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Exponential Stabilization of Discrete Nonlinear Time-Varying Systems

Adam Czornik * **Evgenii Makarov** ** **Michał Niezabitowski** *
Svetlana Popova *** **Vasilii Zaitsev** ***

* *Faculty of Automatic Control, Electronics and Computer Science, Silesian University of Technology, Gliwice, 44-100 Poland (e-mail: adam.czornik@polsl.pl, michal.niezabitowski@polsl.pl).*
** *Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, 220072 Belarus (e-mail: jcm@im.bas-net.by).*
*** *Udmurt State University, Izhevsk, 426034 Russia (e-mail: udsu.popova.sn@gmail.com, verba@udm.ru)*

Abstract: We consider a discrete nonlinear control time-varying system $x(k+1) = f\big(k, x(k), u(k)\big)$, $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$. A control process of this system is a pair $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ consisting of a control $\big(\widehat{u}(k)\big)_{k\in\mathbb{N}}$ and some solution $\big(\widehat{x}(k)\big)_{k\in\mathbb{N}}$ of the system with this control. We assume that the control process is defined for all $k \in \mathbb{N}$. We have obtained sufficient conditions for uniform and non-uniform (with respect to the initial moment) exponential stabilization of the control process with any pregiven decay of rate. Exponential convergence to zero of the deviation of both the state vector and the control vector is guaranteed. The result is based on the property of uniform complete controllability (in the sense of Kalman) for a system of linear approximation.

Keywords: discrete nonlinear time-varying system, controllability, stabilization.

1. INTRODUCTION

The problem of stability by the linear approximation has been intensively investigated for continuous-time systems since the fundamental Lyapunov's paper (Lyapunov (1956)). He proved that if the system of the first approximation is regular and all its Lyapunov exponents are negative, then the solution of the original system is asymptotically stable. In 1930, it was stated by O. Perron that the requirement of regularity of the first approximation is substantial. He constructed an example of the second-order system of the first approximation, which has negative characteristic exponents along a zero solution of the original system but, at the same time, this zero solution of the original system is Lyapunov unstable. Furthermore, in a certain neighborhood of this zero solution almost all solutions of the original system have positive characteristic exponents. A very exhausting review of results obtained in this area may be found in Izobov (2001). Another approach to this problem is presented in Dai (2006). For discrete-time systems in (Agarwal, 2000, Theorem 5.6.2), it is shown that uniform exponential stability of

$$x(k+1) = A(k)x(k), \quad k \in \mathbb{N},\ x \in \mathbb{R}^n,$$

implies uniform exponential stability of

$$y(k+1) = A(k)y(k) + f(k, y(k)),$$

for all $f \in \bigcup_{m>1} F_m$, where the class F_m consists of all functions $g\colon \mathbb{N}\times\mathbb{R}^n \to \mathbb{R}^n$ for which there exists a constant C_g such that

$$\|g(k,x)\| \le C_g \|x\|^m$$

for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. The converse theorem has been shown in (Gyori and Pituk, 2001, Theorem 4). A natural continuation of this research is a problem of stabilizing a nonlinear time-varying control system on the base of the linearized system. This area is, however in very early stage and some preliminary results and approaches may by found in Cai et al. (2012); Byrnes et al. (1993); Fu and Abed (1991). In this paper we investigate a discrete nonlinear time-varying system and analyzing its linear approximation. We obtain sufficient conditions for uniform and non-uniform exponential stabilization with any given rate of decay. The paper is organized as follows. In the next section we introduce necessary notations and definitions. The main result is formulated in Section 3. In Section 4 an illustrative example is presented.

2. DEFINITIONS

Let us introduce some denotations. Suppose $\mathbb{R}^n = \big\{x = \mathrm{col}(x_1,\dots,x_n)\colon x_i \in \mathbb{R}\big\}$ is the linear space of column vectors over $\mathbb{R}$; $\|x\| = \sqrt{x^T x}$ is the (Euclidean) norm in $\mathbb{R}^n$, where T denotes the transposition; $\mathrm{B}^n_h(\widehat{x}) := \{x \in \mathbb{R}^n\colon \|x - \widehat{x}\| < h\}$; $\mathbb{R}^{n\times m}$ denotes the space of all real $n \times m$-matrices with the spectral norm, i.e., with the operator norm induced in $\mathbb{R}^{n\times m}$ by the Euclidean norms in $\mathbb{R}^n$ and $\mathbb{R}^m$; $I \in \mathbb{R}^{n\times n}$ is the identity matrix. For any sequence $\psi = \big(\psi(k)\big)_{k\ge k_0} \subset \mathbb{R}$ which is not equal to zero finally, denote by $\lambda[\psi] = \limsup_{k\to\infty} k^{-1}\ln|\psi(k)|$ *the Lyapunov exponent* of ψ, by $\beta[\psi] = \limsup_{k,s\to\infty} s^{-1}\ln|\psi(k+s)/\psi(k)|$ denote *the Bohl exponent* of ψ.

Consider a linear discrete time-varying system

$$x(k+1) = A(k)x(k), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}^n. \tag{1}$$

Denote by $X(k,s)$ the transition matrix of (1), i.e.,

$$X(s,s) = I,$$
$$X(k,s) = A(k-1)\cdots A(s) \quad \text{for} \quad k > s.$$

Additionally, when $A = \big(A(k)\big)_{k\in\mathbb{N}}$ consists of invertible matrices we define $X(s,k) := \big(X(k,s)\big)^{-1}$ for $k > s$.

A bounded sequence $\big(D(k)\big)_{k\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ of invertible matrices such that $\big(D^{-1}(k)\big)_{k\in\mathbb{N}}$ is bounded is called *a Lyapunov sequence.* If $\big(D(k)\big)_{k\in\mathbb{N}}$ is a Lyapunov sequence, then the transformation which transforms a sequence $\big(x(k)\big)_{k\in\mathbb{N}} \subset \mathbb{R}^n$ into a sequence $\big(y(k)\big)_{k\in\mathbb{N}} \subset \mathbb{R}^n$ according to the formula

$$y(k) = D(k)x(k)$$

is called *a Lyapunov transformation.* It reduces (1) to

$$y(k+1) = C(k)y(k), \quad k \in \mathbb{N}, \quad y \in \mathbb{R}^n, \tag{2}$$

where

$$C(k) = D(k+1)A(k)D^{-1}(k). \tag{3}$$

Systems (1) and (2) connected by (3) where $\big(D(k)\big)_{k\in\mathbb{N}}$ is a Lyapunov sequence are called *dynamically equivalent*, see Popova (2018) (or kinematically similar, see Gohberg et al. (1996)).

Consider a discrete nonlinear time-varying system

$$x(k+1) = f\big(k, x(k), u(k)\big), \quad k \in \mathbb{N}. \tag{4}$$

Here $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^r$ is a control vector. Suppose that for any $k \in \mathbb{N}$ the function $(x,u) \mapsto f(k,x,u)$ and its derivatives with respect to x and with respect to u are continuous on $\mathbb{R}^n \times \mathbb{R}^r$ (or, at least, in some neighborhood of the admissible control process, see below).

Definition 1. We call by *an admissible control process of system* (4) any sequence $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ such that $\big(\widehat{u}(k)\big)_{k\in\mathbb{N}} \subset \mathbb{R}^r$ is some control sequence, and $\big(\widehat{x}(k)\big)_{k\in\mathbb{N}} \subset \mathbb{R}^n$ is a solution sequence of the system

$$x(k+1) = f\big(k, x(k), \widehat{u}(k)\big), \quad k \in \mathbb{N},$$

with some initial condition.

Let an admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ of system (4) be fixed. We consider the problem of exponential stabilization of this process by state feedback control $u(k) = \mathfrak{u}\big(k, x(k)\big)$.

Definition 2. We say that the admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ of system (4) is *exponentially stabilizable with the decay rate $\varkappa > 0$ by state feedback control* if for any $k_0 \in \mathbb{N}$ there exist $\delta > 0$, $c > 0$ such that for any $x_0 \in \mathrm{B}^n_\delta\big(\widehat{x}(k_0)\big)$ there exists a control $\widetilde{u} = \big(\widetilde{u}(k)\big)_{k\geq k_0} = \big(\widetilde{\mathfrak{u}}\big(k, x(k)\big)\big)_{k\geq k_0}$ satisfying inequality

$$\|\widetilde{u}(k) - \widehat{u}(k)\| \leq c e^{-\varkappa(k-k_0)}, \quad k \geq k_0,$$

and such that the solution $\widetilde{x}(k)$ of the system (4) with $u(k) = \widetilde{u}(k)$ and with the initial condition $x(k_0) = x_0$ satisfies the inequality

$$\|\widetilde{x}(k) - \widehat{x}(k)\| \leq c e^{-\varkappa(k-k_0)}, \quad k \geq k_0.$$

If $\delta > 0$ and $c > 0$ do not depend on k_0, then we say that the admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ of system (4) is *uniformly exponentially stabilizable with the decay rate $\varkappa > 0$ by state feedback control.*

Remark 3. Note that along with exponential stability of the state vector, the definition requires exponential stability of the control vector. This explains why we say about exponential stabilization of *the control process.*

In problems of stabilization of linear time-varying systems, the property of uniform complete controllability plays an important role. Consider a linear control system

$$y(k+1) = A(k)y(k) + B(k)v(k), \quad k \in \mathbb{N}, \tag{5}$$

here $y \in \mathbb{R}^n$ is a state vector, $v \in \mathbb{R}^r$ is a control vector. Denote by $Y(k,s)$ the transition matrix of the corresponding free system

$$y(k+1) = A(k)y(k), \quad k \in \mathbb{N}.$$

For system (5), let us construct the following gramian

$$W(k,\tau) = \sum_{s=\tau}^{k-1} Y(k,s+1)B(s)B^T(s)Y^T(k,s+1). \tag{6}$$

Definition 4. System (5) is said to be *uniformly completely controllable* (Kwakernaak and Sivan, 1972, Definition 6.3) if there exists a $\vartheta \in \mathbb{N}$ and there exist $\alpha_i = \alpha_i(\vartheta) > 0$, $i = 1,2,3,4$, such that for all $\tau \in \mathbb{N}$ the following inequalities hold:

$$W(\tau+\vartheta,\tau) > 0,$$
$$0 < \alpha_1 I \leq W^{-1}(\tau+\vartheta,\tau) \leq \alpha_2 I,$$
$$0 < \alpha_3 I \leq Y^T(\tau+\vartheta,\tau)W^{-1}(\tau+\vartheta,\tau)Y(\tau+\vartheta,\tau) \leq \alpha_4 I.$$

This definition goes back to the definition of Kalman (1960) for linear systems with continuous time. The following criterion holds (the proof is given, e.g., in (Zaitsev et al., 2014, Theorem 4)).

Proposition 5. System (5) is uniformly completely controllable iff the following conditions hold:

(a) $A = \big(A(k)\big)_{k\in\mathbb{N}}$ is a Lyapunov sequence;

(b) $B = \big(B(k)\big)_{k\in\mathbb{N}}$ is bounded, i.e., $\sup_{k\in\mathbb{N}} \|B(k)\| < \infty$;

(c) there exist a natural ϑ and a positive ℓ such that for any $\tau \in \mathbb{N}$ and for any $x_1 \in \mathbb{R}^n$ there exists a control $u(k)$, $k = \tau, \ldots, \tau+\vartheta-1$, that transfers the solution of system (5) from the point $x(\tau) = 0$ into the point $x(\tau+\vartheta) = x_1$, and the inequality $\|u(k)\| \leq \ell\|x_1\|$ holds for all $k = \tau, \ldots, \tau+\vartheta-1$.

Remark 6. In Babiarz et al. (2017), the property of uniform complete controllability was used in the sense of fulfilling conditions (a), (b), (c). Due to Proposition 5 all statements of Babiarz et al. (2017) holds for the definition of uniform complete controllability in the sense of Definition 4.

3. MAIN RESULT

The main result of the paper is Theorem 9. The similar result (on non-uniform exponential stabilization) was proved for continuous-time systems in Zaitsev et al. (2010).

First, let us give some auxiliary propositions. The following lemma is a discrete analog of the Bihari Lemma (see Bihari (1956)).

Lemma 7. Let $p(k)$, $q(k)$, $k = 0,1,2,\ldots$, be non-negative number sequences such that $0 \leq p(0) \leq \sigma$,

$$p(k) \le \sigma + \sum_{j=0}^{k-1} q(j)\omega\big(p(j)\big), \quad k \in \mathbb{N},$$

where $\sigma > 0$, $\omega(\tau)$ is continuous on $[0,\infty)$, monotonically increasing and positive function for $\tau > 0$. Suppose that $\sum_{j=0}^{k-1} q(j) < \Omega(\infty)$ for all $k \in \mathbb{N}$, where $\Omega(\tau) = \int_\sigma^\tau \frac{ds}{\omega(s)}$. Then $p(k) \le \Omega^{-1}\left(\sum_{j=0}^{k-1} q(j)\right)$, here $\Omega^{-1}(s)$ is the inverse function to $\Omega(\tau)$.

The proof of Lemma 7 is given, e.g., in (Gaishun, 2001, § 11). Let us give a corollary from Lemma 7 for the function $\omega(\tau) = \tau^m$, $m > 1$ (see Demidovich (1969)). In that case

$$\Omega(\tau) = \frac{\sigma^{1-m} - \tau^{1-m}}{m-1}, \quad \Omega(\infty) = \frac{\sigma^{1-m}}{m-1},$$

$$\Omega^{-1}(s) = \sigma(1-(m-1)\sigma^{m-1}s)^{\frac{1}{1-m}}, \quad s \in \Big[0, \frac{\sigma^{1-m}}{m-1}\Big).$$

Corollary 8. Let $p(k)$, $q(k)$, $k = 0,1,2,\ldots$, be non-negative number sequences such that

$$0 \le p(0) \le \sigma,$$
$$p(k) \le \sigma + \sum_{j=0}^{k-1} q(j)\big(p(j)\big)^m, \quad k \in \mathbb{N}, \tag{7}$$

where $\sigma > 0$, $m > 1$. Suppose that

$$\sum_{j=0}^{\infty} q(j) < \frac{1}{(m-1)\sigma^{m-1}}. \tag{8}$$

Then

$$p(k) \le \sigma \left[1 - (m-1)\sigma^{m-1}\sum_{j=0}^{k-1} q(j)\right]^{\frac{1}{1-m}}, \quad k \in \mathbb{N}.$$

Theorem 9. Let $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ be an admissible control process of system (4) such that:

(1) the system (5) with $A(k) := \left.\frac{\partial f(k,x,u)}{\partial x}\right|_{(\widehat{x}(k),\widehat{u}(k))}$, $B(k) := \left.\frac{\partial f(k,x,u)}{\partial u}\right|_{(\widehat{x}(k),\widehat{u}(k))}$ is uniformly completely controllable;

(2) the following equality holds:

$$\begin{aligned} f\big(k, \widehat{x}(k)+y, \widehat{u}(k)+v\big) - f\big(k, \widehat{x}(k), \widehat{u}(k)\big) \\ = A(k)y + B(k)v + \varphi(k,y,v), \end{aligned} \tag{9}$$

where

$$\|\varphi(k,y,v)\| \le \psi(k)\left\|\begin{pmatrix} y \\ v\end{pmatrix}\right\|^m, \tag{10}$$

for all $k \in \mathbb{N}$, $(y,v) \in \mathrm{B}_h^n(0) \times \mathrm{B}_h^r(0)$, where $\lambda[\psi] \le 0$, $m > 1$.

Then for an arbitrary $\varkappa > 0$ the admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ is exponentially stabilizable with the decay rate $\varkappa$ by state feedback control.

Additionally, if $\beta[\psi] \le 0$, then for an arbitrary $\varkappa > 0$ the admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ is uniformly exponentially stabilizable with the decay rate $\varkappa$ by state feedback control.

Proof. Let us consider system (4) in a neighborhood of the admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$. Denote

$$y = x - \widehat{x}, \quad v = u - \widehat{u}.$$

Taking into account (9), we can rewrite system (4) in deviations in the following form:

$$y(k+1) = A(k)y(k) + B(k)v(k) + \varphi\big(k, y(k), v(k)\big) \tag{11}$$

on the set $k \in \mathbb{N}$, $(y,v) \in \mathrm{B}_h^n(0) \times \mathrm{B}_h^r(0)$, where $\varphi(k,y,v)$ satisfies condition (10).

Let an arbitrary $\varkappa > 0$ be given. Using the property of uniform complete controllability of the system (5) of the first approximation for (11) and (Babiarz et al., 2017, Theorem 4.3), we construct the control

$$v(k) = V(k)y(k), \quad k \in \mathbb{N}, \tag{12}$$

such that $\sup_{k\in\mathbb{N}} \|V(k)\| \le v_0$ and the closed-loop system

$$y(k+1) = \big(A(k) + B(k)V(k)\big)y(k)$$

is dynamically equivalent to the system

$$z(k+1) = e^{-\varkappa} z(k)$$

by means of some Lyapunov transformation

$$z(k) = D(k)y(k), \tag{13}$$

where $\sup_{k\in\mathbb{N}} \|D(k)\| \le d$, $\sup_{k\in\mathbb{N}} \|D^{-1}(k)\| \le d$ for some $d \ge 1$.

Denote $\varphi_1\big(k, y(k)\big) = \varphi\big(k, y(k), V(k)y(k)\big)$. The system (11) with the control (12) has the form

$$y(k+1) = \big(A(k) + B(k)V(k)\big)y(k) + \varphi_1\big(k, y(k)\big). \tag{14}$$

Let $k_0 \in \mathbb{N}$ be some initial time. Set $h_1 = \min\{h, h/v_0\}$. Let $k \ge k_0$, $\|y(k)\| \le h_1$. Then $\|y(k)\| \le h$ and $\|V(k)y(k)\| \le v_0 h_1 \le h$, therefore from (10) we obtain

$$\begin{aligned} &\|\varphi_1\big(k, y(k)\big)\| = \|\varphi\big(k, y(k), V(k)y(k)\big)\| \le \\ &\psi(k)\left\|\begin{pmatrix} y(k) \\ V(k)y(k)\end{pmatrix}\right\|^m = \psi(k)\big(\|y(k)\|^2 + \|V(k)y(k)\|^2\big)^{m/2} \\ &\le \psi(k)\Big(\big(1+v_0^2\big)\|y(k)\|^2\Big)^{m/2} = \psi(k)\big(1+v_0^2\big)^{m/2}\|y(k)\|^m \\ &=: \psi_1(k)\|y(k)\|^m. \end{aligned}$$

It follows that $\lambda[\psi_1] = \lambda[\psi]$, $\beta[\psi_1] = \beta[\psi]$.

Let us apply the transformation (13) to the system (14). We obtain the following system:

$$z(k+1) = e^{-\varkappa} z(k) + g\big(k, z(k)\big), \tag{15}$$

where $g\big(k, z(k)\big) = D(k+1)\varphi_1\big(k, D^{-1}(k)z(k)\big)$. Set $h_2 := h_1/d$. Let $k \ge k_0$, $\|z(k)\| \le h_2$. Then $\|D^{-1}(k)z(k)\| \le dh_2 = h_1$. Hence, we have

$$\begin{aligned} &\|g\big(k, z(k)\big)\| = \|D(k+1)\varphi_1\big(k, D^{-1}(k)z(k)\big)\| \le \\ &\le d\psi_1(k)\|D^{-1}(k)z(k)\big)\|^m \le d^{m+1}\psi_1(k)\|z(k)\|^m =: \\ &=: \phi(k)\|z(k)\|^m. \end{aligned}$$

It follows that $\lambda[\phi] = \lambda[\psi_1]$, $\beta[\phi] = \beta[\psi_1]$.

Let us choose the number $\epsilon > 0$ so small that the inequality $\epsilon < \varkappa(m-1)$ holds. Denote $\gamma := \varkappa(m-1) - \epsilon > 0$. Since $\lambda[\phi] = \lambda[\psi_1] = \lambda[\psi] \le 0$, it follows that there exists a $c_1 > 0$ (which, in general, depend on k_0) such that for all $j = 0,1,2,\ldots$ the inequality

$$\phi(k_0 + j) \le c_1 e^{\epsilon j} \tag{16}$$

holds. Moreover, if $\beta[\psi] \le 0$, then $\beta[\phi] \le 0$, and therefore there exists a $c_1 > 0$, which does not depend on k_0, such that (16) holds for all $j = 0, 1, 2, \ldots$. We set

$$\delta = \min\left\{ \frac{h_2}{d}, \frac{1}{2d}\left(\frac{1-e^{-\gamma}}{(m-1)c_1 e^{\varkappa}}\right)^{\frac{1}{m-1}}, \frac{1}{d}\left(\frac{(1-(1/2)^{m-1})}{c_1 e^{\varkappa}} \cdot \frac{1-e^{-\gamma}}{m-1}\right)^{\frac{1}{m-1}} \right\}. \quad (17)$$

Then the following inequalities hold

$$d\delta \le h_2,$$

$$d^{m-1}\delta^{m-1} < \frac{1-e^{-\gamma}}{(m-1)c_1 e^{\varkappa}}, \quad (18)$$

$$(m-1)d^{m-1}\delta^{m-1}\frac{c_1 e^{\varkappa}}{1-e^{-\gamma}} \le 1 - \left(\frac{1}{2}\right)^{m-1}. \quad (19)$$

Let $x_0 \in \mathrm{B}^n_\delta\big(\widehat{x}(t_0)\big)$. Set $y_0 = x_0 - \widehat{x}(t_0)$. Then $y_0 \in \mathrm{B}^n_\delta(0)$. Take $z_0 = D(k_0)y_0$. Then $\|z_0\| \le d\delta \le h_2$. Consider the initial value problem for the system (15) with the initial condition $z(k_0) = z_0$. By virtue of the Cauchy formula (Gaishun, 2001, p. 20), for any $k > k_0$, we have

$$z(k) = e^{-\varkappa(k-k_0)}z_0 + \sum_{j=0}^{k-k_0-1} e^{-\varkappa(k-k_0-1-j)} g\big(k_0+j, z(k_0+j)\big). \quad (20)$$

Denote $s := k - k_0$, $\nu(j) := z(k_0+j)$, $\phi_1(j) := \phi(k_0+j)$, $g_1\big(j,\nu(j)\big) := g\big(k_0+j, z(k_0+j)\big)$. Then we have

$$\|g_1\big(j,\nu(j)\big)\| = \|g\big(k_0+j, z(k_0+j)\big)\| \le \le \phi(k_0+j)\|z(k_0+j)\|^m = \phi_1(j)\|\nu(j)\|^m,$$

and $\phi_1(j) \le c_1 e^{\epsilon j}$, $j = 0, 1, 2, \ldots$. By (20), we have

$$\nu(s) = e^{-\varkappa s}z_0 + \sum_{j=0}^{s-1} e^{-\varkappa(s-1-j)} g_1\big(j,\nu(j)\big)$$

for all $s \in \mathbb{N}$. Hence

$$\|\nu(s)\| \le e^{-\varkappa s}\|z_0\| + \sum_{j=0}^{s-1} e^{-\varkappa(s-1-j)}\|g_1\big(j,\nu(j)\big)\| \le \le e^{-\varkappa s}\|z_0\| + \sum_{j=0}^{s-1} e^{-\varkappa(s-1-j)} c_1 e^{\epsilon j}\|\nu(j)\|^m. \quad (21)$$

Multiplying (21) by $e^{\varkappa s}$, we obtain

$$\|\nu(s)\|e^{\varkappa s} \le \|z_0\| + \sum_{j=0}^{s-1} c_1 e^{\varkappa} e^{j(\epsilon-(m-1)\varkappa)}\left(\|\nu(j)\|e^{\varkappa j}\right)^m.$$

Denoting $\xi(j) = \|\nu(j)\|e^{\varkappa j}$, we have

$$\xi(s) \le \|z_0\| + \sum_{j=0}^{s-1} c_1 e^{\varkappa} e^{-\gamma j}\big(\xi(j)\big)^m$$

for all $s \in \mathbb{N}$. Moreover,

$$\xi(0) = \|\nu(0)\| = \|z(k_0)\| = \|z_0\|. \quad (22)$$

We see that conditions (7) of Corollary 8 are fulfilled for $p(j) = \xi(j)$, $\sigma = \|z_0\|$, and

$$q(j) = c_1 e^{\varkappa} e^{-\gamma j}. \quad (23)$$

It follows from (23) that

$$\sum_{j=0}^{\infty} q(j) = c_1 e^{\varkappa}\sum_{j=0}^{\infty} e^{-\gamma j} = \frac{c_1 e^{\varkappa}}{1-e^{-\gamma}}. \quad (24)$$

Using (18), we have

$$\|z_0\|^{m-1} = \|D(k_0)y_0\|^{m-1} \le \le d^{m-1}\delta^{m-1} < \frac{1-e^{-\gamma}}{(m-1)c_1 e^{\varkappa}}.$$

Hence,

$$\frac{c_1 e^{\varkappa}}{1-e^{-\gamma}} < \frac{1}{(m-1)\|z_0\|^{m-1}}. \quad (25)$$

From (24) and (25) it follows that

$$\sum_{j=0}^{\infty} q(j) < \frac{1}{(m-1)\|z_0\|^{m-1}}.$$

It follows that condition (8) of Corollary 8 is fulfilled. By Corollary 8, for all $s \in \mathbb{N}$, we have

$$\xi(s) \le \|z_0\|\left[1 - (m-1)\|z_0\|^{m-1}\sum_{j=0}^{s-1} q(j)\right]^{\frac{1}{1-m}}. \quad (26)$$

Using (24) and (19), for all $s \in \mathbb{N}$, we have

$$(m-1)\|z_0\|^{m-1}\sum_{j=0}^{s-1} q(j) \le \le (m-1)d^{m-1}\delta^{m-1}\frac{c_1 e^{\varkappa}}{1-e^{-\gamma}} \le 1 - (1/2)^{m-1}. \quad (27)$$

From (26) and (27) it follows that

$$\xi(s) \le 2\|z_0\| \quad (28)$$

for all $s \in \mathbb{N}$. By (22), inequality (28) holds for $s = 0$ as well. Making reverse replacements from ξ to ν and then to z, we obtain

$$\|z(k)\| \le 2\|z_0\|e^{-\varkappa(k-k_0)} \le 2h_2 e^{-\varkappa(k-k_0)}, \quad k \ge k_0.$$

Then the solution $y(\cdot)$ of the initial value problem for the system (14) with the initial condition $y(k_0) = y_0$ is defined by the equality $y(k) = D^{-1}(k)z(k)$, and the following estimation holds:

$$\|y(k)\| \le \|D^{-1}(k)\|\|z(k\| \le 2dh_2 e^{-\varkappa(k-k_0)}, \quad k \ge k_0.$$

Finally, the solution $x(\cdot)$ of the initial value problem for the system (4) with the initial condition $x(k_0) = x_0$ and with control $u(k) = \widehat{u}(k) + v(k)$ is defined by the equality $x(k) = \widehat{x}(k) + y(k)$ and

$$\|x(k) - \widehat{x}(k)\| \le 2dh_2 e^{-\varkappa(k-k_0)}, \quad k \ge k_0.$$

For the control sequence $u(\cdot)$, we have the estimation $\|u(k) - \widehat{u}(k)\| = \|v(k)\| \le \|V(k)\|\|y(k\| \le 2v_0 dh_2 e^{-\varkappa(k-k_0)}$, $k \ge k_0$. Setting $c = 2dh_2\max\{1, v_0\}$, we obtain the required inequalities. The constant $c > 0$ does not depend on k_0. If $\beta[\psi] \le 0$, then c_1 does not depend on k_0, therefore $\delta > 0$ defined by (17) does not depend on k_0. The theorem is proved. □

4. EXAMPLE

Example 1. Let $\big(B(k)\big)_{k\in\mathbb{N}} \subset \mathbb{R}$ be an arbitrary Lyapunov sequence; $\sup_{k\in\mathbb{N}}|B(k)| \le \ell$, $\sup_{k\in\mathbb{N}}|B^{-1}(k)| \le \ell$, where $\ell \ge 1$. Consider a scalar nonlinear equation

$$x(k+1) = \frac{1}{x(k)} + B(k)u(k), \quad k \in \mathbb{N}. \quad (29)$$

Let us choose $\widehat{u}(k) = -B^{-1}(k)$, $k \in \mathbb{N}$. The equation (29) with $u(k) = \widehat{u}(k)$ takes the form

$$x(k+1) = \frac{1}{x(k)} - 1, \quad k \in \mathbb{N}. \tag{30}$$

Note that this equation has two equilibria: the unstable one $x_1 = \dfrac{\sqrt{5}-1}{2}$ and the locally asymptotically stable one $x_2 = -\dfrac{\sqrt{5}+1}{2}$. For an arbitrary $x_0 \neq x_1$ the solution $x(k)$ of (30) with the initial condition $x(1) = x_0$ is either defined for all $k \in \mathbb{N}$, and in this case $\lim\limits_{k\to\infty} x(k) = x_2$, or defined not for all $k \in \mathbb{N}$.

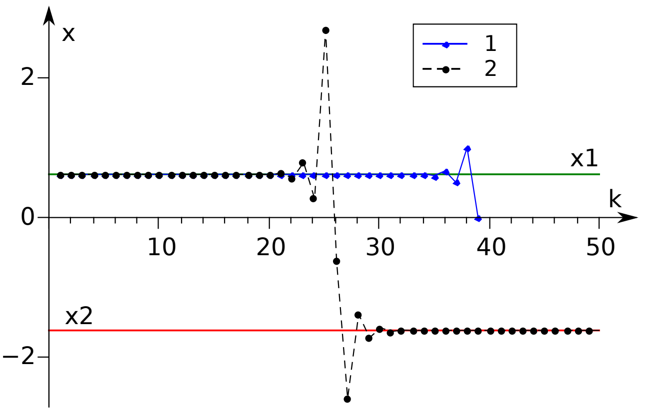

Fig. 1. Solutions of equation (30). 1 is a breaking solution with the initial condition $x_0 = w_{38}$. 2 is a non-breaking solution with the initial condition $x_0 = x_1 + 10^{-8}$.

We say that a solution of (30) is *breaking* if it is defined not for all $k \in \mathbb{N}$. Initial conditions of breaking solutions are numbers of the form $w_s = \dfrac{P_s}{Q_s}$, $s \in \mathbb{N}$, where $P_s = z_s$, $Q_s = z_{s+1}$, and $z_1, z_2, z_3, \ldots$ is the sequence of Fibonacci numbers, i.e., $z_1 = 1$, $z_2 = 1$, $z_s = z_{s-2} + z_{s-1}$ for $s \geq 3$. A solution $x(k)$ of (30) with the initial condition $x(1) = w_s$ is defined for $k = 1, 2, \ldots, s+1$, and

$$x(k) = \begin{cases} w_{s+1-k}, & \text{for } k = 1, \ldots, s, \\ 0, & \text{for } k = s+1, \end{cases}$$

and the solution does not exist for $k \geq s+2$.

The sequence w_s converges to x_1, and it is known from the number theory (see, e.g., (Vorobiev, 2002, p. 107, Legendre's Theorem)) that $|x_1 - w_s| \leq \dfrac{1}{Q_s Q_{s+1}} = \dfrac{1}{z_{s+1} z_{s+2}}$. From this estimation, it follows that for any $\delta > 0$ the neighborhood $\mathrm{B}^1_\delta(x_1)$ contains infinitely many initial conditions of breaking solutions of (30).

The only solution of the equation (30) satisfying condition $\lim\limits_{k\to\infty} x(k) = x_1$ is the equilibrium x_1. Let us set $\widehat{x}(k) \equiv x_1$, $k \in \mathbb{N}$. Thus, we have the admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ for the equation (29). From the above it follows that this process is unstable.

We pose the problem of exponential stabilization of this process. Denote $f(k, x, u) := \dfrac{1}{x} + B(k)u$. Let us construct

$$A := \frac{\partial f(k,x,u)}{\partial x}\bigg|_{\left(\widehat{x}(k),\widehat{u}(k)\right)} = -\frac{1}{x_1^2} = -\frac{3+\sqrt{5}}{2}.$$

The equation of the first approximation for (29) in the neighborhood of the control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ has the form

$$y(k+1) = Ay(k) + B(k)v(k), \quad k \in \mathbb{N}. \tag{31}$$

Let us consider the gramian (6) for the equation (31):

$$W(k+1, k) = B^2(k), \quad k \in \mathbb{N}.$$

Since $\big(B(k)\big)_{k\in\mathbb{N}}$ is a Lyapunov sequence, it follows that conditions of Definition 4 are fulfilled, i.e., the equation (31) is uniformly completely controllable. Thus, condition (1) of Theorem 9 is fulfilled. Now check condition (2) of Theorem 9. Take $h = x_1/2$. Then for all $k \in \mathbb{N}$ and $(y, v) \in \mathrm{B}^1_h(0) \times \mathrm{B}^1_h(0)$ the residual

$$\varphi(k, y, v) := f\big(k, \widehat{x}(k) + y, \widehat{u}(k) + v\big) - \\ - f\big(k, \widehat{x}(k), \widehat{u}(k)\big) - Ay - B(k)v$$

satisfies inequality

$$\|\varphi(k,y,v)\| = \Big|\frac{1}{x_1 + y} + B(k)\big(\widehat{u}(k) + v\big) - \\ - \frac{1}{x_1} - B(k)\widehat{u}(k) + \frac{y}{x_1^2} - B(k)v\Big| = \\ = \frac{y^2}{x_1^2(x_1+y)} \leq \frac{2y^2}{x_1^3} \leq \frac{2(y^2+v^2)}{x_1^3}.$$

Hence, inequality (10) holds, where $\psi(k) = \dfrac{2}{x_1^3}$, $m = 2$. Note that $\lambda[\psi] = \beta[\psi] = 0$. Thus, all conditions of Theorem 9 are fulfilled. From Theorem 9 it follows that for an arbitrary $\varkappa > 0$ the admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ is uniformly exponentially stabilizable with the decay rate $\varkappa$ by state feedback control. Let us construct stabilizing control.

Let an arbitrary $\varkappa > 0$ be given. Let us construct control $v(k)$ in the form (12), where $V(k) = \dfrac{e^{-\varkappa} - A}{B(k)}$. Then $|V(k)| \leq \ell\left(1 + \dfrac{3+\sqrt{5}}{2}\right) = \dfrac{\ell(5+\sqrt{5})}{2} =: v_0$. Set $h_1 = h/v_0 = x_1/(2v_0)$. Take $\psi_1(k) = \psi(k)(1 + v_0^2) = \dfrac{2(1+v_0^2)}{x_1^3} =: c_1$. We have $\phi(k) = \psi_1(k)$ and $h_2 = h_1$. Note that estimation (16) holds for $\epsilon = 0$, hence we can take $\gamma = \varkappa(m-1) = \varkappa$. From (17) it follows that $\delta = \min\left\{h_1, \dfrac{1-e^{-\varkappa}}{2c_1 e^{\varkappa}}\right\}$. It is easy to check that $\delta = \dfrac{1-e^{-\varkappa}}{2c_1 e^{\varkappa}}$ for any $\varkappa > 0$. Let $x_0 \in \mathrm{B}^1_\delta(x_1)$ and $k_0 \in \mathbb{N}$. Then control

$$\widetilde{u}(k) = \widetilde{\mathfrak{u}}(k, x(k)) := \widehat{u}(k) + V(k)\big(x(k) - x_1\big), \ k \geq k_0,$$

provides inequalities $\|\widetilde{x}(k) - \widehat{x}(k)\| \leq 2h_1 e^{-\varkappa(k-k_0)}$ and $\|\widetilde{u}(k) - \widehat{u}(k)\| \leq 2v_0 h_1 e^{-\varkappa(k-k_0)}$ for all natural $k \geq k_0$, where $\widetilde{x}(\cdot)$ is the solution of (29) with $u(k) = \widetilde{u}(k)$ with the initial condition $x(k_0) = x_0$. Hence, the admissible control process $\big(\widehat{x}(k), \widehat{u}(k)\big)_{k\in\mathbb{N}}$ is uniformly exponentially stabilized by means of constructed control $\widetilde{u}(\cdot)$.

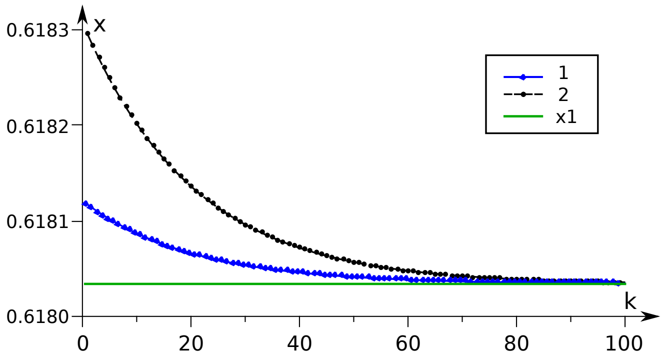

Fig. 2. Solutions of equation (29), where $B(k) = 1$, $u(k) = -1+V(k)\big(x(k)-x_1\big)$, $V(k) = e^{-\varkappa}-A$, $A = -\dfrac{3+\sqrt{5}}{2}$, $\varkappa = 0.05$, corresponding $\delta = 2.8\cdot 10^{-4}$. 1 is a solution with the initial condition $x_0 = w_{10}$. 2 is a solution with the initial condition $x_0 = x_1 + 2.5\cdot 10^{-4}$.

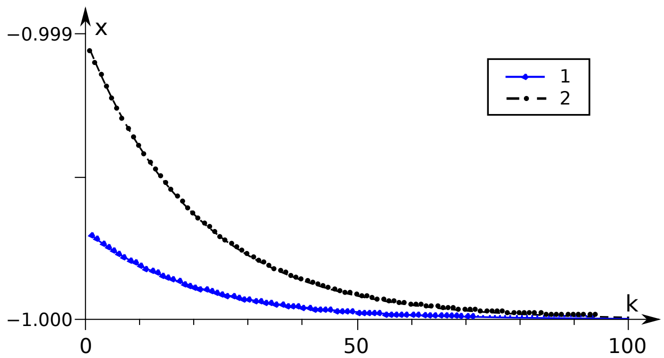

Fig. 3. Graphs of controls for equation (29), where $B(k) = 1$, $u(k) = -1+V(k)\big(x(k)-x_1\big)$, $V(k) = e^{-\varkappa}-A$, $A = -(3+\sqrt{5})/2$, $\varkappa = 0.05$: line 1 is a stabilizing control for solution with the initial condition $x_0 = w_{10}$; line 2 is a stabilizing control for solution with the initial condition $x_0 = x_1 + 2.5\cdot 10^{-4}$.

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