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An LMI Approach for Structured $\mathcal{H}_\infty$ State Feedback Control ⋆

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Abstract: In this paper we consider the problem of designing optimal $\mathcal{H}_\infty$ static state feedback control in the presence of structural constraints on the feedback gain. This problem arises in many applications, such as Network Decentralized Control and Overlapping Control, where the controller is constrained to have a specific nonzero patterns. Building upon previous results on S-variable approach for LMI-based robust control, we derive a novel solution to the design of $\mathcal{H}_\infty$ state feedback controllers when the controller gain is constrained to belong to a given linear space. Through numerical examples we demonstrate the simplicity of the method and performance of the optimal control law.

Keywords: Distributed/decentralized control, Linear Matrix inequalities, Linear systems, Structured control design, State feedback optimal control methods

1. INTRODUCTION

Coordination and control of large-scale systems, composed by interconnected independent subsystems, is a relevant problem in many applications, such as transportation networks (Ataşlar and İftar, 1998), flow networks (Bauso et al., 2013; Bauso et al., 2010), communication networks (Ephremides and Verdu, 1989; Moreno Banos and Papageorgiou, 1995), and smart-grids (Ayar et al., 2017), to mention just a few. Dealing with these large, complex, and distributed systems poses new challenges. In fact, the implementation of centralized controllers is in general not a viable approach for this kind of systems, and controllers must be designed in a decentralized/distributed way in order to be robust and scalable (Ikeda et al., 1984). Using only information about a subset of components to produce control laws imposes several restrictions to the structure of the overall control schemes, and the complexity of the controller depends both on the number of the states and on the particular architecture (Ikeda et al., 1984). In (Delvenne and Langbort, 2006) the performance degradation using only local information is quantified and compared with the global optimum obtained when the global information of the system is available in synthesis. In (Tanaka and Langbort, 2011) distributed and scalable control methods are proposed for positive systems but the optimality is no longer guaranteed.

The quadratic invariance principle (Lessard and Lall, 2016) provides a powerful tool to identify control problem with structural constraints that are convex. However, also in this case, the problem of finding the optimal distributed control can often be intractable.

Besides decentralized/distributed control where the controller is required to have a block-diagonal structure, more complicated constraints arise in sparse optimal control (Polyak et al., 2014), where the constraints are in the form of sparsity requirements for the feedback matrix (Lin et al., 2011) and the optimization problem becomes nonconvex. A large portion of literature proposes to solve directly the nonconvex problem. For instance, in (Lin et al., 2011) the authors propose to tackle the optimization via augmented Lagrangian and alternating direction method of multipliers (ADMM), or sequential convex-programming (Fardad and Jovanovic, 2014). Although these methods are very flexible and general, the convergence to the optimal solution can not be guaranteed.

In this paper, we study structured $\mathcal{H}_\infty$ optimal control design, namely we aim at designing a state feedback obeying to specified *structural constraints*, so that the system is stabilized while minimizing the gain of the disturbance-to-output channel. This problem has attracted a large interest in the past years, and several results are available in the literature. However, it should be remarked that most of the approaches to this topic are restricted to *very specific* classes of systems and problems. These assumptions allow one to derive closed-form solution of the control law, but substantially restrict their practical applicability. For instance, closed-form expressions have been derived be for linear time-invariant discrete time systems with symmetric and Schur state matrix (Lidström et al., 2017; Rantzer et al., 2017), or for internally-positive systems, for which the closed-loop state space matrix needs is Met-

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zler (Tanaka and Langbort, 2011). Other approaches, as already discussed, directly tackle the ensuing nonconvex problem (Lian et al., 2018), or resort to dynamic feedback, see e.g. (Langbort et al., 2004) and references therein.

Specifically, in this work we assume the feedback matrix is constrained to belong to a given linear space. We remark that the formulation of the structural constraints in terms of linear space is very general, and, as extensively discussed in our previous work (Ferrante et al., 2020), it captures important problems such as network decentralized/distributed control, overlapping control, and sparse control as particular cases. Our methodology is mainly based on a linear matrix inequalities (LMI) formulation of the problem. In this setting, the design of stabilizing structured static feedback controllers is recast into the solutions to some linear matrix inequalities with structured decision variables. Those matrix inequalities naturally stem from Lyapunov/dissipation inequalities. One of the challenging features of this problem is that stability conditions generally take the form of bilinear matrix inequalities in the controller gain and the matrix defining the underlying Lyapunov/storage functions. Hence, those cannot be directly employed for controller design. An approach to overcome this problem consists of performing linearizing change of variables preserving the structure of the resulting controller gain. This is done, e.g., in (Blanchini et al., 2015), by considering a specific structure of the underlying Lyapunov function. However, imposting a predetermined structure to Lyapunov functions is in general a major source of conservatism. In our recent paper, (Ferrante et al., 2020), we show how by relying on an approach that is reminiscent of the S-variable approach (Ebihara et al., 2015), one can obtain less conservative conditions for structured feedback control design. The key idea consists of resorting to structured "dilated" stability conditions (Pipeleers et al., 2009) to remove the need of imposing a specific structure to the underlying Lyapunov function. The idea of using dilated conditions to release contraints on the Lyapunov matrix when the feedback gain has a certain structure was originally illustrated in (De Oliveira et al., 2002) in some specific examples for the case of discrete-time systems. The main novelty in (Ferrante et al., 2020) consists of providing a systematic and computationally affordable approach to design stabilizing controllers fulfilling general geometric constraints.

This paper extends the results in (Ferrante et al., 2020) to $\mathcal{H}_\infty$ static feedback control design. The methodology we propose relies on the solution to some linear matrix inequalities coupled to a line search on a scalar parameter and turns out to be less conservative than previous methods presented in literature. Through numerical examples we demonstrate the simplicity of the method and performance of the optimal control law.

1.1 Outline

We formally present the problem of structured $\mathcal{H}_\infty$ state feedback control in Section 2 and review some preliminary results that are useful for subsequent developments in Section 3. The main theoretical contribution of this paper is presented and discussed in Section 4. Then, the effectiveness of the proposed approach is shown in examples retrieved from literature on Decentralized Control and Overlapping Control (see Section 5). Finally, Section 6 collects some concluding remarks and discussions on future directions of research.

1.2 Notation

We denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. The symbols $\mathbb{R}^{n\times m}$ and $\mathbb{S}^n_p$ represent the set of $n\times m$ real matrices and the set of real $n\times n$ symmetric positive definite matrices, respectively. The symbol $\mathcal{R}^n$ stands for the set of $n\times n$ nonsingular real matrices. Given $A\in\mathbb{R}^{m\times n}$, we denote its transpose by A^T, and, when $n=m$, we define $\mathrm{He}(A)=A+A^\mathsf{T}$. The notation $\mathbf{I}$ denotes the identity matrix whose size is determined from context. For a symmetric matrix A, negative and positive definiteness are denoted, respectively, by $A\prec 0$ and $A\succ 0$. The symbol $\star$ stands for symmetric block in symmetric matrices. The symbol $\otimes$ is used for Kronecker product between matrices, respectively. Given a linear time-invariant continuous-time system with state x, input w, and output z of the form:

$$\Sigma:\begin{cases}\dot{x}=Ax+Ew\\ z=Cx+Dw.\end{cases}$$

we denote by $\|\Sigma\|_\infty$ the $\mathcal{H}_\infty$ norm of the transfer function of Σ from w to z.

2. PROBLEM STATEMENT

We consider a linear continuous-time plant of the form:

$$\begin{aligned}\dot{x}&=Ax+Bu+Ew\\ z&=Cx+Dw\end{aligned}\tag{1}$$

where $x\in\mathbb{R}^{n_x}$ is the plant state, $u\in\mathbb{R}^{n_u}$ is the control input, $w\in\mathbb{R}^{n_w}$ is an exogenous perturbation, and $z\in\mathbb{R}^{n_z}$ is a regulated output. We assume that matrices $A\in\mathbb{R}^{n_x\times n_x}, B\in\mathbb{R}^{n_x\times n_u}, C\in\mathbb{R}^{n_z\times n_x}$, $E\in\mathbb{R}^{n_x\times n_w}$, and $D\in\mathbb{R}^{n_z\times n_w}$ are constant and given. The problem we consider in this paper consists of designing a structured state-feedback control law $u=Kx$ for (1) ensuring closed-loop stability and an $\mathcal{H}_\infty$ performance from the input w to the output z. More precisely, let $\mathcal{S}\subset\mathbb{R}^{n_u\times n_x}$ be a linear space and

$$\Sigma:\begin{cases}\dot{x}=(A+BK)x+Ew\\ z=Cx+Dw.\end{cases}\tag{2}$$

be the closed-loop system. The problem we solve can be formalized as follows:

Problem 1. (Structured $\mathcal{H}_\infty$ State Feedback). Given $\gamma>0$, design $K\in\mathcal{S}$ such that the closed-loop system (2) is asymptotically stable and the following bound holds:

$$\|\Sigma\|_\infty\leq\gamma.$$

◇

3. PRELIMINARY RESULTS

For a given controller gain, consider the closed-loop system (2). With the objective of streamlining the design of the controller, as in (Ebihara et al., 2015), we consider the dual of closed-loop system (2), which reads as follows

$$\Sigma_D:\begin{cases}\dot{\chi}=(A+BK)^\mathsf{T}\chi+C^\mathsf{T}v\\ f=E^\mathsf{T}\chi+D^\mathsf{T}v.\end{cases}\tag{3}$$

that has state $\chi \in \mathbb{R}^{n_x}$, input $v \in \mathbb{R}^{n_z}$, and output $f \in \mathbb{R}^{n_w}$. In the result given next, we recall that $\|\Sigma\|_\infty = \|\Sigma_D\|_\infty$.

Lemma 1. (Dual Representation). Let Σ and Σ_D be defined respectively in (2) and (3). Then, the following identity holds:

$$\|\Sigma\|_\infty = \|\Sigma_D\|_\infty.$$

Proof. Let G_Σ and G_{Σ_D} be the transfer functions of Σ and Σ_D, respectively. Then, the result follows directly from the following identity:

$$G_{\Sigma_D}(s) = G_\Sigma(s)^\mathsf{T} \quad \forall s \in \mathbb{C}$$

∎

It is well-know from $\mathcal{H}_\infty$ theory (see, e.g., (Boyd et al., 1994, Chapter 2, page 26)) that $\|\Sigma_D\|_\infty < \gamma$ if and only if there exists $P \in \mathbb{S}_p^{n_p}$ such that the following matrix inequality holds:

$$\begin{bmatrix} \mathrm{He}((A+BK)P) & PC^\mathsf{T} & E \\ \star & -\gamma\mathbf{I} & D \\ \star & \star & -\gamma\mathbf{I} \end{bmatrix} \prec 0. \tag{4}$$

Building upon that, we state a "dilated" version of (4), which will be used to formulate the main result of this paper. This condition is based on the general results in (Pipeleers et al., 2009; Ebihara et al., 2015).

Theorem 1. Let $P \in \mathbb{S}_p^{n_p}$ and $K \in \mathbb{R}^{n_u \times n_p}$ be given. The following items are equivalent.

(i) Inequality (4) holds;
(ii) There exists $X \in \mathbb{R}^{n_p \times n_p}$ and $\alpha > 0$ such that

$$\mathrm{He}\left(\begin{bmatrix} (A+BK)X & \alpha(A+BK)X+P & 0 & E \\ -X & -\alpha X & 0 & 0 \\ CX & \alpha CX & -\frac{\gamma}{2}\mathbf{I} & D \\ 0 & 0 & 0 & -\frac{\gamma}{2}\mathbf{I} \end{bmatrix}\right) \prec 0 \tag{5}$$

Proof. Following the results in (Pipeleers et al., 2009, Section 5, Extension IV), it follows that (4) holds if and only if there exists X and $\alpha > 0$ such that:

$$\mathrm{He}\left(\begin{bmatrix} (A+BK)X & \alpha(A+BK)X & 0 \\ -X & -\alpha X & 0 \\ CX & \alpha CX & 0 \end{bmatrix}\right) + \begin{bmatrix} \gamma^{-1}EE^\mathsf{T} & P & \gamma^{-1}ED^\mathsf{T} \\ \star & 0 & 0 \\ \star & \star & \gamma^{-1}DD^\mathsf{T} - \gamma\mathbf{I} \end{bmatrix} \prec 0 \tag{6}$$

(7)

Thus, using the fact that $\gamma > 0$, via Schur's complement, it follows that (6) is equivalent to (5). This concludes the proof. ∎

Building on the above result, the following "design oriented" equivalent condition can be obtained. The proof of the result follows directly from Theorem 1, hence it is omitted.

Corollary 1. Assume that there exist a nonsingular matrix $X \in \mathbb{R}^{n_p \times n_p}$, $P \in \mathbb{S}_p^{n_p}$, $R \in \mathbb{R}^{n_u \times n_p}$, and $\alpha > 0$ such that

$$\mathrm{He}\left(\begin{bmatrix} AX+BR & \alpha(AX+BR)+P & 0 & E \\ -X & -\alpha X & 0 & 0 \\ CX & \alpha CX & -\frac{\gamma}{2}\mathbf{I} & D \\ 0 & 0 & 0 & -\frac{\gamma}{2}\mathbf{I} \end{bmatrix}\right) \prec 0 \tag{8}$$

$$RX^{-1} \in \mathcal{S}. \tag{9}$$

Then, $K = RX^{-1}$ solves Problem 1.

4. MAIN RESULT

We are now in the position of stating the main result of this paper.

Theorem 2. Let $\{S_1, S_2, \ldots, S_k\}$ be a basis of $\mathcal{S}$ and

$$L := [\ S_1 \mid S_2 \mid \ \ldots \ \mid S_k\ ]$$

Define the following *structure set*

$$\Upsilon := \{Q \in \mathbb{R}^{n_p \times n_p} : \exists \Lambda \in \mathcal{R}^k \text{ s.t. } L(\mathbf{I}_k \otimes Q) = L(\Lambda \otimes \mathbf{I}_{n_p})\}.$$

Assume that there exist $P \in \mathbb{S}_p^{n_p}$, $R \in \mathrm{span}\{S_1, S_2, \ldots, S_k\}$, $\alpha > 0$, and $X \in \mathbb{R}^{n_p \times n_p}$ such that:

$$X \in \Upsilon, \tag{10a}$$

$$\mathrm{He}\left(\begin{bmatrix} AX+BR & \alpha(AX+BR)+P & 0 & E \\ -X & -\alpha X & 0 & 0 \\ CX & \alpha CX & -\frac{\gamma}{2}\mathbf{I} & D \\ 0 & 0 & 0 & -\frac{\gamma}{2}\mathbf{I} \end{bmatrix}\right) \prec 0. \tag{10b}$$

Then, X is nonsingular and $K = RX^{-1}$ solves Problem 1.

Sketch of the proof. Nonsingularity of X follows directly from (10b). Using inequality (10b) and applying Corollary 1, we deduce that $K = RX^{-1}$ is such that $\|\Sigma_D\| < \gamma$. The proof can be completed by observing that, as shown in (Ferrante et al., 2020, proof of Theorem 2), $R \in \mathrm{span}\{S_1, S_2, \ldots, S_k\}$ and (10a) imply that $RX^{-1} \in \mathcal{S}$. ∎

5. APPLICATIONS AND EXAMPLES

In this section, we compare the performance show the effectiveness of the proposed approach in two examples. The first example is retrieved from the literature of Network Decentralized Control. The second example pertains to the so-called Overlapping Control problem.

5.1 Network Decentralized Control

We analyze the setting proposed in (Blanchini et al., 2015), in which a collection of N decoupled linear systems is interconnected through a set of local controller nodes, each having access only to local state information. We revisit the example in (Blanchini et al., 2015) where the networked system is composed of $N = 5$ subsystems and the overall dynamics can be written as in (1) with

$$A = \mathrm{diag}(A_1, A_2, \ldots, A_5),$$

$$B = \begin{bmatrix} B_u & -B_d & 0 & 0 & 0 & 0 \\ 0 & B_u & -B_d & 0 & 0 & -B_d \\ 0 & 0 & B_d & -B_u & 0 & 0 \\ 0 & 0 & 0 & B_d & B_u & 0 \\ 0 & 0 & 0 & 0 & B_u & B_d \end{bmatrix}, E = \mathbf{I}$$

where for all $i = 1, 2, \dots, 5$

$$A_i = \begin{bmatrix} -\xi_i & \beta_i & 0 \\ \xi_i & -\beta_i & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_d = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, B_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In particular, for all $i \in \{1, 2, 4\}$ we select $\beta_i = 0$, $\xi_1 = 15, \xi_2 = 20, \xi_3 = 16, \xi_4 = 16.7, \xi_5 = 14, \beta_3 = 12$, and $\beta_5 = 22$. In addition, we assume $C = \mathbf{I}$ and $D = 0$.

As illustrated in (Blanchini et al., 2015; Ferrante et al., 2020), in this setting, the proposed control strategy can be performed by considering a standard feedback controller with a block-structured feedback gain. In particular, the feedback gain needs to have the same zero-block structure of B^T. It can be shown [1] that the space of matrices with the same zero-blocks of B^T is a linear space of $\mathbb{R}^{6\times 15}$ with dimension equal to 33. We use Theorem 2 to design a controller fulfilling the prescribed structural constraints and ensuring a guaranteed level of $\mathcal{H}_\infty$ performance. Let Υ be defined as in Theorem 2. Then, simple manipulations in this case show that any matrix $X \in \Upsilon$ has the following structure:

$$X = \operatorname{diag}(X_1, X_2, X_3, X_4, X_5) \tag{12}$$

where $X_i \in \mathbb{R}^{3\times 3}$, $i = 1, \dots, 5$, are nonsingular arbitrary matrices.

Therefore, Theorem 2 can be applied with X structured as in (12) and R with the same zero-blocks of B^T, that is:

$$R = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & r_{2,5} & r_{2,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{3,4} & r_{3,5} & r_{3,6} & r_{3,7} & r_{3,8} & r_{3,9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_{4,7} & r_{4,8} & r_{4,9} & r_{4,10} & r_{4,11} & r_{4,12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{5,10} & r_{5,11} & r_{5,12} & r_{5,13} & r_{5,14} & r_{5,15} \\ 0 & 0 & 0 & r_{6,4} & r_{6,5} & r_{6,6} & 0 & 0 & 0 & 0 & 0 & 0 & r_{6,13} & r_{6,14} & r_{6,15} \end{bmatrix}$$

We solve Problem 1 for a minimal value of [2] γ:

$$\inf_{P,X,R,\alpha,\gamma} \gamma$$
$$\text{subject to}$$
$$\text{(10a)}, \text{(10b)}, P \succ 0, \alpha > 0.$$

Once the structure of the set Υ is made explicit, the above optimization problem can be solved via standard SDP solvers, with the caveat of performing a line search on the positive scalar α. Numerical experiments show that the selection of α is not critical to ensure the feasibility of the problem, but it has an impact on the resulting solution. To show the influence of the variable α on the design of the controller, in Fig. 1 we report the norm of the gain K against the value of α. With the objective of selecting a good tradeoff between control effort and disturbance rejection, in this example we select $\alpha = 0.1554$, which leads to $\gamma = 1.7887$, and K and P as in (11). From (11a) one can understand the main novelty of the proposed approach: we observe that K has indeed the desired zero-block structure, while the matrix P is completely free and does not present any zero-block structure; see (11b). This degree of freedom in the choice of P renders the proposed design much less conservative with respect to other approaches.

[1] Code available at `https://github.com/f-ferrante/IFAC2020_FerranteDabbeneRavazzi`

[2] To avoid the occurrence of overly large control gains, a specific lower bound on (10b) is considered in the solution to the optimization problem.

5.2 *Overlapping Control*

We consider the *control with overlapping information structure constraints* problem in (Zecevic and Siljak, 2010, Chapter 2) and we revisit (Zecevic and Siljak, 2010, Example 2.16). Let

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, D = 0, C = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathcal{S} = \left\{ K \in \mathbb{R}^{2\times 3} \colon K = \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix} \mid (a, b, c, d) \in \mathbb{R}^4 \right\}. \tag{13}$$

The setup analyzed in this example is schematically represented in Fig. 2. In particular, our example falls in the so-called Type II problem defined in (Zecevic and Siljak, 2010, Fig. 2.6), since the control inputs do not directly affect the "shared" state x_2, which however directly influenced by the disturbance w. This is recognized to be a critical situation, considerably more challenging than the Type I overlapping control, in which the shared state is directly influenced by the control inputs. Notice that as a performance output we take $y = x_2$, i.e., the shared state. Now we show how our approach hinging upon Theorem 2 can be successfully adopted in this case to solve Problem 1. It can be easily shown that the set $\mathcal{S}$ in (13) is a linear space of $\mathbb{R}^{2\times 3}$. In particular, as illustrated in (Ferrante et al., 2020), in this case the set Υ in Theorem 2 can be written as:

$$\Upsilon = \left\{ Q \in \mathcal{R}^3 : Q = \begin{bmatrix} q_{11} & q_{12} & 0 \\ 0 & q_{22} & 0 \\ 0 & q_{32} & q_{33} \end{bmatrix}, q_{ij} \in \mathbb{R} \right\}.$$

As such, Problem 1 can be solved by solving (10b) with

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ 0 & r_{22} & r_{23} \end{bmatrix}, \qquad X = \begin{bmatrix} x_{11} & x_{12} & 0 \\ 0 & x_{22} & 0 \\ 0 & x_{23} & x_{33} \end{bmatrix}$$

Also in this case, with the objective to minimize the effect of the disturbance w onto the performance output y, we recast Problem 1 as the solution to (13). In particular, solving in (13) in Matlab® using the YALMIP package (Lofberg, 2004) combined with the solver MOSEK (Andersen and Andersen, 2000), one gets:

$$P = \begin{bmatrix} 16.68 & -1.615 & 3.867 \\ -1.615 & 0.8624 & -3.231 \\ 3.867 & -3.231 & 22.48 \end{bmatrix}, \alpha = 0.09$$

$$K = \begin{bmatrix} -27.23 & -65.88 & 0 \\ 0 & -121.7 & -29.22 \end{bmatrix}, \gamma \approx 0.13724$$

To further emphasize the benefit of the proposed design strategy in achieving disturbance rejection despite the structural contraints on the control gain K, we compare the response of the closed-loop system for the optimal gain K here above and for the (nonoptimal) gain

$$K_s = \begin{bmatrix} -3.831 & -5.744 & 0 \\ 0 & -5.059 & -8.922 \end{bmatrix}$$

proposed in (Ferrante et al., 2020, Section IV.b) to achieve closed-loop asymptotic stability for the same system under the same overlapping control constraint. For the sake of comparison, in Fig. 3 we report the response of the plant output to an energy bounded disturbance and of the control energy for the two controllers, i.e., the "optimal controller" $u = Kx$ and the stabilizing control law $u =$

$$K = \begin{bmatrix} -47.49 & -56.26 & -77.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13.9 & 11.86 & 20.37 & -2.986 & -19.25 & -21.72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6.634 & 7.203 & 7.685 & -29.58 & -17.31 & -38.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 19.46 & 26.65 & 40.18 & -24.55 & -26.26 & -21.46 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.654 & -9.293 & -5.273 & 19.41 & -0.1235 & 75.85 \\ 0 & 0 & 0 & 4.089 & 5.598 & 8.68 & 0 & 0 & 0 & 0 & 0 & 0 & -36.25 & -40.41 & -302.8 \end{bmatrix} \quad (11a)$$

$$P = \begin{bmatrix} 20.62 & -5.958 & -0.3951 & 1.428 & 0.641 & -0.03923 & 1.128 & -2.918 & 0.05114 & 0.396 & -1.376 & 0.03363 & 0.2131 & -0.2668 & -0.02757 \\ -5.958 & 14.32 & -1.242 & -5.747 & -0.9599 & -0.03144 & -1.333 & 3.865 & -0.05901 & -0.5604 & 1.648 & -0.06324 & -0.1386 & 0.2075 & 0.03432 \\ -0.3951 & -1.242 & 0.8886 & 0.07157 & -0.06799 & 0.1175 & -0.04786 & 0.1937 & -0.01306 & 0.01138 & 0.0757 & 0.02661 & -0.01253 & -0.0001176 & 0.01002 \\ 1.428 & -5.747 & 0.07157 & 8.359 & -1.929 & -0.2793 & 0.7549 & -1.07 & -0.01014 & -0.0384 & -1.588 & 0.03661 & 0.05197 & -1.353 & -0.008679 \\ 0.641 & -0.9599 & -0.06799 & -1.929 & 14.27 & -1.117 & -2.434 & -3.82 & -0.01675 & 0.4633 & 1.621 & -0.06153 & 0.1542 & -1.96 & 0.01498 \\ -0.03923 & -0.03144 & 0.1175 & -0.2793 & -1.117 & 1.068 & 0.08113 & 0.03432 & -0.02792 & 0.1705 & -0.01836 & 0.0289 & 0.06043 & -0.02749 & 0.01761 \\ 1.128 & -1.333 & -0.04786 & 0.7549 & -2.434 & 0.08113 & 10.4 & -0.6518 & -0.2026 & -4.102 & -0.3342 & 0.05546 & 0.9172 & -1.083 & -0.1001 \\ -2.918 & 3.865 & 0.1937 & -1.07 & -3.82 & 0.03432 & -0.6518 & 8.064 & -0.7811 & 1.496 & -4.171 & -0.01638 & -2.236 & 1.177 & 0.01466 \\ 0.05114 & -0.05901 & -0.01306 & -0.01014 & -0.01675 & -0.02792 & -0.2026 & -0.7811 & 0.5749 & 0.04017 & 0.06079 & -0.02013 & 0.03448 & -0.02585 & 0.003337 \\ 0.396 & -0.5604 & 0.01138 & -0.0384 & 0.4633 & 0.1705 & -4.102 & 1.496 & 0.04017 & 9.063 & -5.656 & 0.1422 & -2.302 & 2.852 & -0.1741 \\ -1.376 & 1.648 & 0.0757 & -1.588 & 1.621 & -0.01836 & -0.3342 & -4.171 & 0.06079 & -5.656 & 12.18 & -0.8017 & -0.2452 & -1.943 & 0.05757 \\ 0.03363 & -0.06324 & 0.02661 & 0.03661 & -0.06153 & 0.0289 & 0.05546 & -0.01638 & -0.02013 & 0.1422 & -0.8017 & 0.7858 & -0.01265 & 0.009939 & 0.00158 \\ 0.2131 & -0.1386 & -0.01253 & 0.05197 & 0.1542 & 0.06043 & 0.9172 & -2.236 & 0.03448 & -2.302 & -0.2452 & -0.01265 & 13.86 & -3.468 & -0.4921 \\ -0.2668 & 0.2075 & -0.0001176 & -1.353 & -1.96 & -0.02749 & -1.083 & 1.177 & -0.02585 & 2.852 & -1.943 & 0.009939 & -3.468 & 10.3 & -0.6915 \\ -0.02757 & 0.03432 & 0.01002 & -0.008679 & 0.01498 & 0.01761 & -0.1001 & 0.01466 & 0.003337 & -0.1741 & 0.05757 & 0.00158 & -0.4921 & -0.6915 & 0.1659 \end{bmatrix} \quad (11b)$$

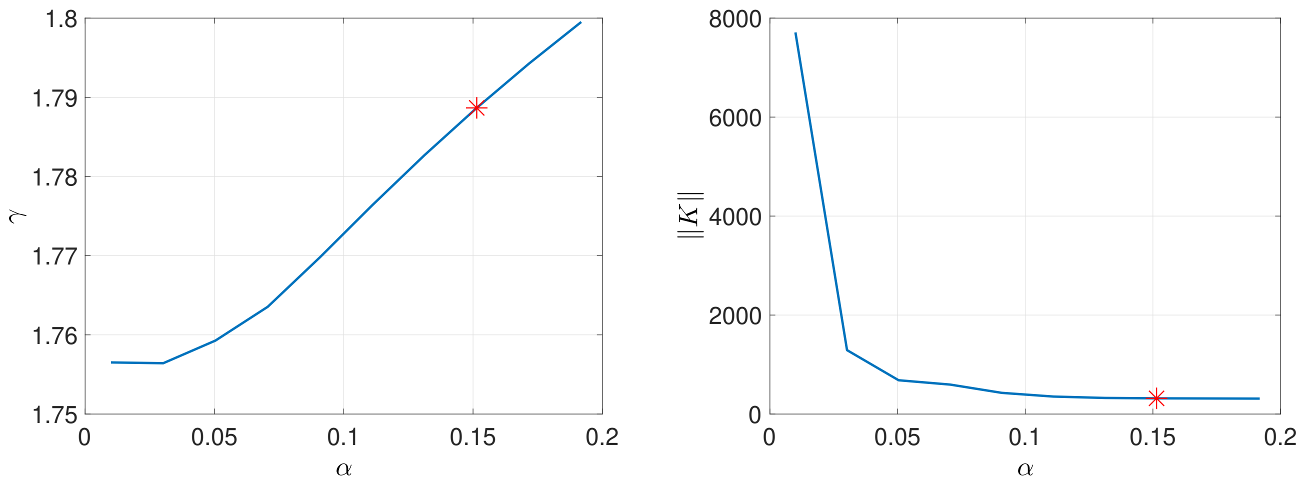

Fig. 1. Variation of γ (left) and $\|K\|$ (right) versus α. The red asterisk denotes the solution selected in this example.

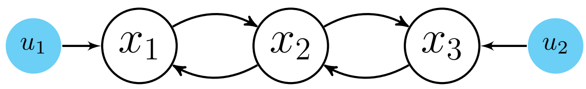

Fig. 2. Topology of the overlapping control problem in Example 5.2.

$K_s x$. Simulations clearly show the benefit of the proposed design strategy. In particular, it is interesting to notice that, despite the constrained structure of the control gain, the proposed optimal design leads to improved disturbance rejection and this with reduced control effort with respect to the nonoptimal feedback law $u = K_s x$.

6. CONCLUSIONS

This paper proposes a very general approach for the design of state feedback $\mathcal{H}_\infty$ controllers obeying to structural constraints. The constraints are expressed in terms of LMIs depending on a design matrix R which is constrained to belong to a pre-defined structure, while the Lyapunov matrix P is not forced to obey any structural constraint but it is instead left completely free. This approach is suitable for a large range of possible applications, as also shown by our numerical simulations. We are now working into extending the proposed structured design approach for cases even more realistic, involving for instance nonlinear terms and/or different costs, as $\mathcal{H}_2$ or guaranteed-cost design.

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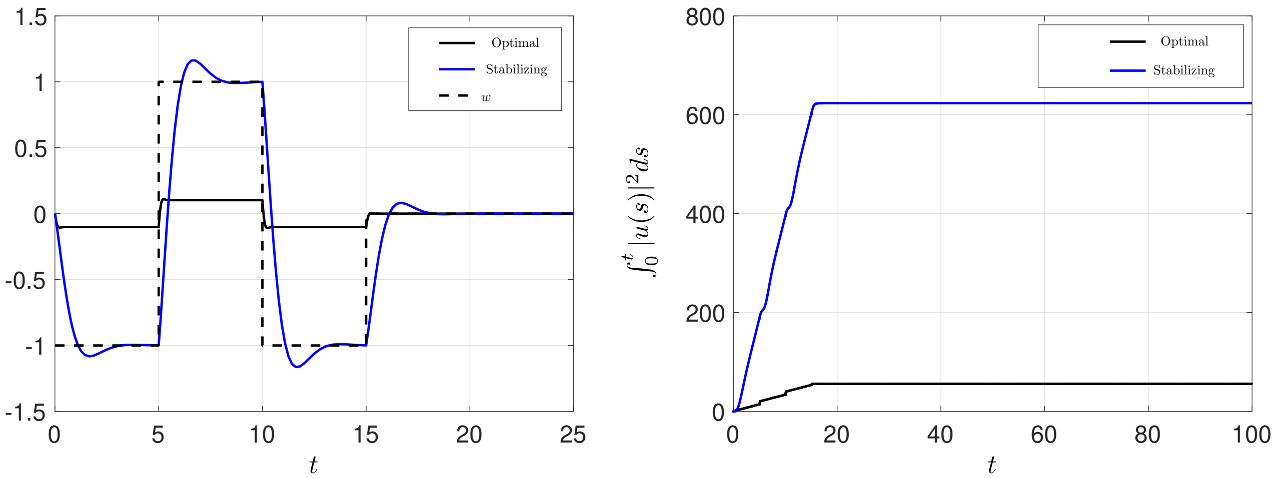

Fig. 3. Plant response (left) and associated control energy (right) for optimal and nonoptimal feedback.

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