are constant system matrices. If the control input u is constant during the sampling interval, the discrete time system can be given as

$$x^*_{i+1} = A_d x^*_i + B_d u_i, \tag{24}$$

where the matrices are given by

$$A_d = \exp(Ad\tau), \tag{25}$$

$$B_d = A^{-1}(\exp(Ad\tau) - I)B, \tag{26}$$

where I is an identity matrix. If the original state vector is included in the augmented vector as $x^* = [x^T, g(x)^T]^T$, the same cost function and constraint can be used and the optimization problem becomes

$$\min J = \bar{\varphi}(x^*_N(t)) + \sum_{i=0}^{N-1} \bar{L}(x^*_i(t), u_i(t))d\tau \tag{27}$$

$$s.t. \quad x^*_{i+1}(t) = A_d x^*_i(t) + B_d u_i(t), \tag{28}$$

$$x^*_0(t) = [x(t)^T, g(x(t))^T]^T, \tag{29}$$

$$\bar{C}(x^*_i(t), u_i(t)) \leq 0. \tag{30}$$

In this paper, we assume that $\bar{\varphi}(x^*), \bar{L}(x^*, u)$ contain quadratic terms of $x^* - x^*_d$ and control input or difference of control input. x^*_d is a constant desired state which is compatible to x_d. We also assume that $\varphi(x), L(x, u)$ are quadratic form of the augmented state variables. This criterion function may have a higher order terms of the original states than the original one. Further, we assume that the constraint $\bar{C}(x^*_i, u_i) \leq 0$ is given by a linear equation for the augmented states in the form

$$\bar{C}x^*_i + \bar{D}u_i \leq \bar{E}, \tag{31}$$

where $\bar{C}, \bar{D}, \bar{E}$ are constant matrices.

By assumptions, the original optimization can be transformed to the quadratic optimization problem as

$$\min J = \frac{1}{2}U^T Q^*_1 U + {Q^*_2}^T U + {x^*_0}^T Q^*_3 x^*_0, \tag{32}$$

$$s.t. \quad x^*_0 = [x(t)^T, g(x(t))^T]^T, \tag{33}$$

$$C^* x^*_0 + D^* U \leq E^*, \tag{34}$$

where $U = [u_1^T, u_2^T, \cdots, u_{N-1}^T]^T$, and matrices above can be calculated from the original problem. The quadratic problem can be solved quickly and the optimality is ensured. Please note that the initial augmented states for the optimization problem are calculated every time based on the observed $x(t)$ using $g(x)$ as in eq. (33). The augmented linear system has some approximation error and thus, long term prediction accumulate large errors; however, the prediction in the receding horizon is suppressed since the augmented initial state are reset to the true state.

4. PROPOSED METHOD

In the DFL method, we assume that dynamics of the systems are given in principle though the equations are not used directly but corresponding to signals are collected. However, there exist uncertainties or parasitic dynamics in practice. So we should consider such uncertainties when constructing lifting linearizations. In order to consider such unmodeled terms, we propose introducing basis functions, which may be thin plate spline radial basis functions as in Korda and Mezić (2018), in addition to the auxiliary vector in DFL. Let us assume that a vector γ consists of such basis functions as its elements, then we consider the following augmented state vector x^* as $x^* := [x^T, \eta^T, \gamma^T]^T$ and its dynamics is approximated by

$$\dot{x} = A_x x + A_\eta \eta + A_\gamma \gamma + B_x u, \tag{35}$$
$$\dot{\eta} = H_x x + H_\eta \eta + H_\gamma \gamma + H_u u, \tag{36}$$
$$\dot{\gamma} = F_x x + F_\eta \eta + F_\gamma \gamma + F_u u, \tag{37}$$

where $A_\cdot, H_\cdot, F_\cdot, B_\cdot$ are constant system matrices of appropriate dimensions, and the those system matrices can be estimated using the observed data in the same way as eq. (12) in DFL.

Furthermore, if there may exist dynamic uncertainties whose states are independent to x, we assume that the observed x^* can be determined as a summation of $\hat{x}^*$ plus an output of the parasitic dynamics y_p as

$$x^* = \hat{x}^* + y_p, \tag{38}$$

where $\hat{x}^*$ is calculated from the definition of the augmented state vector using x. The parasitic dynamics can be determined from the data of $(x^* - \hat{x}^*, u)$ as

$$\dot{x}_p = A_p x_p + B_p u, \tag{39}$$
$$y_p = C_p x_p + D_p u, \tag{40}$$

using any identification method Oyama et al. (2016). Please note that if x_p is introduced, it must be estimated online using the identified model. If the dimensions of x^*, x_p are very high, they can be reduced by any model reduction technique.

5. NUMERICAL SIMULATION

We compare performances and robustness for open loop approximation and MPC of the proposed method to other lifting linearization methods using numerical simulations. In this numerical simulation, only uncertainties of the nonlinear terms are considered. All simulations were done using Matlab. For the static nonlinear optimization, 'fmincon' is used and for the quadratic optimization, 'quadprog' is used, and a computer used for the numerical simulations has Intel Core i7 CPU, 3.40 GHz with 8 GB RAM.

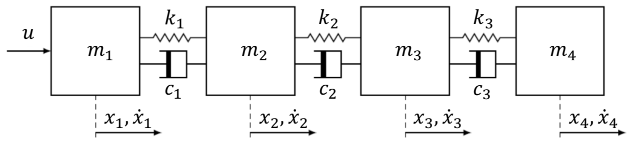

Fig. 2. Target model

5.1 Target Model

We consider a nonlinear system in Fig. 2 as the target system in which 4 masses are connected with nonlinear springs and dampers. The weight of the mass is 1 and the origin of the state vector is the equilibrium state, and x, $\dot{x}$ are position and velocity vectors of the masses, respectively, and control force is applied to mass m_1 and the control force vector is denoted u. The dynamic equation of the system is given as

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} = \begin{bmatrix} u - (e_{R_1} + e_{C_1} + \omega_1) \\ (e_{R_1} + e_{C_1} + \omega_1) - (e_{R_2} + e_{C_2} + \omega_2) \\ (e_{R_2} + e_{C_2} + \omega_2) - (e_{R_3} + e_{C_3} + \omega_3) \\ (e_{R_3} + e_{C_3} + \omega_3) \end{bmatrix}, \tag{41}$$

where $e_{C_i}, e_{R_i},\ (i = 1, 2, 3)$ are nonlinear spring and damper terms which are given as

$$e_{C_1} = a_1(x_1 - x_2) + b_1(x_1 - x_2)^3, \tag{42}$$
$$e_{C_2} = a_2(x_2 - x_3) + b_2(x_2 - x_3)^3, \tag{43}$$
$$e_{C_3} = a_3(x_3 - x_4) + b_3(x_3 - x_4)^3, \tag{44}$$
$$e_{R_1} = c_1(\dot{x}_1 - \dot{x}_2)^2 \ \mathrm{sign}(\dot{x}_1 - \dot{x}_2), \tag{45}$$
$$e_{R_2} = c_2(\dot{x}_2 - \dot{x}_3)^2 \ \mathrm{sign}(\dot{x}_2 - \dot{x}_3), \tag{46}$$
$$e_{R_3} = c_3(\dot{x}_3 - \dot{x}_4)^2 \ \mathrm{sign}(\dot{x}_3 - \dot{x}_4), \tag{47}$$

and $\omega_i,\ (i = 1, 2, 3)$ are unmodeled terms which are given as

$$\omega_1 = d_1(x_1 - x_2)^5, \tag{48}$$
$$\omega_2 = d_2(x_2 - x_3)^5, \tag{49}$$
$$\omega_3 = d_3(x_3 - x_4)^5, \tag{50}$$

where $a_1, a_2, \cdots, d_3$ are constant parameters and they are all 1 in this simulation. The original state vector for the nonlinear system is defined as

$$x = [x_1, \dot{x}_1, x_2, \dot{x}_2, x_3, \dot{x}_3, x_4, \dot{x}_4]^T. \tag{51}$$

5.2 Linearization Method

To compare the proposed method with other linearization methods, a linear model was calculated using the same data. Observation data to determine the system matrices are generated by a grid point sampling so that x has its value in $[-0.5, 0.5]^8$ and u has its value in [-1,1].

1) Koopman linearization
For comparison with DFL, the size is the same as that of the proposed method and this method does not use a model of the system. The observation function is the original state x and 16 thin plate spline radial basis functions of $g(x)$. It is defined as $\psi(x) = \|x - x_0\|^2 \log(\|x - x_0\|)$, the center positions of the basis functions x_0 are generated randomly in $[-0.5, 0.5]^8$. The augmented vector

is $x^* = [x^T, g(x)^T]^T$, and the dimension of the augmented linear system becomes 24, the same as the proposed method.

2) DFL
The auxiliary variables η are e_R, e_C from eq. (42) to eq. (47). The unmodeled terms ω_i are not ued and not included in the linear system. The linear system with η is given as

$$\dot{x} = A_x x + A_\eta \eta + B_x u, \tag{52}$$

where $\eta = [e_{C_1}, e_{C_2}, e_{C_3}, e_{R_1}, e_{R_2}, e_{R_3}]^T$, and system matrices are given as

$$A_x = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \tag{53}$$

$$A_\eta = \begin{bmatrix} 0&0&0&0&0&0\\ -1&0&0&-1&0&0\\ 0&0&0&0&0&0\\ 1&-1&0&1&-1&0\\ 0&0&0&0&0&0\\ 0&1&-1&0&1&-1\\ 0&0&0&0&0&0\\ 0&0&1&0&0&1 \end{bmatrix}, \tag{54}$$

$$B_x = \begin{bmatrix} 0&1&0&0&0&0&0&0 \end{bmatrix}^T. \tag{55}$$

The dynamics of η is determined as

$$\dot{\eta} = H_x x + H_\eta \eta + H_u u, \tag{56}$$

and the system parameters are determined using $\dot{\eta}$ calculated from the same data used in Koopman linearization.

3) Proposed method
The auxiliary variables η are the same as DFL. The additional auxiliary variable γ are 10 thin plate spline radial basis functions. The system matrices from eq. (35) to eq. (37) can be estimated based on the same observation data.

5.3 Numerical calculation

We compared the error between the actual nonlinear model and the discretized models. The discretization state for $N = 10$ steps was taken. The initial state x_0 has its value in $[-0.5, 0.5]^8$ and input u has its value in $[-1, 1]$. We compared the discretized linearization model obtained by the three linearization methods with the discretized nonlinear system by the Euler method with and without the unmodeled terms. The root mean square (RMS) error with the actual model is shown in Fig. 3. By incorporating unmodeled elements, it is confirmed that the proposed method is better than discretization of nonlinear systems without the unmodeled terms.

5.4 Simulation result

Control results by MPC are compared where the control interval is $dt = 0.01$[sec] and the prediction step is $N = 10$. The initial position is randomly generated in $[0, 0.5]^4$ and the velocities are all set 0. The desired state is $x_d = [0, 0, 0, 0, 0, 0, 0, 0]^T$. In this simulation with no constraints is considered. The criterion function is a quadratic form of the original state and input as follows.

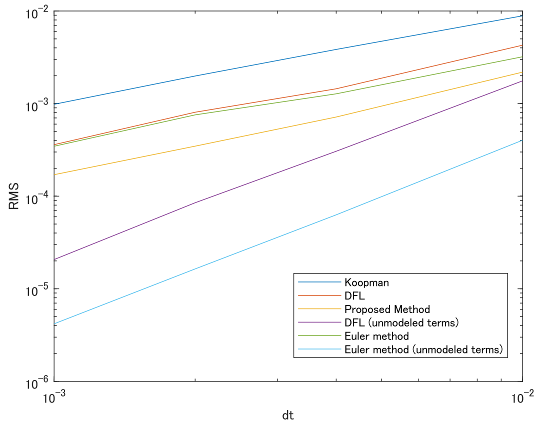

Fig. 3. Discretization error

Table 1. Mse and calculation time

	MSE	Calculation time[sec]
Koopman	0.0123	0.00106
DFL	0.0145	0.00095
Proposed Method	0.0095	0.00101
NP	0.0141	0.06978

$$\varphi(x) = (x - x_d)^T S_f (x - x_d), \tag{57}$$
$$L(x, u) = (x - x_d)^T Q (x - x_d) + u^T R u \tag{58}$$

Parameters in the criterion function are set as $S_f = 2000\ I_8, Q = 1000\ I_8, R = 1$. To compare MPC performances, control input is determined by the following methods:

- Quadratic optimization using Koopman linearization
- Quadratic optimization using DFL
- Quadratic optimization using Proposed method
- Nonlinear optimizaton (from eq. (19) to (22))

In Fig. 4, responses for the control input determined by each method are shown and the Mean Squared Error (MSE) of the deviation between actual state and the desired state, and the average computational times are shown on Table 1. From the table, it can be seen that the MSE of the three linearization methods are similar to that of the nonlinear optimization but the computation times are very small, and the MSE of the proposed method is the smallest among the linearization methods.

6. CONCLUSIONS

In this paper, we propose a method to robustify the Dual Faceted Linearization (DFL) for uncertainties of the plants. By giving additional auxiliary variables to DFL, unmodeled terms can be incorporated into the linearization model from the observed data. Numerical simulations confirmed that the proposed method outperforms other linearization methods.

As future work, we will investigate the basis functions and sizes of additional auxiliary variables.

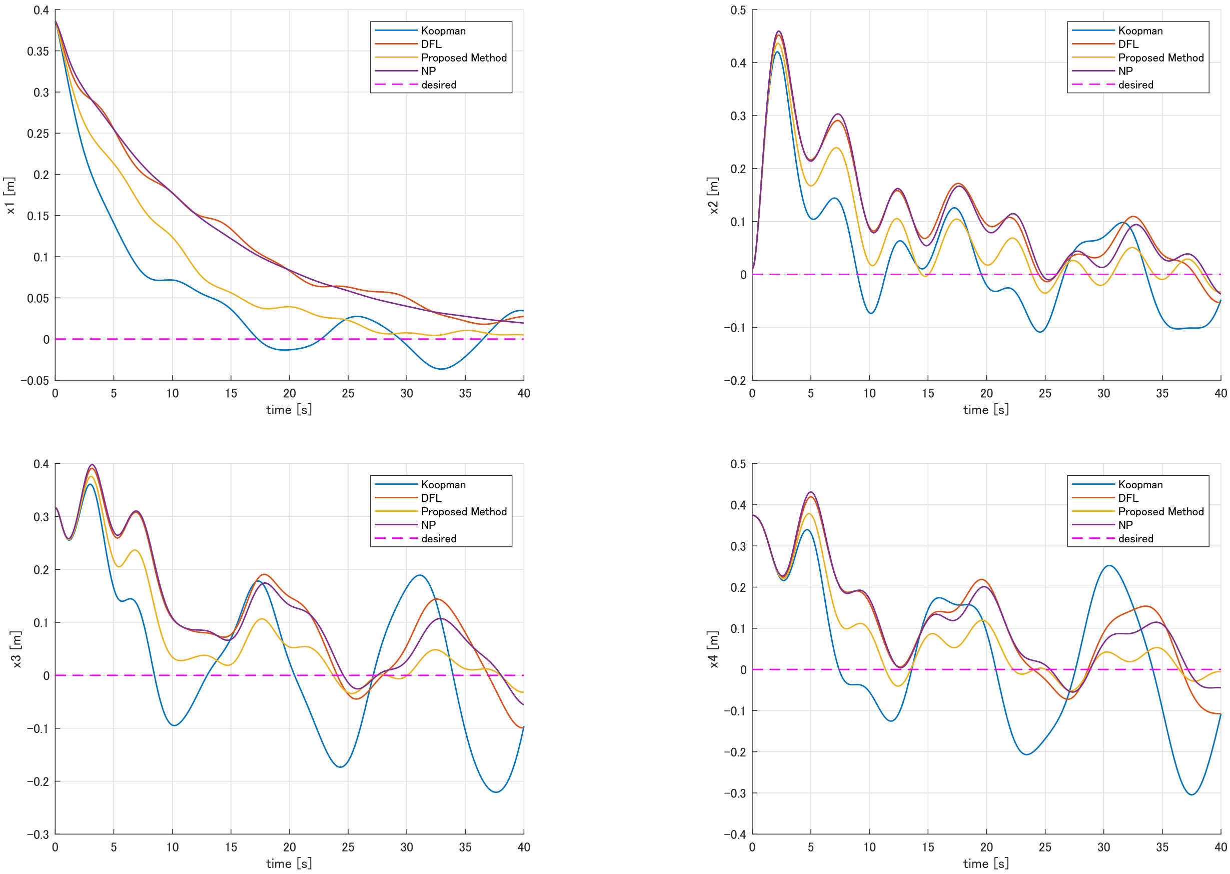

Fig. 4. Simulation result

REFERENCES

B.O.Koopman (1931). Hamiltonian Systems and Transformation in Hilbert Space. Proc. Natl. Acad. Sci. USA, Vol. 17, No. 5, pp. 315318.

H. Harry Asada, Filippos E. Sotiropoulos (2019). Dual Faceted Linearization of Nonlinear Dynamical Systems Based on Physical Modeling Theory. ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 141, 021002.

W.-H. Steeb and F. Wilhelm (1980). Non-Linear Autonomous Systems of Differential Equations and Carleman Linearization Procedure. Journal of Mathematical Analysis and Applications, Vol. 77, Issue 2, pp. 601-611.

Bethany Lusch, J. Nathan Kutz and Steven L. Brunton (2018). Deep learning for universal linear embeddings of nonlinear dynamics. NATURE COMMUNICATIONS, 9, Article number 4950.

Milan Korda and Igor Mezić (2018). Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control. Automatica, Vol. 93, pp. 149-160.

Amit Surana (2016). Koopman Operator Based Observer Synthesis for Control-Affine Nonlinear Systems. IEEE Conference on Decision and Control (CDC), Las Vegas, USA.

Hassan Arbabi, Milan Korda and Igor Mezić (2018). A data-driven Koopman model predictive control framework for nonlinear partial differential equations. IEEE Conference on Decision and Control (CDC), Miami Beach, FL, USA.

Hiroyuki Oyama, Masaki Yamakita and H. Harry Asada (2016). Approximated Stochastic Model Predictive Control using Statistical Linearization of Nonlinear Dynamical System in Latent Space. IEEE Conference on Decision and Control (CDC), Las Vegas, USA.

Yusuke Igarashi, Masaki Yamakita, Jerry Ng, and H. Harry Asada (2020). MPC Performances for Nonlinear Systems Using Several Linearization Models. American Control Conference (ACC), Denver, USA.

Matthew O. Williams, Ioannis G. Kevrekidis and Clarence W. Rowley (2015). A DataDriven Approximation of the Koopman Operator: Extending Dynamic Mode Decomposition. Journal of Nonlinear Science, Vol. 25, pp. 13071346.

Enoch Yeung, Soumya Kundu, and Nathan Hodas (2019). Learning deep neural network representations for koopman operators of nonlinear dynamical systems, American Control Conference (ACC), Philadelphia, PA, USA.

Naoya Takeishi, Yoshinobu Kawahara, and Takehisa Yairi (2017). Learning Koopman Invariant Subspaces for Dynamic Mode Decomposition. In Advances in Neural Information Processing Systems (NIPS 2017), Long Beach, CA, USA.

Toshiyuki Ohtsuka (2004). A continuation/GMRES method for fast computation of nonlinear receding horizon control. Automatica, Vol. 40, No. 4, pp. 563-574.