Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Distributed Observer Design for Achieving Asymptotical Omniscience over Time-variant Disconnected Networks ⋆

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Abstract: This paper investigates the distributed observer design for linear system under time-variant disconnected communication network. By constructing basic eigenvectors of 0-eigensubspace of disconnected Laplacian Matrix and using LMIs method, we prove the distributed observer cannot achieve omniscience asymptotically under switching topology without constraining the system matrix or alternative topologies set. To deal with this problem, this paper investigates three kinds of constraints and the system matrix only needs to satisfy any one of them. Then a group of sufficient conditions corresponding to the asymptotic omniscience of distributed observer under switching topology are proved by Lyapunov analysis. Finally, a numerical simulation shows the validity of our method.

Keywords: Distributed observer, Continuous time system estimation, Consensus, Sensor networks, Switching topology

1. INTRODUCTION

The distributed observer problem has attracted a lot of attention to recent years. The dynamical system is monitored by a network of local observers. Each local observer, or named node, is required to estimate the whole states of system using its own (limited) measurements and via information exchange in network.

The main difficult of the distributed observer problem come from the limitation that no single observer can estimate the whole states. Therefore, it is necessary to introduce the coupling term appearing in the multi-agent problem into the classical observer dynamics, such as distributed Luenberger observer Zhu et al. (2014); Kim et al. (2016), distributed augmented state observer Park and Martins (2017), and distributed high-gain observer Xu and Wang (2020). Han et al. (2019) and Han et al. (2018) extend the distributed observer from the perspective of directed graph and minimum order observer respectively. Silm et al. (2019) studies the conditions of distributed finite-time observer, and Xu and Wang (2019) gives a group of sufficient conditions for distributed nonlinear observer.

However, all of these papers are based on fixed communication topology, and assume that the communication network is connected (undirected graph) or strongly connected (directed graph). According to author's knowledge, there is no systematic research on distributed observer designed under switching topology.

Despite all this, the research on the consensus of multi-agent systems under switching topology can be traced back to several decades. In the early years, the research on switching topology mainly focused on the first-order multi-agent system, for which the corresponding compact system matrix equals to negative Laplacian matrix. This property enables a non increasing Lyapunov function along each subsystem, so the consensus of multi-agent system can be guaranteed under switching topology if all the alternative topologies are balanced digraph (Reza and Richard (2004); Ren and Beard (2005)). The problem becomes more challenging when the agent dynamics becomes to second-order integrators or more complex system dynamics, because unstable poles will appear in the system matrices. Some results can be founded in Hong et al. (2007); Lin and

⋆ This work is supported by National Natural Science Foundation of China (No.61533013, 61633019), Key projects from Ministry of Science and Technology (No.2017ZX07207003, 2017ZX07207005), Shaanxi Provincial Key Project (2018ZDXM-GY-168), Shanghai Project (17DZ1202704) and MIIT Intelligent Manufacturing Special Project List of Y2017(No.54).

Jia (2010) but they add at least one connected graph in alternative topologies set.

During the recent years, few scholars have chosen to include connected graphs (undirected graph) or strongly connected graphs (directed graph) in the alternative topologies set of devices. They are more inclined to assume that all the topologies in the set can satisfy the average connectivity (Münz et al. (2011)), uniformly quasi-strong connectivity (Monshizadeh et al. (2016); Reza and Richard (2004)) or joint connectivity, see (Shi and Hong (2009); Yang et al. (2015); Casadei et al. (2017)). Under these conditions, if the subsystems themselves are stable or marginally stable, the multi-agent system can achieve consensus (Wei and Cheng (2010)). But for the case where the subsystems themselves contains unstable poles, some special technologies are needed to achieve consensus, such as input-state stable (Khan et al. (2019); Zhu et al. (2016)) and hierarchical control framework (Liu et al. (2017)). However, the contribution of these kind of researches mainly based on the designing and consensus of the reference models. But in the end, the reference models are still designed to be marginally stable. Similar cases can also be seen in Su and Jie (2012); Peng and Jia (2010).

In this paper, we will study the asymptotic omniscience (equivalent to the consensus of error system) of distributed linear observers under switched disconnected topologies by referring to the existing literature on switched systems or multi-agent systems under switching topologies. The most challenging of this work mainly comes from the weighted matrix in coupling term, which is significantly different from the multi-agent system. This weight matrix is indispensable for the stability of error dynamic between distributed observer and real system, but it will prevent us from getting non increasing Lyapunov function as easily as in the multi-agent problem.

The contributions of this paper mainly include: (i) We prove by LMIs methods that the distributed observer with switching (disconnected) topologies fails to achieve omniscience asymptotically without adding constraints on system matrix or alternative topologies set. (ii) By focusing on adding constraints on the system matrix, three kinds of constraints are proposed and the system matrix is required to fulfilled at least one of them. (iii) A group of sufficient conditions are proved by Lyapunov analysis to guarantee the distributed observer designed on time-variant disconnected topology to achieve omniscience asymptotically under the proposed system matrix constraints.

This paper is organized as follows. Section 2 summarizes the notations used throughout the paper and formulates the problem. Section 3 analyzes the conditions and constraints for the existence of distributed observer under switching topology. Section 4 is used to prove the result of Section 3, and the sufficient conditions for the asymptotic omniscience of distributed observer under switching topology are proposed and proved in Section 5. Section 6 verifies our method with an example.

2. PROBLEM FORMULATION

Notations used throughout this paper are summarized here. Denote I_n and 0_n as n-dimensional identity matrix and zero matrix respectively. $col\{a_1, \cdots, a_n\}$ represents the stack of column $a_i, i = 1, \cdots, n$. $diag\{M_1, \cdots, M_n\}$ is defined as a block diagonal matrix. A symmetric sign $sym\{M\}$ represents $M + M^T$. The rank of matrix M is $rank\{M\}$. We denote $\lambda(M)$ as the characteristic polynomial of M and denote $\Lambda(M)$ as the spectral set of M. Denote an index set as $\mathcal{P} = \{1, 2, \cdots, \varpi\}$, and $\sigma : [0, \infty) \to \mathcal{P}$ is a switching signal whose value at time t belongs to $\mathcal{P}$. $card(\cdot)$ represents the cardinality of set. This paper denotes disconnected and undirected communication network at time t as $\mathcal{G}_{\sigma(t)} = (\mathcal{V}_{\sigma(t)}, \mathcal{E}_{\sigma(t)}, \mathcal{A}_{\sigma(t)})$, in which $\mathcal{V}_{\sigma(t)}$, $\mathcal{E}_{\sigma(t)}$, and $\mathcal{A}_{\sigma(t)} = [a_{ij}^{\sigma(t)}] \in \mathbb{R}^{N\times N}$ are the set of nodes, set of edges, and the adjacent matrix respectively. $a_{ij}^{\sigma(t)} = 1$ if there is an edge between node i and node j at time t, otherwise, $a_{ij}^{\sigma(t)} = 0$. Laplacian matrix of $\mathcal{G}_{\sigma(t)}$ is calculated by $\mathcal{L}_{\sigma(t)} = \mathcal{D}_{\sigma(t)} - \mathcal{A}_{\sigma(t)}$, where the ith diagonal entry of diagonal matrix $\mathcal{D}_{\sigma(t)}$ is given by $d_i^{\sigma(t)} = \sum_{j=1}^{N} a_{ij}^{\sigma(t)}$.

Consider a linear system

$$\begin{aligned} \dot{x} &= Ax, \\ y &= Cx = [C_1^T, C_2^T, \cdots, C_N^T]^T x, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is state, $y \in \mathbb{R}^p$ is measurement output, and $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ are system matrix and output matrix respectively. Here, the portion $y_i = C_i x$ is the only output information can be obtained by each node. The distributed Luenberger observer is introduced as:

$$\dot{\hat{x}}_i = A\hat{x}_i + H_i(y_i - C_i\hat{x}_i) + \gamma P_i \sum_{i=1}^{N} a_{ij}^{\sigma(t)} (\hat{x}_j - \hat{x}_i), \tag{2}$$

where $\hat{x}_i$ is the state estimation of node i, $a_{ij}^{\sigma(t)}$ is the element of adjacent matrix of the time-dependence communication graph, observer gain H_i, coupling gain γ and weighted matrix P_i are the designing parameters which need to be designed. Note the difference between (2) and classic distributed Luenberger observer is that a_{ij}^{σ} is time-variant. The objective of this paper is

$$\lim_{t\to\infty} \|\hat{x}_i(t) - x(t)\| = 0, \tag{3}$$

which is the property *asymptotical omniscience* of distributed observer. Next, we will give three basic assumptions of this paper.

Assumption 1. We assume that all the graph $\mathcal{G}_k, k \in \mathcal{P}$ are not connected but the union of all communication topologies $\bigcup_{k\in\mathcal{P}} \mathcal{G}_k$ is connected (*jointly connected*, see Definition 2 in Wei and Cheng (2010)).

Assumption 2. Consider an infinite sequence of nonempty, bounded time series $t_0 = 0, t_1, \cdots, t_k, \cdots$ satisfied $t_{k+1} - t_k \le \mathcal{T}$ for $k = 0, 1, \cdots$, and a subsequence in $[t_k, t_{k+1})$ defined as $t_k^0, \cdots, t_k^{\ell_k}$ with $t_k^{j+1} - t_k^j \ge \tau, j = 1, \cdots, \ell_k - 1$, where $\tau > 0$ and $\mathcal{T} > 0$ are two given constants. Then we suppose the interconnection topology in each of $[t_k^j, t_k^{j+1})$ remains unchanged and we further suppose each topology in $\mathcal{P}$ appears at least once in the time intervals $[t_k, t_{k+1})$.

For saving of statement, we define that a *connected branch of $\mathcal{G}_k$ is observable* if the state of the whole system can be observed by the output information about nodes only in this connected branch.

Assumption 3. We assume the pair (C, A) is observable and further assume that (C_i, A) is not completely observable for all $i = 1, 2, \cdots, N$. Moreover, for each topology $\mathcal{G}_k, k \in \mathcal{P}$, we assume there is at least one connected branch of $\mathcal{G}_k$ is unobservable.

3. CONDITIONS OF CONSISTENCY

3.1 Research Method of Distributed Observer under Fixed Topology

This subsection will review the research method of distributed observer under the fixed topology. Note that $card\{\mathcal{P}\} = 1$ when the topology is fixed, so that $\sigma(t)$ is a constant function. For saving of writing, we omit σ in this subsection.

Denote $e_i = \hat{x}_i - x$, then the error dynamic between (1) and (2) can be introduced as

$$\dot{e}_i = (A - H_i C_i) e_i + \gamma P_i^{-1} \sum_{i=1}^{N} a_{ij} (e_j - e_i), \tag{4}$$

By setting $\Psi = diag\{\Psi_1, \cdots, \Psi_N\}$ with $\Psi_i = A - H_i C_i$ and $e = col\{e_1, \cdots, e_N\}$, a compact form of error dynamic can be obtained as follows:

$$\dot{e} = \Psi e - \gamma P^{-1} (\mathcal{L} \otimes I_n) e, \tag{5}$$

where $P = diag\{P_1, \cdots, P_N\}$. For $\forall i \in \mathcal{V}$, let T_i be an orthogonal matrix such that

$$T_i^T A T_i = \begin{bmatrix} A_{io} & 0 \\ A_{ir} & A_{iu} \end{bmatrix}, C_i T_i = [C_{io} \ 0], \tag{6}$$

where $A_{io} \in \mathbb{R}^{v_i \times v_i}$ with v_i being the dimension of observable subspace of (C_i, A), and A_{iu}, A_{ir}, C_{io} are block matrices corresponding to A_{io} with appropriate dimensions. Note that the consensus of distributed observer (2) is equivalent to the stability of error system (5). So what we need to do next is to define $V(e(t)) = \sum_{i=1}^{N} e_i^T P_i e_i$ and prove

$$\dot{V}(t) = e^T \left(\Psi^T P + P\Psi\right) e - 2\gamma e^T (\mathcal{L} \otimes I_n) e < 0. \tag{7}$$

Han et al. (2019) proves that (7) holds if for all $g_i > 0, i = 1, \cdots, N$, there exists $\varepsilon > 0$ such that

$$T^T (\mathcal{L} \otimes I_n) T + G \geq \varepsilon I_{nN}, \tag{8}$$

$$T^T \left(\Psi^T P + P\Psi\right) T + 2\gamma G < \varepsilon I_{nN}, \tag{9}$$

where $T = diag\{T_1, \cdots, T_N\}$, $G = diag\{G_1, \cdots, G_N\}$, and $G_i = diag\{g_i I_{v_i}, 0_{n-v_i}\}$.

3.2 Problems and Solutions in Switching Topology

As for the situation of switching topology, the sufficient conditions of classic distributed observer cannot be fulfilled. In fact, equation (7) can be fulfilled with fixed topology mainly depending on the coupling term. We know from the proof of Lemma 4 of Han et al. (2019) that though the Laplacian matrix of a connected graph is only a semi-positive definite matrix (including a zero eigenvalue), the observability of the whole system makes up for this deficiency and makes equation (8) be fulfilled. In the following lemma, we will describe this problem with a more general perspective.

Lemma 4. Set $\sigma_0 = \sigma(t_0) \in \mathcal{P}$ with a given constant t_0. And denote $\mathcal{L}_{\sigma_0}$ as Laplacian matrix of $\mathcal{G}_{\sigma_0}$. Then we alarm that the necessary condition of

$$T^T (\mathcal{L}_{\sigma_0} \otimes I_n) T + G \geq \varepsilon I_{nN} \tag{10}$$

is that every connected branch of a graph $\mathcal{G}_{\sigma_0}$ is observable.

The proof of this Lemma will be showed in Section 4. Since all connected branches are assumed to be unobservable (Assumption 3), (10) cannot be fulfilled. As a result, coupling term $-2\gamma(\mathcal{L}_k \otimes I_n)$ in (7) cannot compensate for the unstable modes of $\Psi^T P + P\Psi$ by increasing coupling gain. This means that if we want to achieve omniscience asymptotically for the distributed observer under the time-varying disconnected topologies, some constraints must be imposed on the system matrix or alternative topology. For instance, Yang et al. (2015) assumes system matrix to be marginally stable, and Stilwell et al. (2005) assumes the alternative topologies can be switched at any fast frequency. Due to the limitation of space, this paper only considers the constraints on system matrix. Fortunately, in the sense of distributed observer problem, we can allow the system matrix to be unstable under certain conditions, i.e., system matrix need not to be assumed as marginally stable. These conditions will be given in form of theorems in Section 5.

4. PROVE OF LEMMA 4

We know $rank(\mathcal{L}_{\sigma_0}) < N - 1$ owing to $\mathcal{L}_\sigma$ is disconnected and there are m-multiple zero eigenvalue if $rank(\mathcal{L}_{\sigma_0}) < N - 1$ includes m connected branches. Let $\mathcal{I}_k \subset \{1, 2, \cdots, N\}$ be the index set of the kth connected component of $\mathcal{G}_{\sigma_0}$. Thus we know $\sum_{k=1}^{m} card\{\mathcal{I}_k\} = N$. Now m basic eigenvectors of 0-eigensubspace of $\mathcal{L}_{\sigma_0}$ can be designed as:

$$\eta_{ki} = \begin{cases} 1/\sqrt{c_k}, & \text{if } i \in \mathcal{I}_k \\ 0, & \text{if } i \notin \mathcal{I}_k \end{cases}, \tag{11}$$

where $c_k = card\{\mathcal{I}_k\}$, η_{ki} is the ith entry of vector η_k. Note that all the $\eta_k, k = 1, 2, \cdots, m$ are the eigenvectors corresponding to 0. Moreover, they are orthogonal. For saving the expression, we let $\eta = [\eta_1, \cdots, \eta_m]$. Before the proof, we denote $\lambda_1 \geq \cdots \geq \lambda_{N-m} > 0$ as the nonzero eigenvalues of $\mathcal{L}_{\sigma_0}$.

Proof. Equation (10) holds if and only if

$$\begin{aligned} &\left(U^T \otimes I_n\right) (\mathcal{L}_{\sigma_0} \otimes I_n) (U \otimes I_n) \\ &+ \left(U^T \otimes I_n\right) TGT^T (U \otimes I_n) > 0, \end{aligned} \tag{12}$$

where $U = [\eta, V]$ with $V \in \mathbb{R}^{N \times (N-m)}$ is an orthogonal matrix such that

$$U^T \mathcal{L}_{\sigma_0} U = diag\{\underbrace{0, \cdots, 0}_{m}, \lambda_1, \cdots, \lambda_{N-m}\}. \tag{13}$$

Notice that the necessary condition of (12) is

$$\begin{aligned} &\left[\lambda_1 \otimes I_N - \begin{bmatrix} \lambda_1 I_m & 0 \\ 0 & 0_{N-m} \end{bmatrix}\right] \otimes I_n \\ &+ \left(U^T \otimes I_n\right) TGT^T (U \otimes I_n) > 0. \end{aligned} \tag{14}$$

By pre- and post-multiplying $U \otimes I_n$ and $U^T \otimes I_n$ on the both side of (14), we have

$$\lambda_1 I_{nN} - \left\{ [\eta, V] \begin{bmatrix} \lambda_1 I_m & 0 \\ 0 & 0_{N-m} \end{bmatrix} \begin{bmatrix} \eta^T \\ V \end{bmatrix} \right\} \otimes I_n + TGT^T > 0, \tag{15}$$

which is equivalent to

$$\lambda_1 I_{nN} - \lambda_1 T^T (\eta \otimes I_n)(\eta^T \otimes I_n) T + G > 0. \tag{16}$$

The definition of η could deduce the detail of $T^T(\eta \otimes I_n)$:

$$T^T(\eta \otimes I_n) = \begin{bmatrix} \eta_{11}T_1^T & \eta_{21}T_1^T & \cdots & \eta_{m1}T_1^T \\ \eta_{12}T_2^T & \eta_{22}T_2^T & \cdots & \eta_{m2}T_2^T \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{1N}T_N^T & \eta_{2N}T_N^T & \cdots & \eta_{mN}T_N^T \end{bmatrix}. \tag{17}$$

Using Schur complement lemma on equation (16) yields

$$\begin{bmatrix} \Phi_1 & 0 & \cdots & 0 & \eta_{11}T_1^T & \cdots & \eta_{m1}T_1^T \\ 0 & \Phi_2 & \cdots & 0 & \eta_{12}T_2^T & \cdots & \eta_{m2}T_2^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_N & \eta_{1N}T_N^T & \cdots & \eta_{mN}T_N^T \\ \eta_{11}T_1 & \eta_{12}T_2 & \cdots & \eta_{1N}T_N & \frac{1}{\lambda_1}I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_{m1}T_1 & \eta_{m2}T_2 & \cdots & \eta_{mN}T_N & 0 & \cdots & \frac{1}{\lambda_1}I_n \end{bmatrix} > 0, \tag{18}$$

where Φ_i is denoted as:

$$\Phi_i = \lambda_1 I_n + G_i = \begin{bmatrix} (\lambda_1 + g_i)I_{v_i} & 0 \\ 0 & \lambda_1 I_{n-v_i} \end{bmatrix}. \tag{19}$$

Denote $\mathcal{I}_k(\ell)$ as the first ℓ elements of the index set $\mathcal{I}_k$ and further denote

$$\Sigma_{k\ell}^* = \frac{1}{\lambda_1}I_n - \frac{1}{c_k}\sum_{i\in\mathcal{I}_k(\ell)} \frac{1}{\lambda_1 + g_i} T_{i1}T_{i1}^T, \tag{20}$$

$$\Sigma_{k\ell} = \frac{1}{\lambda_1}I_n - \frac{1}{c_k}\sum_{i\in\mathcal{I}_k(\ell)} \frac{1}{\lambda_1 + g_i} T_{i1}T_{i1}^T - \frac{1}{c_k}\sum_{i\in\mathcal{I}_k(\ell)} \frac{1}{\lambda_1} T_{i2}T_{i2}^T. \tag{21}$$

Then by supposing the first observer belongs to index set $\mathcal{I}_{k_0}$, equation (18) can be calculated by using schur complement lemma again:

$$\begin{bmatrix} \lambda_1 I_{n-v_i} & \cdots & 0 & \eta_{11}T_{12}^T & \cdots & \eta_{m1}T_{12}^T \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi_N & \eta_{1N}T_N^T & \cdots & \eta_{mN}T_N^T \\ \eta_{11}T_{12} & \cdots & \eta_{1N}T_N & \Sigma_{k_0 1}^* & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_{m1}T_{12} & \cdots & \eta_{mN}T_N & 0 & \cdots & \frac{1}{\lambda_1}I_n \end{bmatrix} > 0. \tag{22}$$

Continuing to use Schur complement lemma, we can further obtain

$$\begin{bmatrix} \Phi_2 & \cdots & 0 & \eta_{12}T_2^T & \cdots & \eta_{m2}T_2^T \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi_N & \eta_{1N}T_N^T & \cdots & \eta_{mN}T_N^T \\ \eta_{12}T_2 & \cdots & \eta_{1N}T_N & \Sigma_{k_0 1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_{m2}T_2 & \cdots & \eta_{mN}T_N & 0 & \cdots & \frac{1}{\lambda_1}I_n \end{bmatrix} > 0. \tag{23}$$

By using Schur complement lemma repeatedly according to the calculation methods of (22) and (23), formula (16) can be simplified to

$$diag\{\Sigma_{1c_1}, \Sigma_{2c_2}, \cdots, \Sigma_{mc_m}\} > 0. \tag{24}$$

Equation (24) holds if and only if $\Sigma_{kc_k} > 0$ for all $k = 1, \cdots, m$. Because of the similarity of each diagonal block structure, we will only discuss one of them. Actually, we can show

$$\begin{aligned} \Sigma_{kc_k} =& \frac{1}{\lambda_1}I_n - \frac{1}{c_k}\sum_{i\in\mathcal{I}_k(c_k)} \frac{1}{\lambda_1 + g_i} T_{i1}T_{i1}^T \\ &-\frac{1}{c_k}\sum_{i\in\mathcal{I}_k(c_k)} \frac{1}{\lambda_1} T_{i2}T_{i2}^T \\ &-\frac{1}{c_k}\sum_{i\in\mathcal{I}_k(c_k)} \frac{1}{\lambda_1} T_{i1}T_{i1}^T + \frac{1}{c_k}\sum_{i\in\mathcal{I}_k(c_k)} \frac{1}{\lambda_1} T_{i1}T_{i1}^T \\ =&\frac{1}{\lambda_1}I_n - \frac{1}{c_k}\sum_{i\in\mathcal{I}_k(c_k)} \left(T_{i1}T_{i1}^T + T_{i2}T_{i2}^T\right) \\ &+\frac{1}{c_k}\sum_{i\in\mathcal{I}_k(c_k)} \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1 + g_i}\right) T_{i1}T_{i1}^T. \end{aligned} \tag{25}$$

Denote $g_{max} = \max_i\{g_i\}$ and then the necessary condition of $\Sigma_{kc_k} > 0$ is

$$\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1 + g_{max}}\right) \sum_{i\in\mathcal{I}_k(c_k)} T_{i1}T_{i1}^T > 0, \tag{26}$$

which indicates

$$\sum_{i\in\mathcal{I}_k(c_k)} T_{i1}T_{i1}^T > 0. \tag{27}$$

It is equivalent to

$$rank\{\underbrace{[T_{i_1 1}, \cdots, T_{i_{c_k} 1}]}_{i_1,\cdots,i_{c_k}\in\mathcal{I}_k(c_k)}\} = n. \tag{28}$$

Equation (28) is obtained from Han et al. (2019). This means that every connected branch of $\mathcal{G}_{\sigma_0}$ is observable.

5. MAIN RESULTS

This section will prove a set of constraints for the system matrix. Moreover, we prove a group of sufficient conditions for the distributed observer to be asymptotically omniscient for the systems that meets any one of matrix constraints. One now states the following.

Theorem 5. Consider system (1). Assume that Assumptions 1-3 are satisfied. Then $\Psi_i = A - H_iC_i, i = 1, 2, \cdots, N$ is marginally stable if one of the following three conditions hold:

(1) A is marginally stable.
(2) The polynomial $\lambda(A)/\lambda(A_{io})$ has not multiple roots on virtual axis for all $i = 1, 2, \cdots, N$ if A is *polynomial instability*, where A_{io} represents the observable subspace corresponding to C_i.
(3) $\Lambda_u \subset \bigcap_{i=1}^N \Lambda(A_{io})$ and $\lambda(A)/\lambda(A_{io})$ has not multiple roots on virtual axis if A is exponential instability with Λ_u being the set of open right half plane (ORHP) poles of A.

Proof. Due to the limitation of space, we omit this relatively simple proof.

Theorem 6. Consider system (1) and a set of alternative topologies $\mathcal{G}_k, k \in \mathcal{P}$. Assume that Assumptions 1-3 are satisfied and further assume the pair (C, A) satisfies at least one condition of Theorem 5. Then the designed distributed observer (2) can achieve omniscience asymptotically under the switching topologies $\{\mathcal{G}_k : k \in \mathcal{P}\}$ if there exists some constants $g_i \geq 1$, $\varsigma > 0$ and $\gamma > 0$, and symmetric positive definite matrices $P_{io} \in \mathbb{R}^{v_i \times v_i}$, $P_{iu} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}, i = 1, 2, \cdots, N$ such that $P_i = T_i \tilde{P}_i T_i^T$, $\tilde{P}_i = diag\{P_{io}, P_{iu}\}$ and

$$(C1) P_{io}\Psi_{io} + \Psi_{io}^T P_{io} = -2\gamma I_{v_i}, \quad (29)$$

$$(C2) sym\{P_{iu}A_{iu}\} - \frac{1}{\rho_i} P_{iu}A_{ir}A_{ir}^T P_{iu} \leq -\varsigma I_{n-v_i}, \quad (30)$$

where $\rho_i = 2\gamma(g_i - 1) + \varsigma$.

Proof. Since at least one conditions is satisfied by (C, A), observer gain matrix H_i can be designed as $H_i = T_i^T [H_{io}^T, 0]^T$ so that $A_i - H_{io}C_{io}$ is Hurwitz. Similar with the steps in subsection 3.1, the error dynamic and the derivation of Lyapunov function $V(t) = e^T P e$ can be calculated as follows:

$$\dot{e} = \Psi e - \gamma P^{-1} (\mathcal{L}_\sigma \otimes I_n) e, \quad (31)$$

$$\dot{V}(t) = e^T \left(\Psi^T P + P\Psi\right) e - 2\gamma e^T (\mathcal{L}_\sigma \otimes I_n) e. \quad (32)$$

By using Schur complement lemma on (30), we can obtain

$$\begin{bmatrix} 2\gamma(g_i - 1)I_{v_i} + \varsigma I_{v_i} & A_{ir}^T P_{iu} \\ P_{iu}A_{ir} & sym\{P_{iu}A_{iu}\} + \varsigma I_{n-v_i} \end{bmatrix} \leq 0. \quad (33)$$

Then we denote $\Psi_{io} = A_{io} - H_{io}C_{io}$ and there is a positive definite symmetric matrix P_{io} such that $P_{io}\Psi_{io} + \Psi_{io}^T P_{io} = -2\gamma I_{v_i}$. Therefore, (33) is equivalent to

$$\begin{bmatrix} sym\{P_{io}\Psi_{io}\} + 2g_i\gamma I_{v_i} & A_{ir}^T P_{iu} \\ P_{iu}A_{ir} & sym\{P_{iu}A_{iu}\} \end{bmatrix} + \varsigma I_n \leq 0. \quad (34)$$

For saving of compactness, (34) can be rewritten as

$$T^T \left(\Psi^T P + P\Psi\right) T + 2\gamma G \leq -\varsigma I_{nN}. \quad (35)$$

Furthermore, we know from the proof of Lemma 4 that

$$T^T (\mathcal{L}_\sigma \otimes I_n) T + G \geq 0. \quad (36)$$

Now by Substituting (36) and (35) into (32), we get

$$\begin{aligned} \dot{V}(t) &= e^T T T^T \left(\Psi^T P + P\Psi - 2\gamma \mathcal{L}_{\sigma(t)} \otimes I_n\right) T T^T e \\ &= e^T T \left[T^T \left(\Psi^T P + P\Psi\right) T + 2\gamma G \right. \\ &\quad \left. -2\gamma G - 2\gamma T^T \left(\mathcal{L}_{\sigma(t)} \otimes I_n\right) T\right] T^T e \\ &\leq -\varsigma e^T T I_{nN} T^T e = -\varsigma e^T e \leq 0. \end{aligned} \quad (37)$$

Thanks to the proof of Theorem 2 of Wei and Cheng (2010), we can prove $\lim_{t\to\infty} e_i(t) = 0, i = 1, 2, \cdots, N$ from the fact $\dot{V}(t) = -\varsigma \sum_{i=1}^N e_i^T e_i \leq 0$, which indicates the designed distributed observer (2) can achieve omniscience asymptotically. For saving of space, we omit this proof.

6. SIMULATION

Consider the following linear system

$$\dot{x} = Ax, y = Cx,$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 10.2 \\ -1.5 & -2 & 0 & 0 \\ 0 & 0 & -1.5 & 1.63 \\ 3.2 & 0 & 0 & -2.2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that $\Lambda(A) = \{-2, -1.5, 5.33, -6.53\}$. There is a ORHP pole 5.33, thus A is an exponentially unstable matrix. The observable index of C_1, C_2, C_3, C_4 are 2, 3, 3, 2 respectively. By calculating observable decomposition, we can obtain $A_{1u} = diag\{-2, -1.5\}$, $A_{2u} = -1.5$, $A_{3u} = -2$, and $A_{4u} = diag\{-2, -1.5\}$. It means the third condition $\Lambda_u \subset \sum_{i=1}^4 \Lambda(A_{io})$ of Theorem 5 if fulfilled. The alternative communication networks are given in Figure 1.

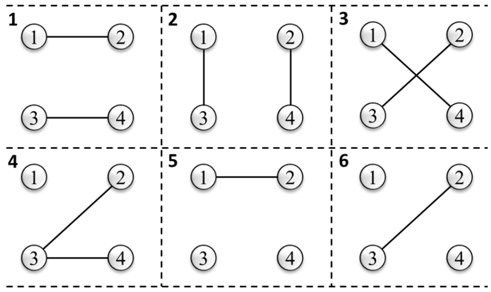

Fig. 1. Alternative communication networks

Let $g_i = 1, i = 1, \cdots, 4$, $\gamma = 1$ and $\varsigma = 1$. Then we can calculate P_1, P_2, P_3, P_4 and H_1, H_2, H_3, H_4. Due to the limitation of the paper, the calculation results of $P_i, i = 1, \cdots, 4$ and the observer gain matrix $H = [H_1, H_2, H_3, H_4]$ are omitted. The initial states of actual system are chosen as $x(0) = [1, 1, 1, 1]^T$ and the initial states of each local observer are chosen randomly. As shown in the Figure 2, the error dynamics between distributed observer and actual system converge to zero even A includes a unstable pole. Figure 3 is the switching sequence of this Example.

7. CONCLUSION

This paper has considered the asymptotic omniscience of distributed linear observer under time-variant disconnected communication network. Firstly, we have proved by LMIs method that the designing distributed observer under time-variant disconnected communication network can only achieve clustered consensus, but not omniscience asymptotically if there are no constraints on system matrix and alternative topologies set. Moreover, we have also analyzed the requirements of the system matrix to meet the omniscience asymptotically. Based on the analysis, three kinds of constraints on system matrix and a group of sufficient conditions for guaranteeing the omniscience asymptotically have been proposed and proved. Finally, we have completed a numerical simulation that verified the validity of our method.

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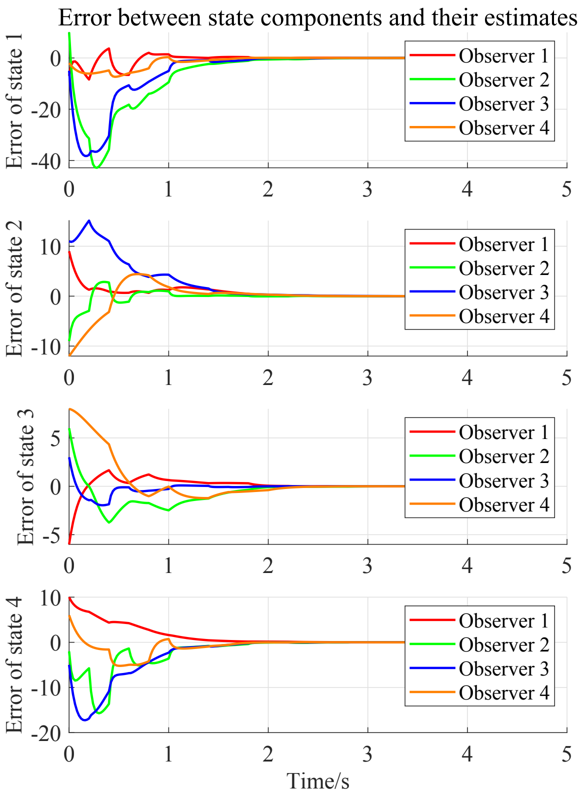

Fig. 2. Error dynamics of distributed observers

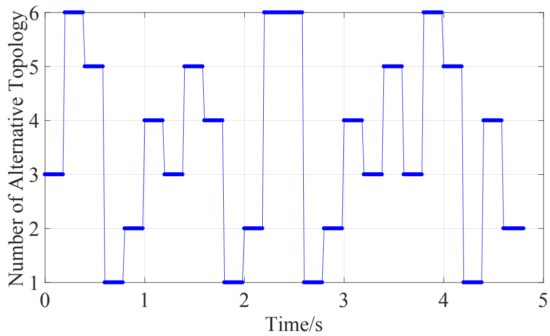

Fig. 3. A switching signal describing time-vary communication topology

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