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Reduced-Order Observers for Linear Metzlerian Systems

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Abstract: For linear time-invariant Metzlerian systems this paper proposes an original approach required to reflect structural system constraints and positiveness in solving the problem of reduced-order Metzlerian observer design. Three forms of design conditions are proposed, all of them exploiting a set of common system parameter constraint representation in the form of linear matrix inequalities, while a systematic H_∞ norm performance strategy is focused on to guarantee the observer asymptotic stability. To serve as a potential estimator of the plant state vector, the impact of strictness and non strictness Metzler matrix structure in design is clarified and reflected in adaptation of the design conditions. A numerical example is included to assess the feasibility of the technique and its applicability.

Keywords: linear Metzlerian systems, positive linear systems, diagonal stabilization, linear matrix inequalities, reduced-order state observers.

1. INTRODUCTION

Positive systems represent one class of dynamical systems whose states and outputs are positive whenever system inputs, acting under coincidence with system initial states, are nonnegative and so they indicate processes whose variable magnitudes do not have a real meaning unless they are positive (Nikaido (1968), Smith (1995)). In this context, theory of Metzler matrices play fundamental role in description of positive systems and in their analysis (Berman et al. (1989), Carnicer et al. (1998)) and linear continuous-time positive systems are therefore often referred to as Metzlerian. The solved problem of Metzler linear observers usually deals with positive system state estimation, external inputs estimation and fault detection and isolation in positive systems.

Observer synthesis for Metzlerian systems means the construction of a model algorithm, driven in time by a nonnegative difference between real output process of the system and an estimated output in a such way that the estimation error is asymptotically stable. Since existence of Metzlerian observer structures depends on acceptable limitation in nonnegativity of the system and observer parameters, constrained design approaches are proposed to solve the design task. Although Metzlerian estimators received considerable attention (Härdin and van Schuppen (2007)), used design principles have not been studied as extensively as that of standard linear systems. The algebraic formulation is introduced in Back and Astolfi (2008), where using the coordinate transformation, design conditions are conditioned by Silvester equation to reflect suitable transform matrix.

The need for design techniques relying on feasibility of the set of linear matrix inequalities (LMI) is reflected by Shu et al. (2008), where LMI design conditions are proposed, but the considered observer is not of Luenberger type. Preferring linear programming (LP) approach, a way to set non-symmetrical bounds in LP constraints, to impose positiveness in estimated states, is proposed by Ait Rami and Tadeo (2006). An interconnection of the observer synthesis methods for linear Metzlerian continuous-time systems and positive discrete-time systems is investigated by Liu et al. (2018), specific relations in positive unknown-input observers design for linear positive systems can find in (Shafai et al. (2015)), but the first, purely LMI-based observer synthesis method for linear Metzlerian systems is proposed by Krokavec and Filasová (2018).

Adapting the authors' results in full-order Metzlerian observer synthesis to a reduced-order Metzlerian observer structure, as well as their potential extensions, are the main issues of this paper. Preferring LMI formulation in parametric constraints definition together with observer asymptotic stability, presented theorems use standard arguments based on Lyapunov function combined with H_∞ approach to obtain design conditions requiring to solve only LMIs. Because using diagonal positive matrix variables, the approach is referred to as the diagonal stabilisation principle, not more difficult than checking the stability of a square Metzler matrix.

To the best author's knowledge, the proposed LMI formulations of the design conditions for reduced-order Metzlerian observer structure were not fully addressed yet in the previous works.

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Following introduction in Sec. 1, the design fundamentals related to linear Metzlerian observers for strictly linear Metzlerian systems are briefly described in Sec. 2. Section Sec. 3 gives the reduced-order observer equations, imparts the reduced-order observer autonomous dynamics to state the design conditions by use of LMIs and some practice adaptations. The design task is illustrated by a numerical solution in Sec. 4 and Sec. 5 draws some conclusions.

Throughout the paper, the following notations are used: $\boldsymbol{x}^{\mathrm{T}}$, $\boldsymbol{X}^{\mathrm{T}}$ denotes the transpose of the vector $\boldsymbol{x}$, and the matrix $\boldsymbol{X}$, respectively, diag $[\cdot]$ marks a (block) diagonal matrix, $\rho(\boldsymbol{X})$ indicates eigenvalue spectrum of the square matrix $\boldsymbol{X}$, for a square symmetric matrix $\boldsymbol{X} \prec 0$ means that $\boldsymbol{X}$ is negative definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $\mathbb{R}$ ($\mathbb{R}_+$) qualifies the set of (nonnegative) real numbers, $\mathbb{R}^{n\times r}$ ($\mathbb{R}_+^{n\times r}$) refers to the set of $n\times r$ (nonnegative) real matrices and $\mathbb{M}_{-+}^{n\times n}$ means the set of (strictly) Metzler square matrices.

2. LINEAR METZLERIAN OBSERVERS

Continuous-time and time-invariant strictly linear Metzlerian SISO systems admit the state-space description

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t), \quad \boldsymbol{v}(t) = \boldsymbol{C}_v\boldsymbol{q}(t) = [\,\boldsymbol{I}_m\ \boldsymbol{0}\,]\,\boldsymbol{q}(t) \tag{2}$$

where $\boldsymbol{q}(t) \in \mathbb{R}_+^n$, $\boldsymbol{u}(t) \in \mathbb{R}^r$, $\boldsymbol{v}(t) \in \mathbb{R}_+^m$ are system state vector, control input and measurable output and $\boldsymbol{y}(t) \in \mathbb{R}_+^m$ is the output of the system to be controlled. In the above model the nonnegative matrix parameters $\boldsymbol{B} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_+^n$ imply from the Metzlerian system structure definition, while a signum indefinite matrix $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$ is considered as strictly Metzler matrix with negative diagonal elements and positive non-diagonal elements (Berman et al. (1989)). Analysis and synthesis of systems with strictly Metzlerian structures is so confronted with n^2 boundaries implying from the Metzler matrix structural constraints

$$a_{ii} < 0 \ \forall\, i = 1,\dots n, \quad a_{ij,i\neq j} > 0 \ \forall\, i,j = 1,\dots n \tag{3}$$

This just means in consequence that Metzlerian systems are diagonally stabilizable, while the derived structural constraints can be implemented through linear matrix inequalities (Krokavec and Filasová (2018)). If non-diagonal elements of a Metzler matrix $\boldsymbol{A}$ are non-negative, the matrix is non-strictly Metzler matrix.

Considering the Luenberger continuous-time observer to strictly Metzler system (1), (2) in the form

$$\dot{\boldsymbol{q}}_e(t) = \boldsymbol{A}\boldsymbol{q}_e(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{J}_E\boldsymbol{C}(\boldsymbol{q}(t) - \boldsymbol{q}_e(t)) \tag{4}$$

$$\boldsymbol{y}_e(t) = \boldsymbol{C}\boldsymbol{q}_e(t) \tag{5}$$

where $\boldsymbol{q}_e(t) \in \mathbb{R}_+^n$, $\boldsymbol{y}_e(t) \in \mathbb{R}_+^m$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} \boldsymbol{c}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{c}_m^{\mathrm{T}} \end{bmatrix}, \ \boldsymbol{c}_k^{\mathrm{T}} = [c_{k1} \ \cdots \ c_{kn}] \tag{6}$$

$$\boldsymbol{B} = [\boldsymbol{b}_1 \ \cdots \ \boldsymbol{b}_r],\ \boldsymbol{J}_E = [\boldsymbol{j}_{E1} \cdots \boldsymbol{j}_{Em}],\ \boldsymbol{j}_{Ek} = \begin{bmatrix} j_{1Ek} \\ \vdots \\ j_{nEk} \end{bmatrix} \tag{7}$$

then it has to be satisfied for elements of strictly positive observer system matrix $\boldsymbol{A}_E = \boldsymbol{A} - \boldsymbol{J}_E\boldsymbol{C} \in \mathbb{M}_{-+}^{n\times n}$ and for a strictly positive matrix $\boldsymbol{J}_E \in \mathbb{R}_+^{n\times m}$

$$a_{ll} - \sum_{k=1}^{m} j_{lEk}c_{kl} < 0 \text{ for all } l \in \langle 1,\dots,n\rangle \tag{8}$$

$$a_{hl} - \sum_{k=1}^{m} j_{hEk}c_{kl} > 0 \text{ for all } h,l,h\neq l, \in \langle 1,\dots,n\rangle \tag{9}$$

To design respecting these constraints the following theorem is available.

Theorem 1. (Krokavec and Filasová (2018)) Using observer of Luneberger type (4), (5) in state estimation of Metzlerian system (1), (2), then observer system matrix $\boldsymbol{A}_e$ is strictly Metzler and Hurwitz if for given strictly Metzler matrix $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$ and non-negative matrix $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$ there exist positive definite diagonal matrices $\boldsymbol{V}, \boldsymbol{W}_k \in \mathbb{R}_+^{n\times n}$ such that for $j = 1,2,\dots n$, $h = 1,2,\dots n-1$, $k = 1,2,\dots m$,

$$\boldsymbol{V} \succ 0, \qquad \boldsymbol{W}_k \succ 0 \tag{10}$$

$$\boldsymbol{V}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{V} - \sum_{k=1}^{m}(\boldsymbol{W}_k\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{C}_{dk} + \boldsymbol{C}_{dk}\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{W}_k) \prec 0 \tag{11}$$

$$\boldsymbol{V}\boldsymbol{A}(j,j)_{(1\leftrightarrow n)/n} - \sum_{k=1}^{m}\boldsymbol{W}_k\boldsymbol{C}_{dk} \prec 0 \tag{12}$$

$$\boldsymbol{V}\boldsymbol{T}^h\boldsymbol{A}(j+h,j)_{(1\leftrightarrow n)/n}\boldsymbol{T}^{hT} - \sum_{k=1}^{m}\boldsymbol{W}_k\boldsymbol{T}^h\boldsymbol{C}_{dk}(\boldsymbol{T}^T)^h \succ 0 \tag{13}$$

where

$$\boldsymbol{C}_{dk} = \text{diag}\,[\boldsymbol{c}_k^{\mathrm{T}}] = \text{diag}\,[\,c_{k1}\ c_{k2}\ \dots\ c_{kn}\,] \tag{14}$$

$$\boldsymbol{A}(j+h,j)_{(1\leftrightarrow n)/n} = \text{diag}\,[a_{1+h,1}\cdots a_{n,n-h}\,a_{1,n-h+1}\cdots a_{hn}] \tag{15}$$

$$\boldsymbol{l} = [\,1\ 1\ \cdots\ 1\,], \quad \boldsymbol{T} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_{n-1} & \boldsymbol{0} \end{bmatrix} \tag{16}$$

$\boldsymbol{T} \in \mathbb{R}_+^{n\times n}$, $\boldsymbol{l} \in \mathbb{R}_+^n$.

When these conditions hold, the positive observer gain matrix $\boldsymbol{J}_E \in \mathbb{R}_+^{n\times m}$ is given as

$$\boldsymbol{J}_{Edk} = \boldsymbol{V}^{-1}\boldsymbol{W}_k,\ \boldsymbol{j}_{Ek} = \boldsymbol{J}_{Edk}\boldsymbol{l},\ \boldsymbol{J}_E = [\,\boldsymbol{j}_{E1}\ \cdots\ \boldsymbol{j}_{Em}\,] \tag{17}$$

Note, (11) implies from standard Lyapunov matrix inequality and guaranties Hurwitz $\boldsymbol{A}_E$ if a solution exists.

3. REDUCED ORDER METZLERIAN OBSERVERS

The problem of interest is to design a reduced-order Metzlerian observer to the Metzlerian system (1), (2).

With the prescribed structure of $\boldsymbol{v}(t)$ then (1), (2) can be partitioned in the following way

$$\dot{\boldsymbol{q}}(t) = \begin{bmatrix} \dot{\boldsymbol{q}}_1(t) \\ \dot{\boldsymbol{q}}_2(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_1(t) \\ \boldsymbol{q}_2(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}\boldsymbol{u}(t) \tag{18}$$

$$\boldsymbol{v}(t) = [\,\boldsymbol{I}_m\ \boldsymbol{0}\,]\,\boldsymbol{q}(t) = [\,\boldsymbol{I}_m\ \boldsymbol{0}\,]\begin{bmatrix} \boldsymbol{q}_1(t) \\ \boldsymbol{q}_2(t) \end{bmatrix} \tag{19}$$

where $\boldsymbol{q}_1(t) \in \mathbb{R}_+^m$, $\boldsymbol{q}_2(t) \in \mathbb{R}_+^{n-m}$, $\boldsymbol{v}(t) \in \mathbb{R}_+^m$ and $\boldsymbol{B}_1 \in \mathbb{R}_+^{m\times r}$, $\boldsymbol{B}_2 \in \mathbb{R}_+^{(n-m)\times r}$, $\boldsymbol{A}_{11} \in \mathbb{M}_{-+}^{m\times m}$, $\boldsymbol{A}_{22} \in \mathbb{M}_{-+}^{(n-m)\times(n-m)}$, $\boldsymbol{A}_{12} \in \mathbb{R}_+^{m\times(n-m)}$, $\boldsymbol{A}_{21} \in \mathbb{R}_+^{(n-m)\times m}$.

Apart of the structure of $v(t) = [\,\boldsymbol{I}_m\ \boldsymbol{0}\,]\,\boldsymbol{q}(t)$, the system coordinate transform cannot depend on any factorization of $\boldsymbol{C}$ which exploits a matrix inversion principle, because it leads to disruption of Metzlerian description structure. Only transformation using a permutation matrix is allowed, if it is necessary to create the block structure (18), (19) for measurable non-first m state variables.

Definition 1. The reduced-order observer to (1), (2) takes the strictly Metzler form

$$\dot{\boldsymbol{p}}_{2e}(t) = \boldsymbol{A}_e\boldsymbol{p}_{2e}(t) + \boldsymbol{A}_v^{\circ}\boldsymbol{v}(t) + \boldsymbol{B}_e\boldsymbol{u}(t) \tag{20}$$

$$\boldsymbol{q}_{2e}(t) = \boldsymbol{p}_{2e}(t) + \boldsymbol{J}\boldsymbol{v}(t) \tag{21}$$

where

$$\boldsymbol{A}_v^{\circ} = \boldsymbol{A}_v + \boldsymbol{A}_e\boldsymbol{J}, \quad \boldsymbol{B}_e = \boldsymbol{B}_2 - \boldsymbol{J}\boldsymbol{B}_1 \tag{22}$$

$$\boldsymbol{A}_e = \boldsymbol{A}_{22} - \boldsymbol{J}\boldsymbol{A}_{12}, \quad \boldsymbol{A}_v = \boldsymbol{A}_{21} - \boldsymbol{J}\boldsymbol{A}_{11} \tag{23}$$

if for positive observer gain matrix $\boldsymbol{J} \in \mathbb{R}_+^{r\times n}$ the matrix $\boldsymbol{A}_e \in \mathbb{M}_{-+}^{(n-m)\times(n-m)}$ is strictly Metzler and the matrix $\boldsymbol{A}_v \in \mathbb{R}_+^{(n-m)\times m}$ is a nonnegative matrix, while the vector $\boldsymbol{q}_{2e}(t) \in \mathbb{R}_+^{n-m}$ is an estimation of the unmeasurable part of $\boldsymbol{q}(t)$ and $\boldsymbol{p}_{2e}(t) \in \mathbb{R}_+^{n-m}$ is the state vector of the reduced-order observer.

Proposition 1. Since (18) implies the following partitions

$$\dot{\boldsymbol{q}}_1(t) = \boldsymbol{A}_{11}\boldsymbol{v}(t) + \boldsymbol{A}_{12}\boldsymbol{q}_2(t) + \boldsymbol{B}_1\boldsymbol{u}(t) \tag{24}$$

$$\dot{\boldsymbol{q}}_2(t) = \boldsymbol{A}_{21}\boldsymbol{v}(t) + \boldsymbol{A}_{22}\boldsymbol{q}_2(t) + \boldsymbol{B}_2\boldsymbol{u}(t) \tag{25}$$

and an immediate consequence of (24) is

$$\boldsymbol{A}_{12}\boldsymbol{q}_2(t) = \dot{\boldsymbol{q}}_1(t) - \boldsymbol{A}_{11}\boldsymbol{v}(t) - \boldsymbol{B}_1\boldsymbol{u}(t) \tag{26}$$

the reduced-order Metzlerian observer can be defined as follows

$$\begin{aligned}\dot{\boldsymbol{q}}_{2e}(t) = {} & \boldsymbol{A}_{21}\boldsymbol{v}(t) + \boldsymbol{A}_{22}\boldsymbol{q}_{2e}(t) + \boldsymbol{B}_2\boldsymbol{u}(t) + \\ & + \boldsymbol{J}\left((\dot{\boldsymbol{q}}_1(t) - \boldsymbol{A}_{11}\boldsymbol{v}(t) - \boldsymbol{B}_1\boldsymbol{u}(t)) - \boldsymbol{A}_{12}\boldsymbol{q}_{2e}(t)\right)\end{aligned} \tag{27}$$

where (2) implies

$$\dot{\boldsymbol{q}}_1(t) = \dot{\boldsymbol{v}}(t) \tag{28}$$

Then (27) can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{q}}_{2e}(t) - \boldsymbol{J}\dot{\boldsymbol{v}}(t) = {} & \boldsymbol{A}_{21}\boldsymbol{v}(t) + \boldsymbol{B}_2\boldsymbol{u}(t) - \boldsymbol{J}\boldsymbol{B}_1\boldsymbol{u}(t) + \\ & + \boldsymbol{A}_{22}(\boldsymbol{q}_{2e}(t) - \boldsymbol{J}\boldsymbol{v}(t) + \boldsymbol{J}\boldsymbol{v}(t)) - \\ & - \boldsymbol{J}\boldsymbol{A}_{11}\boldsymbol{v}(t) - \boldsymbol{J}\boldsymbol{A}_{12}\boldsymbol{q}_{2e}(t)\end{aligned} \tag{29}$$

and writing (21) in the form

$$\boldsymbol{p}_{2e}(t) = \boldsymbol{q}_{2e}(t) - \boldsymbol{J}\boldsymbol{v}(t) \tag{30}$$

then (29), (30) yields the simpler structure

$$\begin{aligned}\dot{\boldsymbol{p}}_{2e}(t) = {} & (\boldsymbol{A}_{22} - \boldsymbol{J}\boldsymbol{A}_{12})\boldsymbol{p}_{2e}(t) + \\ & + (\boldsymbol{A}_{21} - \boldsymbol{J}\boldsymbol{A}_{11})\boldsymbol{v}(t) + \\ & + (\boldsymbol{A}_{22} - \boldsymbol{J}\boldsymbol{A}_{12})\boldsymbol{J}\boldsymbol{v}(t)) \\ & + (\boldsymbol{B}_2 - \boldsymbol{J}\boldsymbol{B}_1)\boldsymbol{u}(t)\end{aligned} \tag{31}$$

and with specific choices given in (22), (23) then (31) reduces to (20). If the problem is solvable, then there exists an $\boldsymbol{J}$ solving it.

Moreover, formally, (19) can be interpreted as

$$\boldsymbol{v}(t) = \boldsymbol{q}_1(t) = \boldsymbol{p}_{1e}(t) \tag{32}$$

to construct full order vector $\boldsymbol{p}_e(t)$. To obtain positive $\boldsymbol{q}_e(t)$ for positive $\boldsymbol{q}(t)$ also $\boldsymbol{p}_e(t)$ has to be positive. This defines the Metzler structure to the reduced observer for a Metzler system.

Considering the autonomous regime of (20), the result of Theorem 1 can be used directly to state the design conditions respecting stability and structural constraints of $\boldsymbol{A}_e \in \mathbb{M}_{-+}^{(n-m)\times(n-m)}$ as stated below, where by definition, the matrix $\boldsymbol{A}_{22} \in \mathbb{M}_{-+}^{(n-m)\times(n-m)}$ plays the role of $\boldsymbol{A}$ and $\boldsymbol{A}_{12} \in \mathbb{M}_{-+}^{m\times(n-m)}$ the role of $\boldsymbol{C}$ and the dimensionality of matrix variables is adjusted accordingly. For this reason, proof of the theorem is omitted.

Theorem 2. The matrix $\boldsymbol{A}_e \in \mathbb{M}_{-+}^{(n-m)\times(n-m)}$ of reduced-order observer (20) is strictly Metzler and Hurwitz if there exist positive diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_k \in \mathbb{R}_+^{(n-m)\times(n-m)}$ such that for $j = 1, 2, \dots n-m$, $h = 1, 2, \dots n-m-1$, $k = 1, 2, \dots n-m$

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{R}_k \succ 0 \tag{33}$$

$$\boldsymbol{P}\boldsymbol{A}_{22} + \boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{n-m}(\boldsymbol{R}_k\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{dk} + \boldsymbol{C}_{dk}\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_k) \prec 0 \tag{34}$$

$$\boldsymbol{P}\boldsymbol{A}_{22}(j,j)_{(\triangle)} - \sum_{k=1}^{n-m}\boldsymbol{R}_k\boldsymbol{C}_{dk} \prec 0 \tag{35}$$

$$\boldsymbol{P}\boldsymbol{T}^h\boldsymbol{A}_{22}(j+h,j)_{(\triangle)}\boldsymbol{T}^{h\mathrm{T}} - \sum_{k=1}^{m-m}\boldsymbol{R}_k\boldsymbol{T}^h\boldsymbol{C}_{dk}\boldsymbol{T}^{h\mathrm{T}} \succ 0 \tag{36}$$

$(\triangle) = (1 \leftrightarrow (n-m))/(n-m)$, $\boldsymbol{T} \in \mathbb{R}_+^{(n-m)\times(n-m)}$, $\boldsymbol{l} \in \mathbb{R}_+^{(n-m)}$

$$\boldsymbol{A}_{12} = \begin{bmatrix} \boldsymbol{a}_{121}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{a}_{12m}^{\mathrm{T}} \end{bmatrix}, \ \boldsymbol{A}_{22} = \begin{bmatrix} \boldsymbol{a}_{22,1}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{a}_{22,n-m}^{\mathrm{T}} \end{bmatrix} \tag{37}$$

$$\boldsymbol{C}_{dk} = \mathrm{diag}\left[\, a_{12k1} \ a_{12k2} \ \dots \ a_{12k,n-m} \,\right] \tag{38}$$

$$\begin{aligned}&\boldsymbol{A}_{22}(j+h,j)_{(\triangle)} = \mathrm{diag} \\ &[a_{22,1+h,1} \cdots a_{22,n-m,n-m-h} \ a_{22,1,n-m-h+1} \cdots a_{22,h,n-m}]\end{aligned} \tag{39}$$

$$\boldsymbol{l}^{\mathrm{T}} = [\,1 \ 1 \ \cdots \ 1\,], \quad \boldsymbol{T} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_{n-m-1} & \boldsymbol{0} \end{bmatrix} \tag{40}$$

When these conditions hold, the corresponding positive observer gain matrix $\boldsymbol{J} \in \mathbb{R}_+^{(n-m)\times m}$ is given as

$$\boldsymbol{J}_{dk} = \boldsymbol{P}^{-1}\boldsymbol{R}_k, \quad \boldsymbol{j}_k = \boldsymbol{J}_{dk}\boldsymbol{l}, \quad \boldsymbol{J} = [\,\boldsymbol{j}_1 \ \cdots \ \boldsymbol{j}_m\,] \tag{41}$$

It is natural to restrict the H_∞ norm of the observer transfer function. This takes the following formulation.

Theorem 3. The reduced-order Metzler observer (20) to strictly Metzler system (1), (2) is asymptotically stable with the quadratic performance γ if there exists diagonal positive definite matrices $\boldsymbol{P}, \boldsymbol{R}_k \in \mathbb{R}_+^{(n-m)\times(n-m)}$ and a positive scalar $\gamma \in \mathbb{R}_+$ such that for $j = 1, 2, \dots n-m$, $h = 1, 2, \dots n-m-1$, $k = 1, 2, \dots n-m$

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{R}_k \succ 0, \quad \gamma > 0 \tag{42}$$

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_{22} + \boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{P} - \sum_{k=1}^{n-m}(\boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{dk} + \boldsymbol{C}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_k) & * & * \\ \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{P} - \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}} & -\gamma\boldsymbol{I}_r & * \\ \boldsymbol{A}_{12} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{43}$$

$$\boldsymbol{P}\boldsymbol{A}_{22}(j,j)_{(\triangle)} - \sum_{k=1}^{n-m}\boldsymbol{R}_k\boldsymbol{C}_{dk} \prec 0 \tag{44}$$

$$\boldsymbol{P}\boldsymbol{T}^h\boldsymbol{A}_{22}(j+h,j)_{(\triangle)}\boldsymbol{T}^{h\mathrm{T}} - \sum_{k=1}^{m-m}\boldsymbol{R}_k\boldsymbol{T}^h\boldsymbol{C}_{dk}\boldsymbol{T}^{h\mathrm{T}} \succ 0 \tag{45}$$

while

$$\boldsymbol{R} = [\,\boldsymbol{R}_1 \ \cdots \ \boldsymbol{R}_m\,], \quad \boldsymbol{L} = \mathrm{diag}\,[\,\boldsymbol{l} \ \cdots \ \boldsymbol{l}\,] \tag{46}$$

where $\boldsymbol{R} \in \mathbb{R}_+^{(n-m)\times(n-m)^2}$ is structured matrix variable, $\boldsymbol{L} \in \mathbb{R}_+^{(n-m)^2\times(n-m)}$ and the remaining design parameter are defined in (37)-(40).

When the above conditions hold, the positive observer gain matrix $\boldsymbol{J} \in \mathbb{R}_+^{(n-m)\times m}$ is given as in (41).

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. Defining the Lyapunov function candidate as follows

$$
\begin{aligned}
v(\boldsymbol{p}_{2e}(t)) = {} & \boldsymbol{p}_{2e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{p}_{2e}(t) + \\
& + \gamma^{-1}\int_0^t (\boldsymbol{y}_p^{\mathrm{T}}(\tau)\boldsymbol{y}_p(\tau) - \gamma^2\boldsymbol{u}^{\mathrm{T}}(\tau)\boldsymbol{u}(\tau))\mathrm{d}\tau
\end{aligned}
\tag{47}
$$

where

$$
\boldsymbol{y}_p(t) = \boldsymbol{A}_{12}\boldsymbol{p}_{2e}(t) \tag{48}
$$

$\gamma \in I\!R_+$ is an upper bound of H_∞ norm of the transfer functions between $\widehat{\boldsymbol{y}}_p(s)$ and $\widehat{\boldsymbol{u}}(s)$, then the time derivative of (48) along any trajectory is

$$
\begin{aligned}
\dot{v}(\boldsymbol{p}_{2e}(t)) = {} & \boldsymbol{p}_{2e}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{p}}_{2e}(t) + \dot{\boldsymbol{p}}_{2e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{p}_{2e}(t) + \\
& + \gamma^{-1}\boldsymbol{p}_{2e}^{\mathrm{T}}(t)\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{A}_{12}\boldsymbol{p}_{2e}(t) - \gamma\boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{u}(t)) \\
& < 0
\end{aligned}
\tag{49}
$$

Considering $\boldsymbol{v}(t) = \boldsymbol{0}$ and substituting then (20) into (49) gives

$$
\begin{aligned}
\dot{v}(\boldsymbol{p}_{2e}(t)) = {} & \boldsymbol{p}_{2e}^{\mathrm{T}}(t)(\boldsymbol{P}\boldsymbol{A}_e + \boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{P} + \gamma^{-1}\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{A}_{12})\boldsymbol{p}_{2e}(t) + \\
& + \boldsymbol{p}_{2e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{B}_e\boldsymbol{u}(t) + \boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{P}\boldsymbol{p}_{2e}(t) - \\
& - \gamma\boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{u}(t) \\
& < 0
\end{aligned}
\tag{50}
$$

Defining the composed vector

$$
\boldsymbol{p}_{2e}^{\bullet\mathrm{T}}(t) = \left[\, \boldsymbol{p}_{2e}^{\mathrm{T}}(t) \;\; \boldsymbol{u}^{\mathrm{T}}(t) \,\right] \tag{51}
$$

the inequality (50) can be rewritten as

$$
\dot{v}(\boldsymbol{p}_{2e}^{\bullet\mathrm{T}}(t)) = \boldsymbol{p}_{2e}^{\bullet\mathrm{T}}(t)\boldsymbol{P}^{\bullet}\boldsymbol{p}_{2e}^{\bullet}(t) < 0 \tag{52}
$$

which corresponds to the condition writable in the linear matrix inequality form

$$
\boldsymbol{P}^{\bullet} = \begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_e + \boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{P} + \gamma^{-1}\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{A}_{12} & \boldsymbol{P}\boldsymbol{B}_e \\ \boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{P} & -\gamma\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{53}
$$

Regrouping terms using the Schur complement property it yields immediately

$$
\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_e + \boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{P} & \boldsymbol{P}\boldsymbol{B}_e & \boldsymbol{A}_{12}^{\mathrm{T}} \\ \boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{P} & -\gamma\boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{A}_{12} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{54}
$$

Since the matrix product $\boldsymbol{P}\boldsymbol{B}_e$ for given $\boldsymbol{B}_e$ is defined by the relation

$$
\boldsymbol{P}\boldsymbol{B}_e = \boldsymbol{P}\boldsymbol{B}_2 - \boldsymbol{P}\boldsymbol{J}\boldsymbol{B}_1 \tag{55}
$$

it yields using (17) and (46)

$$
\begin{aligned}
\boldsymbol{P}\boldsymbol{J} &= \boldsymbol{P}\,[\, \boldsymbol{j}_1 \; \cdots \; \boldsymbol{j}_m \,] \\
&= \boldsymbol{P}\,[\, \boldsymbol{J}_{d1} \; \cdots \; \boldsymbol{J}_{dm} \,]\,\boldsymbol{L} \\
&= [\, \boldsymbol{R}_1 \; \cdots \; \boldsymbol{R}_m \,]\,\boldsymbol{L} \\
&= \boldsymbol{R}\boldsymbol{L}
\end{aligned}
\tag{56}
$$

and so, consequently, collecting these results,

$$
\boldsymbol{P}\boldsymbol{B}_e = \boldsymbol{P}\boldsymbol{B}_2 - \boldsymbol{R}\boldsymbol{L}\boldsymbol{B}_1 \tag{57}
$$

Because the relation $\boldsymbol{P}\boldsymbol{A}_e + \boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{P}$ can be written as (34) then with (57) the inequality (54) implies (43). Thus any feasible solution of $\boldsymbol{P}$ and the set of $\boldsymbol{R}_k$ must satisfy (33)-(36). This completes the proof. ∎

To exploit the affine property of linear Metzlerian models, a slack matrix variable can be incorporated into the set of LMIs. The main result is that the observer matrices are decoupled from the Lyapunov matrix $\boldsymbol{P}$ to reduce conservatism of such solutions.

Theorem 4. The reduced-order Metzler observer (20) to strictly Metzler system (1), (2) is asymptotically stable with the quadratic performance γ if for given positive $\delta \in \mathbb{R}_+$ there exists diagonal positive definite matrices $\boldsymbol{P}, \boldsymbol{S}, \boldsymbol{R}_k \in \mathbb{R}_+^{(n-m)\times(n-m)}$ and a positive scalar $\gamma \in \mathbb{R}_+$ such that for $j = 1, 2, \dots n-m$, $h = 1, 2, \dots n-m-1$, $k = 1, 2, \dots n-m$

$$
\boldsymbol{P} \succ 0, \quad \boldsymbol{S} \succ 0, \quad \gamma > 0 \tag{58}
$$

$$
\begin{bmatrix} \boldsymbol{\Psi}_{11} & * & * & * \\ \boldsymbol{\Psi}_{21} & -2\delta\boldsymbol{S} & * & * \\ (\boldsymbol{S}\boldsymbol{B}_2 - \boldsymbol{R}\boldsymbol{L}\boldsymbol{B}_1)^{\mathrm{T}} & \delta(\boldsymbol{S}\boldsymbol{B}_2 - \boldsymbol{R}\boldsymbol{L}\boldsymbol{B}_1)^{\mathrm{T}} & -\gamma\boldsymbol{I}_r & * \\ \boldsymbol{A}_{12} & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{59}
$$

$$
\boldsymbol{S}\boldsymbol{A}_{22}(j,j)_{(\triangle)} - \sum_{k=1}^{n-m} \boldsymbol{R}_k\boldsymbol{C}_{dk} \prec 0 \tag{60}
$$

$$
\boldsymbol{S}\boldsymbol{T}^h\boldsymbol{A}_{22}(j+h,j)_{(\triangle)}\boldsymbol{T}^{h\mathrm{T}} - \sum_{k=1}^{m-m} \boldsymbol{R}_k\boldsymbol{T}^h\boldsymbol{C}_{dk}\boldsymbol{T}^{h\mathrm{T}} \succ 0 \tag{61}
$$

where

$$
\boldsymbol{\Psi}_{11} = \boldsymbol{S}\boldsymbol{A}_{22} + \boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{S} - \sum_{k=1}^{n-m} (\boldsymbol{R}_k\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{dk} + \boldsymbol{C}_{dk}\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_k) \tag{62}
$$

$$
\boldsymbol{\Psi}_{21} = \boldsymbol{P} - \boldsymbol{S} + \delta\boldsymbol{S}\boldsymbol{A}_{22} - \delta\sum_{k=1}^{n-m} \boldsymbol{R}_k\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{dk} \tag{63}
$$

and the remaining design parameter are defined in (37)-(40), (46).

When the above conditions hold, the estimator gain matrix is given by

$$
\boldsymbol{J}_{dk} = \boldsymbol{S}^{-1}\boldsymbol{R}_k, \quad \boldsymbol{j}_k = \boldsymbol{J}_{dk}\boldsymbol{l}, \quad \boldsymbol{J} = [\, \boldsymbol{j}_1 \; \cdots \; \boldsymbol{j}_m \,] \tag{64}
$$

Proof. Using the relation (20) where $\boldsymbol{v}(t) = \boldsymbol{0}$, then with a positive definite diagonal matrix $\boldsymbol{S} \in \mathbb{R}_+^{(n-m)\times(n-m)}$ and a positive scalar $\delta \in \mathbb{R}_+$ it yields

$$
\boldsymbol{s}_{2e}^{\mathrm{T}}(t)(\boldsymbol{A}_e\boldsymbol{p}_{2e}(t) + \boldsymbol{B}_e\boldsymbol{u}(t) - \dot{\boldsymbol{p}}_{2e}(t)) = \boldsymbol{0} \tag{65}
$$

where

$$
\boldsymbol{s}_{2e}^{\mathrm{T}}(t) = \boldsymbol{p}_{2e}^{\mathrm{T}}(t)\boldsymbol{S} + \delta\dot{\boldsymbol{p}}_{2e}^{\mathrm{T}}(t)\boldsymbol{S} \tag{66}
$$

Thus, adding (65) as well as its transpose to (49) gives

$$
\begin{aligned}
\dot{v}(\boldsymbol{p}_{2e}(t)) = {} & \boldsymbol{p}_{2e}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{p}}_{2e}(t) + \dot{\boldsymbol{p}}_{2e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{p}_{2e}(t) + \\
& + \boldsymbol{s}_{2e}^{\mathrm{T}}(t)(\boldsymbol{A}_e\boldsymbol{p}_{2e}(t) + \boldsymbol{B}_e\boldsymbol{u}(t) - \dot{\boldsymbol{p}}_{2e}(t)) \\
& + (\boldsymbol{A}_e\boldsymbol{p}_{2e}(t) + \boldsymbol{B}_e\boldsymbol{u}(t) - \dot{\boldsymbol{p}}_{2e}(t))^{\mathrm{T}}\boldsymbol{s}_{2e}(t) \\
& + \gamma^{-1}\boldsymbol{p}_{2e}^{\mathrm{T}}(t)\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{A}_{12}\boldsymbol{p}_{2e}(t) - \gamma\boldsymbol{u}^{T}(t)\boldsymbol{u}(t) \\
& < 0
\end{aligned}
\tag{67}
$$

Constructing a new composed representation

$$
\boldsymbol{p}_{2e}^{\circ\mathrm{T}}(t) = \left[\, \boldsymbol{p}_{2e}^{\mathrm{T}}(t) \;\; \dot{\boldsymbol{p}}_{2e}^{\mathrm{T}}(t) \;\; \boldsymbol{u}^{\mathrm{T}}(t) \,\right] \tag{68}
$$

it can now be found rewritten form of (67)

$$
\dot{v}(\boldsymbol{p}_{2e}(t)) = \boldsymbol{p}_{2e}^{\circ T}(t)\boldsymbol{P}^{\circ}\boldsymbol{p}_{2e}^{\circ}(t) < 0 \tag{69}
$$

where

$$
\boldsymbol{P}^{\circ} = \begin{bmatrix} \boldsymbol{S}\boldsymbol{A}_e + \boldsymbol{A}_e^{T}\boldsymbol{S} + \gamma^{-1}\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{A}_{12} & \boldsymbol{P} - \boldsymbol{S} + \delta\boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{S} & \boldsymbol{S}\boldsymbol{B}_e \\ \boldsymbol{P} - \boldsymbol{S} + \delta\boldsymbol{S}\boldsymbol{A}_e & -2\delta\boldsymbol{S} & \delta\boldsymbol{S}\boldsymbol{B}_e \\ \boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{S} & \delta\boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{S} & -\gamma\boldsymbol{I}_r \end{bmatrix} \tag{70}
$$

and the corresponding negative definite matrix inequality implying from (70) is

$$\begin{bmatrix} \boldsymbol{S}\boldsymbol{A}_e+\boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{S} & \boldsymbol{P}-\boldsymbol{S}+\delta\boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{S} & \boldsymbol{S}\boldsymbol{B}_e & \boldsymbol{A}_{12}^{\mathrm{T}} \\ \boldsymbol{P}-\boldsymbol{S}+\delta\boldsymbol{S}\boldsymbol{A}_e & -2\delta\boldsymbol{S} & \delta\boldsymbol{S}\boldsymbol{B}_e & \boldsymbol{0} \\ \boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{S} & \delta\boldsymbol{B}_e^{\mathrm{T}}\boldsymbol{S} & -\gamma\boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{A}_{12} & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_m \end{bmatrix} < 0 \quad (71)$$

Since, in analogy with (34), it can consider the following relation expression

$$\begin{aligned} \boldsymbol{\Psi}_{11} &= \boldsymbol{S}\boldsymbol{A}_e+\boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{S} \\ &= \boldsymbol{S}\boldsymbol{A}_{22}+\boldsymbol{A}_{22}^{\mathrm{T}}\boldsymbol{S}-\sum_{k=1}^{n-m}(\boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{dk}+\boldsymbol{C}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{R}_k) \end{aligned} \quad (72)$$

then (62) is obtained from (72), where it has to be redefined that

$$\boldsymbol{R}_k = \boldsymbol{S}\boldsymbol{J}_{dk} \quad (73)$$

Since induced relation implying from (72) is

$$\boldsymbol{S}\boldsymbol{A}_e = \boldsymbol{S}\boldsymbol{A}_{22} - \sum_{k=1}^{n-m}\boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{dk} \quad (74)$$

and plugging this in the denotation

$$\boldsymbol{\Psi}_{21} = \boldsymbol{P}-\boldsymbol{S}+\delta\boldsymbol{S}\boldsymbol{A}_e \quad (75)$$

then (74), (75) imply (63). Finally, supplanting (57) with (73) as

$$\boldsymbol{S}\boldsymbol{B}_e = \boldsymbol{S}\boldsymbol{B}_2 - \boldsymbol{R}\boldsymbol{L}\boldsymbol{B}_1 \quad (76)$$

then (71) implies (59) and (35), (36) are superseded by (60), (61), respectively. This concludes the proof. ∎

Note, the above given theorems, which could be potentially considered as equivalent, give generally different solutions, while conditions formulated in Theorem 4 are often denoted as the enhanced design conditions.

Remark 1. The design objective is to construct a reduced order observer for strictly Metzlerian linear systems such that the associated synthesis conditions are formulated in terms of LMIs. The exploited reduced order observer structure softens requirement on the strictly Metzler structure of the complex systems matrix $\boldsymbol{A}$, because synthesis principle results, if a solution exists, in a strictly positive matrix $\boldsymbol{J}$ of the observer gain already when the $\boldsymbol{A}_{22}$ matrix is strictly Metzler. This potentially makes it possible to permutate the system state variables to obtain only a strictly Metzler matrix $\boldsymbol{A}_{22}$. If such a structure cannot be found for a given Metzler system, it is necessary to create a state description by permutation of the state variables in which at least some rows of the matrix $\boldsymbol{A}_{22}$ do not contain zero elements and look for a solution with nonnegative matrix $\boldsymbol{J}$ using the methodology given in Krokavec and Filasová (2019).

Remark 2. Conditions requiring the strictly Metzler structure are only applied in construction of the Hurwitz matrix $\boldsymbol{A}_e$. Therefore, it is not possible to expect that generally non-square matrices $\boldsymbol{B}_e$, $\boldsymbol{A}_v$, $\boldsymbol{A}_v^{\circ}$ will be positive or nonnegative, although for some cases such solutions may exist and, consequently, only a stable Metzler structure of the matrix $\boldsymbol{A}_e$ can be a solution criterion. The exception is the case when $n-m=1$ when zero elements cannot occur in $\boldsymbol{A}_{22}$ ($\boldsymbol{A}_{22}$ is a negative scalar quantity) and the result is a positive row vector representing observer gain $\boldsymbol{J}$. In the last case, this positivity applies to a strictly as well as non-strictly Metzler structure of the matrix $\boldsymbol{A}$.

4. ILLUSTRATIVE NUMERICAL EXAMPLE

The considered system is represented by the model (18), (19) and the system model parameters

$$\boldsymbol{A} = \left[\begin{array}{cc|cc} -3.3800 & 2.2080 & 4.7150 & 2.6760 \\ 1.8810 & -4.2900 & 2.0500 & 0.6750 \\ \hline 2.0670 & 4.2730 & -6.6540 & 2.8930 \\ 1.1480 & 2.2730 & 1.3430 & -2.1040 \end{array}\right]$$

$$\boldsymbol{B} = \left[\begin{array}{cc} 0.1500 & 0.1888 \\ 0.1679 & 0.1030 \\ \hline 0.1436 & 0.1146 \\ 0.1036 & 0.1701 \end{array}\right], \quad \boldsymbol{C}_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and it is straightforward to separate matrices in blocks such that

$$\boldsymbol{A}_{11} = \begin{bmatrix} -3.3800 & 2.2080 \\ 1.8810 & -4.2900 \end{bmatrix}, \ \boldsymbol{A}_{12} = \begin{bmatrix} 4.7150 & 2.6760 \\ 2.0500 & 0.6750 \end{bmatrix}$$

$$\boldsymbol{A}_{21} = \begin{bmatrix} 2.0670 & 4.2730 \\ 1.1480 & 2.2730 \end{bmatrix}, \ \boldsymbol{A}_{22} = \begin{bmatrix} -6.6540 & 2.8930 \\ 1.3430 & -2.1040 \end{bmatrix}$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 0.1500 & 0.1888 \\ 0.1679 & 0.1030 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0.1436 & 0.1146 \\ 0.1036 & 0.1701 \end{bmatrix}$$

One can verify that the matrix $\boldsymbol{A}$ is strictly Metzler but not Hurwitz, while both sub-matrices $\boldsymbol{A}_{11}$, $\boldsymbol{A}_{22}$ are strictly Metzler and Hurwitz.

To implement in the following, the prescribed design parameter assumptions dictate

$$\boldsymbol{A}_{22}(j,j)_{(1\leftrightarrow 2)/2} = \begin{bmatrix} -6.6540 & 0 \\ 0 & -2.1040 \end{bmatrix}$$

$$\boldsymbol{A}_{22}(j+1,j)_{(1\leftrightarrow 2)/2} = \begin{bmatrix} 1.3430 & 0 \\ 0 & 2.8930 \end{bmatrix}$$

$$\boldsymbol{C}_{d1} = \begin{bmatrix} 4.7150 & 0 \\ 0 & 2.6760 \end{bmatrix}, \quad \boldsymbol{C}_{d2} = \begin{bmatrix} 2.0500 & 0 \\ 0 & 0.6750 \end{bmatrix}$$

$$\boldsymbol{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

With observer gain design according Theorem 2, SeDuMi package produces the solution

$$\boldsymbol{P} = \begin{bmatrix} 0.1582 & 0 \\ 0 & 0.5022 \end{bmatrix}, \ \boldsymbol{R}_1 = \begin{bmatrix} 0.0371 & 0 \\ 0 & 0.0409 \end{bmatrix}$$

$$\boldsymbol{R}_2 = \begin{bmatrix} 0.0635 & 0 \\ 0 & 0.0596 \end{bmatrix}, \ \boldsymbol{J} = \begin{bmatrix} 0.2347 & 0.4016 \\ 0.0814 & 0.1186 \end{bmatrix}$$

$$\boldsymbol{A}_e = \begin{bmatrix} -8.5840 & 1.9938 \\ 0.7161 & -2.4019 \end{bmatrix}, \ \boldsymbol{B}_e = \begin{bmatrix} 0.0410 & 0.0289 \\ 0.0715 & 0.1425 \end{bmatrix}$$

$$\boldsymbol{A}_v = \begin{bmatrix} 2.1051 & 5.4774 \\ 1.2000 & 2.6021 \end{bmatrix}, \ \boldsymbol{A}_v^{\circ} = \begin{bmatrix} 0.2523 & 2.2668 \\ 1.1726 & 2.6048 \end{bmatrix}$$

It can see that with strictly Metzler $\boldsymbol{A}_{22}$ the reduced-order observer gain $\boldsymbol{J}$ is strictly positive and the reduced-order observer matrix $\boldsymbol{A}_e$ is strictly Metzler and Hurwitz, where the eigenvalue spectrum of $\boldsymbol{A}_e$ is

$$\rho(\boldsymbol{A}_e) = \{-8.8069, \ -2.1790\}$$

With the prescribed tuning parameter $\delta = 0.8$ the solution of (58)-(61), obtained using the same solver, is given as

$$\boldsymbol{P} = \begin{bmatrix} 28.4625 & 0 \\ 0 & 24.7150 \end{bmatrix}, \ \boldsymbol{S} = \begin{bmatrix} 4.6436 & 0 \\ 0 & 14.3068 \end{bmatrix}$$

$$\boldsymbol{R}_1 = \begin{bmatrix} 0.7040 & 0 \\ 0 & 0.6617 \end{bmatrix}, \ \boldsymbol{R}_2 = \begin{bmatrix} 3.2671 & 0 \\ 0 & 2.7689 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} 0.1516 & 0.7036 \\ 0.0463 & 0.1935 \end{bmatrix}$$

$$\boldsymbol{A}_e = \begin{bmatrix} -8.8112 & 2.0124 \\ 0.7282 & -2.3584 \end{bmatrix}, \ \boldsymbol{B}_e = \begin{bmatrix} 0.0027 & 0.0135 \\ 0.0642 & 0.1414 \end{bmatrix}$$

$$\boldsymbol{A}_v = \begin{bmatrix} 1.2560 & 6.9565 \\ 0.9403 & 3.0011 \end{bmatrix}, \ \boldsymbol{A}_v^{\circ} = \begin{bmatrix} 0.0132 & 1.1467 \\ 0.9416 & 3.0570 \end{bmatrix}$$

Analyzing this result it is obvious that the reduced-order observer gain matrix $\boldsymbol{J}$ is strictly positive and checking the eigenvalue spectrum

$$\rho(\boldsymbol{A}_e) = \{-9.0308, \ -2.1388\}$$

it can observe that the reduced-order observer matrix $\boldsymbol{A}_e$ is strictly Metzler and Hurwitz. Note, setting the tuning parameter such that $\delta \geq 1$ the resulting matrix $\boldsymbol{A}_v^{\circ}$ is real signum indefinite.

Therewithal, it can see that both solutions match well together and the dynamic performance of the first solution is acceptable although a little bit worse than that achieved by the conditions related to Bounded Real Lemma structure in Theorem 4.

As a comparison, nonnegative reduced-order observer gain matrix can be obtained defining the structured matrix variables

$$\boldsymbol{R}_1 = \begin{bmatrix} r_{11} & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{R}_2 = \begin{bmatrix} r_{21} & 0 \\ 0 & 0 \end{bmatrix}$$

where $r_{11} > 0$, $r_{21} > 0$, $r_{11}, r_{21} \in \mathbb{R}_+$.

Therefore, a feasible and stable solution of the design conditions (42)-(45) is

$$\boldsymbol{P} = \begin{bmatrix} 2.3290 & 0 \\ 0 & 4.9633 \end{bmatrix}, \ \boldsymbol{R}_1 = \begin{bmatrix} 0.6407 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{R}_2 = \begin{bmatrix} 0.8209 & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{J} = \begin{bmatrix} 0.2751 & 0.3525 \\ 0 & 0 \end{bmatrix}$$

$$\boldsymbol{A}_e = \begin{bmatrix} -8.6735 & 1.9190 \\ 1.3430 & -2.1040 \end{bmatrix}, \ \boldsymbol{B}_e = \begin{bmatrix} 0.0432 & 0.0264 \\ 0.1036 & 0.1701 \end{bmatrix}$$

$$\boldsymbol{A}_v = \begin{bmatrix} 2.3338 & 5.1776 \\ 1.1480 & 2.2730 \end{bmatrix}, \ \boldsymbol{A}_v^{\circ} = \begin{bmatrix} -0.0521 & 2.1206 \\ 1.5174 & 2.7463 \end{bmatrix}$$

where $\boldsymbol{J}$ is nonnegative matrix and stable eigenvalue spectrum of $\boldsymbol{A}_e$ is

$$\rho(\boldsymbol{A}_e) = \{-9.0448, \ -1.7327\}$$

Comparing

$$\boldsymbol{A}_{22} = \begin{bmatrix} -6.6540 & 2.8930 \\ 1.3430 & -2.1040 \end{bmatrix}, \ \boldsymbol{A}_e = \begin{bmatrix} -8.6735 & 1.9190 \\ 1.3430 & -2.1040 \end{bmatrix}$$

it can see that the solution is independent on the second row of $\boldsymbol{A}_{22}$. Then, evidently, it can summarize that applying the methodology presented in Krokavec and Filasová (2019) this also yields for the same observer gain $\boldsymbol{J}$ so that adequately

$$\boldsymbol{A}_{22} = \begin{bmatrix} -6.6540 & 2.8930 \\ 0 & -2.1040 \end{bmatrix}, \ \boldsymbol{A}_e = \begin{bmatrix} -8.6735 & 1.9190 \\ 0 & -2.1040 \end{bmatrix}$$

if the last mentioned Metzler matrix $\boldsymbol{A}_{22}$ contains a zero item in the second row. In this case

$$\rho(\boldsymbol{A}_e) = \{-8.6735, \ -2.1040\}$$

and such resulting $\boldsymbol{A}_e$ is Metzler and Hurwitz, too.

5. CONCLUDING REMARKS

The problem of asymptotic stability of the reduced-order observers for linear continuous-time Metzlerian systems is investigated in the paper. The novelty of this work lies in LMI definition of the observer parametric constraints and in the partly exploited H_∞ approach in observer stability sustaining. The proposed design method guarantees that, if exists, the observer gain matrix is positive (nonnegative) and the reduced-order dynamics is autonomous while maintaining the Metzler structure of the reduced-order observer system matrix.

The achievable performance within the proposed design conditions is guaranteed no worse than that derived from standard Lyapunov formulation. Presented illustrative example documents the design approaches provide an efficient and systematic way for the synthesis of reduced-order observers for linear continuous-time Metzlerian systems.

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