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Plug-and-play Solvability of the Power Flow Equations for Interconnected DC Microgrids with Constant Power Loads

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Abstract: In this paper we study a power grid which consists of interconnected DC microgrids. These microgrids are equipped with constant-power loads, and the lines in the grid are assumed to be purely resistive. For this setup we present a sufficient condition which ensures that the power flow equations are feasible. The novelty of this condition is its plug-and-play property: If one would connect a new microgrid to the grid, there is a test for the new microgrid to guarantee that the interconnection satisfies this condition, and hence is feasible.

Keywords: Smart grids, Optimal operation and control of power systems, Power systems stability

1. INTRODUCTION

The increasing use and demand of electricity is pushing our power grids to their limits. In addition, the incorporation of renewable energy sources is steadily introducing more uncertainty into our energy sources. It has therefore become a necessity to better understand the fundamental limits of our power grid, as well as to come up with smart ways to generate and distribute power.

To address these issues, there has been an increasing interest in the study of microgrids. A microgrid is a small power grid within a greater power grid. One of their key features is their plug-and-play capability: a microgrid can disconnect from the main grid to operate in island mode, and reconnect whenever necessary. This allows for leveling out the fluctuations of power generation and consumption, as well as for a microgrid to disconnect when the main grid suffers from a power outage.

There has been an increasing interest in DC (direct current) microgrids. A large portion of the day-to-day energy consumption is converted to DC. Together with the rise of photo-voltaic cells and advances in battery storage, it is sensible to implement DC power grids on a larger scale.

The matching of supply and demand of power in a power grid is governed by its power flow equations. If these equations have no solution, long-term voltage stability is lost and phenomena such as blackouts and voltage collapse occur (Chiang et al. [1990]). The introduction of constant-power loads leads to non-linearities for which the power flow equations may not be solvable. Conditions which guarantee their solvability are known, but are considered conservative (Shivakumar and Chew [1974], Bolognani and Zampieri [2015], Wang et al. [2016]).

There has been some research on the control of DC microgrids. A distributed control scheme was proposed in Zhao and Dörfler [2015], and a plug-and-play control scheme was proposed in Tucci et al. [2016], although neither paper consider constant-power loads. To the best of our knowledge, no theoretical advances have been made towards power flow solvability where new sources and/or constant-power loads are introduced, and the best available approach is to recalculate known conditions for the altered power grid.

To this end, the goal of this paper is to formulate a condition $\mathcal{C}$ for a DC power grid and a to-be-attached microgrid, not assuming any control scheme, such that

(i) the condition $\mathcal{C}$ implies the power flow equations of the power grid are solvable; and
(ii) there is a test $\mathcal{T}$ to guarantee that condition $\mathcal{C}$ also holds for the interconnection of the power grid with the microgrid, which makes the power flow equations of the interconnection solvable.

The main advantage of this approach is that, when new microgrids are attached to the power grid, reverification of condition $\mathcal{C}$ for the interconnected power grid is not necessary, which is therefore computationally cheaper. We refer to this as the *plug-and-play property*.

The novelty of the presented approach is that it uses the (block) Cholesky decomposition (BCD) as a theoretical tool to analyze the power flow equations. Commonly, whenever a new load is introduced to a power grid, conditions for the feasibility of the power grid should be re-

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evaluated. The BCD allows for a sequential verification of such a condition, and re-evaluation is not always necessary.

This paper is organized as follows. Section 2 contains the mathematical preliminaries. Section 3 deals with the feasibility of a DC power grid. In Section 4 we analyze the interconnection of multiple micrograms and review the block Cholesky decomposition. From this we derive a sufficient condition for solvability of the power flow equations. In Section 5 we consider the process of interconnecting a microgrid to a power grid, and state our main result. Section 6 concludes the paper.

2. PRELIMINARIES

All vectors and matrices in this paper are real-valued. We denote a diagonal matrix by $[d] := \operatorname{diag}(d_1, \ldots, d_n)$ for a vector d. Note that $[v]w = [w]v$ for any pair of vectors v, w. Inequalities between matrices such as $A > 0$ and $A \geq 0$ are meant element-wise, and the same holds for vectors. The notation $\mathbb{0}$ and $\mathbb{1}$ is used for all-zeros and all-ones vectors, respectively. We let I_n denote the $n \times n$ identity matrix.

We say a matrix A is *order-preserving* if $x < y$ implies $Ax < Ay$, for all vectors x, y. It can be shown that a matrix A is order-preserving if and only if $A \geq 0$ and $A\mathbb{1} > \mathbb{0}$. Any nonnegative invertible matrix is order-preserving, and its inverse is known in the literature as a *monotone matrix* (Berman and Plemmons [1979]). The sum and product of two order-preserving matrices is again order-preserving.

We let it be understood that by a graph we mean an undirected weighted graph with no self-loops. That is, a graph Γ is the tuple $(\mathcal{V}, \mathcal{E}, \mathrm{w})$ with node set $\mathcal{V} = \{1, \ldots, |\mathcal{V}|\}$, edge set $\mathcal{E} \subseteq \{\{i, j\} \mid i, j \in \mathcal{V}, i \neq j\}$, and a map $\mathrm{w} : \mathcal{E} \to \mathbb{R}_{>0}$ of edge weights.

For any $\tilde{\mathcal{V}} \subseteq \mathcal{V}$, the subgraph induced by $\tilde{\mathcal{V}}$ is the tuple $(\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \mathrm{w})$ where $\tilde{\mathcal{E}} = \left\{\{i, j\} \in \mathcal{E} \,\middle|\, i, j \in \tilde{\mathcal{V}}\right\}$.

We say that two nodes $i, j \in \mathcal{V}$ are path-connected with respect to the graph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathrm{w})$ if there exists a path $(v_1, v_2, \ldots, v_l)$ such that $\{v_r, v_{r+1}\} \in \mathcal{E}$ for $r = 1, \ldots, l-1$, $i = v_1$ and $j = v_l$.

The node set $\mathcal{V}$ of a graph can be partitioned into connected components such that two nodes are in the same connected component if and only if they are path-connected with respect to that graph.

We let the Laplacian matrix $\mathrm{L}(\Gamma)$ of the graph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathrm{w})$ be defined by $\mathrm{L}(\Gamma)_{ij} = -\mathrm{w}(\{i, j\})$ if $\{i, j\} \in \mathcal{E}$, $\mathrm{L}(\Gamma)_{ij} = 0$ if $i \neq j$ and $\{i, j\} \notin \mathcal{E}$, and $\mathrm{L}(\Gamma)_{ii} = -\sum_{j \neq i} \mathrm{L}(\Gamma)_{ij}$. Each graph is fully determined by its Laplacian matrix.

A matrix A is an M-matrix if there exists a matrix $B \geq 0$ such that $A = sI - B$ where $s \geq \rho(B)$, the spectral norm of B. It is invertible if $s > \rho(B)$. The diagonal elements of an M-matrix are nonnegative, and its off-diagonal elements are nonpositive.

If an M-matrix is invertible, its inverse is nonnegative. Any Schur complement of an invertible M-matrix is again an invertible M-matrix. [1]

[1] See Berman and Plemmons [1979], Theorem 6.2.3 and Exercise 10.6.1.

3. FEASIBILITY OF DC POWER GRIDS

We consider a purely resistive DC power grid where $\begin{pmatrix} V_L^\top & V_S^\top \end{pmatrix}^\top > \mathbb{0}$ and $\begin{pmatrix} \mathcal{I}_L^\top & \mathcal{I}_S^\top \end{pmatrix}^\top$ denote the voltages potentials and outgoing currents at the nodes, respectively. These vectors are real-valued and partitioned according to nodes being loads (L) or sources (S). The power injection at each node is given by

$$\begin{pmatrix} P_L \\ P_S \end{pmatrix} = \left[\begin{pmatrix} V_L \\ V_S \end{pmatrix}\right] \begin{pmatrix} \mathcal{I}_L \\ \mathcal{I}_S \end{pmatrix},$$

where $P_L < \mathbb{0}$ and $P_S > \mathbb{0}$, indicating that load nodes consume power, while source nodes supply power. We consider the setting where each load is a constant-power load, and therefore take $P_L \equiv -P_c$, for $P_c > \mathbb{0}$ constant. The voltage and current in a resistive circuit are related via the weighted Laplacian (Kirchhoff) matrix $Y = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix}$, where the edge weights corresponds to the admittance of the lines between the nodes in the power grid. [2]

The currents at the load nodes flowing into the power grid are given by $\mathcal{I}_L = Y_{LL}V_L + Y_{LS}V_S < \mathbb{0}$. To satisfy the power demand of the loads, the following equation should be satisfied for some $V_L > \mathbb{0}$, where P_c and V_S are given:

$$[V_L](Y_{LL}V_L + Y_{LS}V_S) + P_c = \mathbb{0}. \tag{1}$$

Equation (1) is known as the (purely resistive) DC power flow equation. Note that (1) does not directly depend on Y_{SS}. To simplify notation, we introduce the following definition.

Definition 1. We let $(Y_{LL}, Y_{LS}V_S, P_c)$ denote the power flow equation (1) of the DC power grid described by Y, $V_S > \mathbb{0}$ and $P_c > \mathbb{0}$. We say $(Y_{LL}, Y_{LS}V_S, P_c)$ is *feasible* if there exists a vector $V_L > \mathbb{0}$ such that (1) is satisfied.

Alternatively, Definition 1 states that $(Y_{LL}, Y_{LS}V_S, P_c)$ is feasible if there exists an operating point for the system such that the constant-power loads are satsified.

Since currents should flow from source to load, a power grid cannot be feasible if some load is not path-connected to a source. We therefore exclude such cases in this section by assuming that every load is path-connected to a source. It can be shown that Y_{LL} is therefore a so-called weakly chained diagonally dominant M-matrix, which implies that Y_{LL} is invertible.

The *open-circuit voltages*, defined as $V_L^* := -Y_{LL}^{-1}Y_{LS}V_S$, correspond to the voltage potentials in the case where there is no power demand, and therefore $\mathcal{I}_L = \mathbb{0}$. The vector satisfies $V_L^* > \mathbb{0}$ since every load node is assumed to be path-connected to a source node.

As (1) is a quadratic equation in V_L, it is not always solvable. The next lemma rewrites (1) to simplify the problem analysis.

Lemma 1. The equation $(Y_{LL}, Y_{LS}V_S, P_c)$ is feasible if and only if there exists a vector $x > \mathbb{0}$ such that

$$[x]([V_L^*]Y_{LL}[V_L^*])^{-1}x + P_c = x.$$

[2] While we do not consider shunts in our power grids, all results presented in this paper hold also for nodes with shunts, for which one would use so-called grounded Laplacian matrices.

Proof. Multiply (1) by $[P_c]([V_L]Y_{LL}[V_L])^{-1}$ and define $x = [V_L^*][V_L]^{-1}P_c$, which satisfies $x > \mathbb{0}$. Since $[P_c]^{-1}[V_L]Y_{LL}[V_L]$ is invertible, the implication follows. The steps can be reversed to deduce equivalence. □

The vector x can be thought of as the power demand at a load, scaled by the ratio between the open-circuit voltage and the voltage potential at the load. Since $P_c > \mathbb{0}$ we have $P_c < x$, a lower bound for x.

The matrix $\frac{1}{4}[V_L^*]Y_{LL}[V_L^*]$ was introduced in Simpson-Porco et al. [2016] as the stiffness matrix corresponding to the loading characteristic of the power grid. In the same paper, a sufficient condition for feasibility was derived:

Theorem 2. (Simpson-Porco et al. [2016]). The equation $(Y_{LL}, Y_{LS}V_S, P_c)$ is feasible if

$$Y_{LL}^{-1}[V_L^*]^{-1}P_L < \tfrac{1}{4}V_L^*. \tag{2}$$

Alternatively, the inequality (2) is represented in Simpson-Porco et al. [2016] by the inequality

$$\left\| \left(\tfrac{1}{4}[V_L^*]Y_{LL}[V_L^*]\right)^{-1} P_L \right\|_\infty < 1.$$

In the remainder of this paper we use Theorem 2 as a sufficient condition for feasibility of a DC power grid.

4. MICROGRIDS WITH A FIXED INTERCONNECTION TOPOLOGY

Throughout the rest of this paper we consider a DC power grid consisting of multiple interconnected DC microgrids. Our aim is to study the related power flow problem, while respecting the topological structure of microgrids. In particular, we phrase a sufficient condition $\mathcal{C}$ for the solvability of (1). This condition is formulated in terms of the individual microgrids, their interconnections, and according to a hierarchical structure. The hierarchical structure is precisely the order in which the microgrids are connected to the power grid. To clarify: microgrids which are lower in the hierarchy were added before the ones that are higher in the hierarchy.

The presented approach only considers the case where sources do not supply power to load nodes which are lower in the hierarchy, and therefore leads to more conservative conditions. On the other hand, the main feature of the approach is that only considering power flow between microgrids in a "specified direction" gives conditions for which the introduction of a new microgrid does not alter the conditions on the original power grid.

The main result of this paper is derived from Theorem 2 and relies on the block Cholesky decomposition (BCD) to describe the hierarchical structure between the microgrids. The BCD is reviewed in Section 4.2.

Throughout the rest of this section we focus on a fixed topology of microgrids and study their feasibility. In Section 5 we consider a dynamic topology of microgrids, by which we mean that we consider the interconnection of an islanded microgrid to a power grid.

4.1 System description

We consider a DC power grid as in Section 3, where again $Y = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix}$. Our power grid is subdivided into k

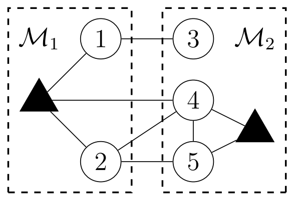

Fig. 1. An example of the interconnection of two microgrids $\mathcal{M}_1$, $\mathcal{M}_2$ satisfying Assumption 3. Source nodes are represented by ▲ and load nodes are represented by ◯.

microgrids. We number our microgrids and denote the i-th microgrid by $\mathcal{M}_i$. The numbering of the microgrids represents the order in which the microgrids were attached to the power grid, which induces a hierarchical structure on the microgrids.

We let $\Gamma = (\mathcal{V}, \mathcal{E}, \mathrm{w})$ to be the graph represented by Y. The nodes of a microgrid $\mathcal{M}_i$ induce a subgraph of Γ. Such a subgraph is denoted by $\Gamma_{\mathcal{M}_i}$.

In order to state the main result of this section, we require the following assumption on the microgrids.

Assumption 3. For each microgrid $\mathcal{M}_i$, each load node in $\mathcal{M}_i$ satisfies at least one the following conditions:

i) The node is path-connected with respect to the graph $\Gamma_{\mathcal{M}_i}$ to a source node in $\mathcal{M}_i$;
ii) The node is path-connected with respect to the graph Γ to a node in $\mathcal{M}_j$ with $j < i$.

These conditions can be interpreted as follows: Condition 3.i is equivalent to the condition that the node is path-connected to a source node in $\mathcal{M}_i$ when the microgrid operates in island mode. Condition 3.ii is equivalent to the condition that every load node is path-connected to a node lower in the hierarchy. Note that Assumption 3 only requires that the first microgrid is able to operate in island mode. In addition, note that we do not assume that the power grid is connected.

Assumption 3 prevents the case where a microgrid is connected to a power grid while some of its load nodes are not path-connected to a source node; such a case can never be feasible. More precisely, it states that for each load node there exists a path $(v_1, \ldots, v_l)$ to a source node such that it descends the hierarchy of microgrids; that is, if $v_i \in \mathcal{M}_s$ and $v_j \in \mathcal{M}_t$, then $i < j$ implies $s \geq t$.

Example 4. Figure 1 represents an interconnection of two microgrids for which Assumption 3 holds: Nodes 1, 2, 4 and 5 satisfy 3.i, while nodes 3, 4 and 5 satisfy 3.ii. Note that node 3 is not path-connected with respect to $\Gamma_{\mathcal{M}_2}$ to a source in $\mathcal{M}_2$. This implies that Assumption 3 would not hold for node 3 if the numbering of $\mathcal{M}_1$ and $\mathcal{M}_2$ would be interchanged. *I.e.*, $\mathcal{M}_2$ cannot operate in island mode.

Assumption 3 implies the following Lemma.

Lemma 5. The matrix Y_{LL} is an invertible M-matrix.

Proof. It follows from Assumption 3 that every load node is path-connected to a source node. This implies that Y_{LL} is weakly chained diagonally dominant, and hence an invertible M-matrix by Corollary 4 of Shivakumar and Chew [1974]. □

4.2 The block Cholesky decomposition

This section reviews the block Cholesky decomposition (BCD) of a symmetric positive definite matrix. More specifically, we consider the BCD of symmetric M-matrices.

Consider a symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$, along with a partition of the rows and columns of A into k nonempty subsets. We obtain $A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$ and $a_{ij} \in \mathbb{R}^{n_i \times n_j}$ such that $\sum_{i=1}^{k} n_i = n$. Note that $a_{ij} = a_{ji}^\top$. We define the matrices

$$A_i := \begin{pmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} \end{pmatrix} \text{ for } i = 1, \dots, k;$$

$$\alpha_i := (a_{i1} \ \cdots \ a_{i,i-1}) \text{ for } i = 2, \dots, k.$$

We have $A = A_k$ and $A_i = \begin{pmatrix} A_{i-1} & \alpha_i^\top \\ \alpha_i & a_{ii} \end{pmatrix}$ for $i = 2, \dots, k$.

Let the matrices C_i, D_i for $i = 1, \dots, k$ be iteratively defined by

$$C_1 := I_{n_1}, \qquad C_i := \begin{pmatrix} C_{i-1} & 0 \\ \alpha_i A_{i-1}^{-1} C_{i-1} & I_{n_i} \end{pmatrix},$$

$$D_1 := A_{11}, \qquad D_i := \begin{pmatrix} D_{i-1} & 0 \\ 0 & a_{ii} - \alpha_i A_{i-1}^{-1} \alpha_i^\top \end{pmatrix}.$$

It follows that $A_i = C_i D_i C_i^\top$ for $i = 1, \dots, k$.
Let $C := C_k$, $D := D_k$, then the *block Cholesky decomposition* (BCD) of A is given by $A = CDC^\top$.

Note that

$$C_i = \begin{pmatrix} C_{i-1} & 0 \\ \alpha_i A_{i-1}^{-1} C_{i-1} & I_{n_i} \end{pmatrix} = \begin{pmatrix} C_{i-1} & 0 \\ \alpha_i C_{i-1}^{-1\top} D_{i-1}^{-1} & I_{n_i} \end{pmatrix};$$

$$C_i^{-1} = \begin{pmatrix} C_{i-1}^{-1} & 0 \\ -\alpha_i A_{i-1}^{-1} & I_{n_i} \end{pmatrix} = \begin{pmatrix} C_{i-1}^{-1} & 0 \\ -\alpha_i C_{i-1}^{-1\top} D_{i-1}^{-1} C_{i-1}^{-1} & I_{n_i} \end{pmatrix},$$

from which we observe that only the diagonal blocks of D have to be inverted to obtain the BCD of A.

By taking $k = n$ we obtain the Cholesky decomposition of A, and the BCD therefore generalizes the Cholesky decomposition.

Suppose that A is an invertible M-matrix, then $\alpha_i \leq 0$ and $A_i^{-1} \geq 0$. Since $a_{ii} - \alpha_i A_{i-1}^{-1} \alpha_i^\top$ is a Schur complement of the invertible M-matrix A_i, it is again an invertible M-matrix, and therefore $(a_{ii} - \alpha_i A_{i-1}^{-1} \alpha_i^\top)^{-1} \geq 0$. Using this fact, it can be shown by induction on i that $D_i^{-1} \geq 0$ and $C_i^{-1} \geq 0$. Since both are invertible, it follows that D_i^{-1} and C_i^{-1} are order-preserving for all i.

4.3 Iterative feasibility condition using the BCD

Continuing Section 4.1, we define $A := Y_{LL}$. We let C, D such that $A = CDC^\top$ is the BCD of A where each block corresponds to a microgrid in the power grid. We will use the notation for submatrices of A as in Section 4.2.

The intuition behind the BCD of Y_{LL} is not straightforward, but can be seen as follows. Each diagonal block of D describes the conductances between load nodes in a microgrid after all load nodes which are lower in the hierarchy are eliminated by Kron reduction. Put differently, for each block it is assumed that there is no current flow at the load nodes which are lower in hierarchy, which means that these nodes do not consume power.

The matrix C describes all weighted paths between load nodes in distinct microgrids which ascend the hierarchy, and is therefore (block) lower triangular. The off-diagonal elements of C are nonpositive, while C has ones on its diagonal. The matrix CD roughly represents the conductances between load nodes in a microgrid, under the assumption that load nodes which are lower in hierarchy draw no current, compensated by the conductances between load nodes in microgrids which are lower in hierarchy and have the same assumption. This compensation is based on paths which descend the hierarchy, giving CD also a block lower triangular structure.

Lemma 6. The load nodes k and l in $\mathcal{M}_i$ are path-connected with respect to the graph induced by the load nodes in $\mathcal{M}_1, \dots, \mathcal{M}_i$ if and only if $(a_{ii} - \alpha_i A_{i-1}^{-1} \alpha_i^\top)_{kl}^{-1} > 0$.

Proof. The connected components of the graph induced by the load nodes in $\mathcal{M}_1, \dots, \mathcal{M}_i$ correspond to the irreducible components of A_i. We can permute the rows and columns of A_i such that A_i is block-diagonal. By the Perron-Frobenius theorem for irreducible matrices, the inverse of each block is positive. It follows that the (k, l)-entry of A_i^{-1} is positive if and only if k and l are path-connected with respect to the graph. It can be shown that $(a_{ii} - \alpha_i A_{i-1}^{-1} \alpha_i^\top)^{-1}$ is the principal submatrix of A_i^{-1} which corresponds to load nodes of $\mathcal{M}_i$. □

The next lemma is the cornerstone of the main theorem in this section, and the reason why we need Assumption 3.

Lemma 7. If Assumption 3 holds, then the vector $C^\top V_L^*$ is positive.

Proof. Recall that

$$V_L^* = -A^{-1} Y_{LS} V_S = -C^{-1\top} D^{-1} C^{-1} Y_{LS} V_S,$$

and therefore $C^\top V_L^* = -D^{-1} C^{-1} Y_{LS} V_S$. Note that $V_S > \mathbb{0}$ and $-Y_{LS} \geq 0$, and therefore $-Y_{LS} V_S \geq \mathbb{0}$.[3] Since A is an invertible M-matrix, we have seen that $C^{-1} \geq 0$ and $D^{-1} \geq 0$. This implies that $-D^{-1} C^{-1} Y_{LS} V_S \geq 0$.

We consider the rows of $C^\top V_L^*$ corresponding to the load nodes in $\mathcal{M}_i$, which are given by

$$-(a_{ii} - \alpha_i A_{i-1}^{-1} \alpha_i^\top)^{-1} \left(-\alpha_i A_{i-1}^{-1} \ I_{n_i} \ 0 \right) Y_{LS} V_S. \qquad (3)$$

Consider a load node k in $\mathcal{M}_i$.

If Assumption 3.i holds for k, then k is path-connect to a node l in $\mathcal{M}_i$ with respect to the graph $\Gamma_{\mathcal{M}_i}$ (or coincides with l) such that l shares an edge with a source node in $\mathcal{M}_i$. This implies that $(a_{ii} - \alpha_i A_{i-1}^{-1} \alpha_i^\top)_{kl}^{-1} (-Y_{LS} V_S)_l > 0$ by Lemma 6 and since $(-Y_{LS} V_S)_l$ is positive if l shares an edge with a source node.

If Assumption 3.ii holds for k, then k is path-connected to a node l in $\mathcal{M}_j$ with respect to Γ such that $j < i$, and l is the only node in the path not in $\mathcal{M}_i$.

[3] All load nodes share an edge with a generator node if and only if $-Y_{LS} V_S > \mathbb{0}$.

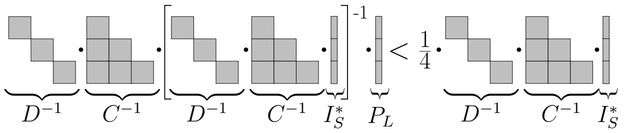

Fig. 2. A schematic representation of inequality (4) and (5) for three microgrids, where $\mathcal{I}_S^* := -Y_{LS}V_S$. Due to the block lower triangular structure of C^{-1} and D^{-1}, the inequality concerning the first two block rows is independent of the third block row of matrices D^{-1}, C^{-1} and vectors $\mathcal{I}_S^*$, P_L. This suggests that the introduction of a fourth block row to D^{-1} and C^{-1} will therefore not compromise the inequality of the first three block rows.

If l is a source node, then again

$$(a_{ii} - \alpha_i A_{i-1}^{-1}\alpha_i^\top)_{kl'}^{-1}(-Y_{LS}V_S)_{l'} > 0$$

where l' is the node in the path which shares an edge with l. If l is a load node, then, by applying induction on i in the above, Assumption 3 implies that l is path-connected to a load node m in $\mathcal{M}_{j'}$ such that $j' \leq j$, which shares an edge with a source node. Hence

$$(-(a_{ii} - \alpha_i A_{i-1}^{-1}\alpha_i^\top)^{-1}\alpha_i)_{kl}(A_{i-1}^{-1})_{lm}(-Y_{LS}V_S)_m > 0$$

by Lemma 6. This implies that each row of (3) is positive. □

The vector $C^\top V_L^*$ is a lower bound for the open circuit voltages V_L^* and roughly represents the effect of the potentials at the sources when there is no current flow at loads and where currents only flow over paths which descend the hierarchy.

The next theorem gives a sufficient condition for feasibility which is more conservative then Theorem 2, but incorporates the topological structure of the microgrids.

Theorem 8. Let $Y_{LL} = CDC^\top$, the block Cholesky decomposition such that the blocks (*i.e.* microgrids) satisfy Assumption 3. If

$$D^{-1}C^{-1}[C^\top V_L^*]^{-1}P_L < \tfrac{1}{4}C^\top V_L^*, \qquad (4)$$

then $Y_{LL}^{-1}[V_L^*]^{-1}P_L < \frac{1}{4}V_L^*$ and $(Y_{LL}, -Y_{LS}V_S, P_L)$ is feasible.

Proof. From Lemma 7 it follows that $[C^\top V_L^*]^{-1}$ is well-defined and nonnegative. The matrix $C^{-1\top}$ has ones on its diagonal, and since $C^{-1\top}$ is nonnegative, it follows that $C^{-1\top} - I_n \geq 0$. This means that $I_n \leq C^{-1\top}$ and we therefore have $C^\top V_L^* \leq C^{-1\top}C^\top V_L^* = V_L^*$. This implies that $[C^\top V_L^*]^{-1}P_L \geq [V_L^*]^{-1}P_L$. The matrix $D^{-1}C^{-1}$ is order-preserving and therefore,

$$D^{-1}C^{-1}[V_L^*]^{-1}P_L \leq D^{-1}C^{-1}[C^\top V_L^*]^{-1}P_L < \tfrac{1}{4}C^\top V_L^*.$$

Multiplication by the order-preserving matrix $C^{-1\top}$ yields

$$C^{-1\top}D^{-1}C^{-1}[V_L^*]^{-1}P_L = Y_{LL}^{-1}[V_L^*]^{-1}P_L < \tfrac{1}{4}V_L^*.$$

Theorem 2 implies that $(Y_{LL}, Y_{LS}V_S, P_L)$ is feasible. □

The condition (4) in Theorem 8 is equivalent to

$$D^{-1}C^{-1}[D^{-1}C^{-1}\mathcal{I}_S^*]^{-1}P_L < \tfrac{1}{4}D^{-1}C^{-1}\mathcal{I}_S^*, \qquad (5)$$

where $\mathcal{I}_S^* := -Y_{LS}V_S$, the current from sources flowing to neighboring loads. The block structure of (5) is schematically represented by Figure 2.

The introduction of a new microgrid gives rise to a new block row of C^{-1} and D^{-1}. Figure 2 illustrates a plug-and-play condition such that, if a power grid satisfies (4), and a microgrid satisfies a condition similar to the last block row in Figure 2, then their interconnection satisfies (4) as well. However, there are some technicalities which prevent us from directly applying Theorem 8 to formulate a plug-and-play condition for feasibility. This is investigated further in Section 5.

5. MICROGRIDS WITH A DYNAMIC INTERCONNECTION TOPOLOGY

In this section we relate the feasibility of a power grid to an associated "virtual power grid", such that Theorem 8 can be applied. From this, a plug-and-play condition $\mathcal{C}$ for feasibility is obtained, along with a test $\mathcal{T}$, which forms the main result of this paper.

Example 9. Consider the power grid in Figure 1 where we assume unit line conductances. We consider the microgrid $\mathcal{M}_1$ in island mode and let Y represent its Laplacian matrix. We have $Y_{LL} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We proceed by connecting $\mathcal{M}_2$ to $\mathcal{M}_1$ in the same fashion as in Figure 1, and let $\widehat{Y}$ represent the Laplacian matrix of the interconnected microgrids. It follows that

$$\widehat{Y}_{LL} = \left(\begin{array}{cc|ccc} 2 & 0 & -1 & 0 & 0 \\ 0 & 3 & 0 & -1 & -1 \\ \hline -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 4 & -1 \\ 0 & -1 & 0 & -1 & 3 \end{array}\right),$$

where the partitioning of the matrix correspond to the two microgrids. Note that the diagonal elements corresponding to node 1 and node 2 have increased.

To formulate a plug-and-play condition as suggested in Section 4.3, it is essential for Theorem 8 that the blocks of the BCD of Y_{LL} remain unaltered when new microgrids are attached to the power grid. However, from Example 9 we see that, when new lines are introduced, the diagonal elements of the matrix $\widehat{Y}_{LL}$ change with respect to Y_{LL}.

5.1 Virtual shunts and virtual power grids

To address the issue above, we increase the diagonal elements of the Laplacian by temporarily introducing shunts [1] at load nodes. These shunts are later replaced (and potentially removed) when more lines are interconnected to a load node. We refer to these shunts as *virtual shunts*. This approach ensures that the diagonal elements of the Laplacian matrix remain constant when new microgrids are attached.

Virtual shunts are not physically present in the power grid and should be seen as placeholders for prospective lines. We refer to a power grid with virtual shunts as a *virtual power grid*. We refer to other power grids as physical power grids, to prevent ambiguity. It should be understood that a virtual power grid does not fully capture the behavior of

[1] A shunt is a component at a node which extracts current proportional to the voltage potential at the node; a resistive line connected to ground.

the associated physical power grid. However, the feasibility of the two is related by the following lemma.

Lemma 10. Let A be an invertible M-matrix, $E \geq 0$ diagonal, $b \geq \mathbb{0}$ and $c > \mathbb{0}$ such that $(A+E)^{-1}b > \mathbb{0}$. Suppose

$$(A+E)^{-1}[(A+E)^{-1}b]^{-1}c < \tfrac{1}{4}(A+E)^{-1}b, \qquad (6)$$

then also $A^{-1}[A^{-1}b]^{-1}c < \frac{1}{4}A^{-1}b$.

Proof. Note that $A^{-1}(A+E) = I + A^{-1}E \geq I$ is order-preserving, which implies that $A^{-1}b \geq (A+E)^{-1}b$, and so $[A^{-1}b]^{-1} \leq [(A+E)^{-1}b]^{-1}$. By multiplying (6) with $A^{-1}(A+E)$, we obtain

$$A^{-1}[(A+E)^{-1}b]^{-1}c < \tfrac{1}{4}A^{-1}b.$$

The inequality $[A^{-1}b]^{-1} \leq [(A+E)^{-1}b]^{-1}$ implies that $A^{-1}[A^{-1}b]^{-1}c < \frac{1}{4}A^{-1}b$, as A^{-1} is order-preserving. □

In the context of Theorem 2, Lemma 10 implies that if $(Y_{LL}+E, Y_{LS}V_S, P_L)$ with diagonal $E \geq 0$ satisfies the condition of Theorem 2, then so does $(Y_{LL}, Y_{LS}V_S, P_L)$, implying that both are feasible. The power grid corresponding to $(Y_{LL}+E, Y_{LS}V_S, P_L)$ is a virtual counterpart to $(Y_{LL}, Y_{LS}V_S, P_L)$, with E representing the virtual shunts.

Note that virtual shunts are not needed for source nodes. This follows directly from the observation that Y_{SS} does not appear in (1). Hence, lines between sources can be altered freely, without compromising feasibility.

It is essential that the virtual shunts are chosen properly. Choosing E large leads to more conservative conditions, whereas choosing E small restricts the possible interconnections with potential microgrids.

However, from a practical point of view, it is reasonable to assume that the conductances of prospective lines are known *a priori*. For example, some lines may already exist but are not in use. Also, there might be limitations on the number of lines a node can connect with, which would give an upper bound for the virtual shunt.

5.2 Main Theorem: Plug-and-play solvability

In this section we present the main result of the paper. The main line of this section is as follows.

We consider a power grid and define a virtual counterpart in the sense of Section 5.1. We define a BCD of the lines of the loads in the virtual power grid. We assume that the BCD satisfies the conditions for Theorem 8. This implies that both the power grid and its virtual counterpart are feasible, by Lemma 10.

We proceed by introducing a microgrid and again define a virtual counterpart. We interconnect both virtual grids, assuming that the physical interconnection satisfies Assumption 3 and does not exceed the virtual shunts. We extend the BCD of the virtual power grid to the virtual interconnection and use this to state the main theorem.

The main theorem gives a sufficient condition such that this interconnected virtual power grid is feasible. Moreover, the theorem states that the physical interconnection of the power grid and the microgrid is therefore also feasible, by Lemma 10.

Consider a DC power grid described by the Laplacian matrix $\begin{pmatrix} Y_{L_1L_1} & Y_{L_1S_1} \\ Y_{S_1L_1} & Y_{S_1S_1} \end{pmatrix}$, together with the virtual shunts E_{L_1} and load voltages V_{L_1}. Let P_{L_1} be the power demand at the load nodes and $V'_{L_1} := -(Y_{L_1L_1}+E_{L_1})^{-1}Y_{L_1S_1}V_{S_1}$ the virtual open-circuit voltages, where V_{S_1} are the source voltages. The vector V'_{L_1} is a lower bound for the open-circuit voltages of the physical power grid.

Let C, D such that $CDC^\top = Y_{L_1L_1}+E_{L_1}$ is a BCD, where the blocks are such that they satisfy Assumption 3.

Assumption 11. The virtual power grid satisfies the condition $\mathcal{C}$, which is given by

$$D^{-1}C^{-1}[C^\top V'_{L_1}]^{-1}P_{L_1} < \tfrac{1}{4}C^\top V'_{L_1}.$$

Assumption 11 implies that $(Y_{L_1L_1}+E_{L_1}, Y_{L_1S_1}V_{S_1}, P_{L_1})$ is feasible by Theorem 8. It follows by Lemma 10 that $(Y_{L_1L_1}, Y_{L_1S_1}V_{S_1}, P_{L_1})$, the physical counterpart, is feasible as well.

In addition, consider a microgrid described by the Laplacian matrix $\begin{pmatrix} Y_{L_2L_2} & Y_{L_2S_2} \\ Y_{S_2L_2} & Y_{S_2S_2} \end{pmatrix}$, together with the virtual shunts E_{L_2} and load voltages V_{L_2}. Let P_{L_2} be the power demand at the load nodes.

Let the physical interconnection of the power grid and the microgrid be given by the Laplacian matrix

$$\begin{pmatrix} Y_{L_1L_1}+\widehat{E}_{L_1} & Y_{L_1L_2} & Y_{L_1S_1} & Y_{L_1S_2} \\ Y_{L_2L_1} & Y_{L_2L_2}+\widehat{E}_{L_2} & Y_{L_2S_1} & Y_{L_2S_2} \\ Y_{S_1L_1} & Y_{S_1L_2} & Y_{S_1S_1}+\widehat{E}_{S_1} & Y_{S_1S_2} \\ Y_{S_2L_1} & Y_{S_2L_2} & Y_{S_2S_1} & Y_{S_2S_2}+\widehat{E}_{S_2} \end{pmatrix},$$

where $\widehat{E}_{L_1}$, $\widehat{E}_{L_2}$, $\widehat{E}_{S_1}$, $\widehat{E}_{S_2}$ are defined as follows:

$$\widehat{E}_{L_1} := -[Y_{L_1L_2}\mathbb{1} + Y_{L_1S_2}\mathbb{1}]; \widehat{E}_{S_1} := -[Y_{S_1L_2}\mathbb{1} + Y_{S_1S_2}\mathbb{1}];$$
$$\widehat{E}_{L_2} := -[Y_{L_2L_1}\mathbb{1} + Y_{L_2S_1}\mathbb{1}]; \widehat{E}_{S_2} := -[Y_{S_2L_1}\mathbb{1} + Y_{S_2S_1}\mathbb{1}].$$

We make the following assumption on the interconnection.

Assumption 12. The interconnection of the power grid and the microgrid is such that $\widehat{E}_{L_1} \leq E_{L_1}$, $\widehat{E}_{L_2} \leq E_{L_2}$ and Assumption 3 holds.

The bounds on $\widehat{E}_{L_1}$ and $\widehat{E}_{L_2}$ imply that the virtual interconnection of both virtual grids does not exceed the virtual shunts. This will allow us to use Lemma 10 in Theorem 13.

Let $\tilde{C}, \tilde{D}$ such that

$$\tilde{C}\tilde{D}\tilde{C}^\top = \begin{pmatrix} Y_{L_1L_1}+E_{L_1} & Y_{L_1L_2} \\ Y_{L_2L_1} & Y_{L_2L_2}+E_{L_2} \end{pmatrix}$$

is a BCD. From Section 4.2 it follows that

$$\tilde{C}^{-1} = \begin{pmatrix} C^{-1} & 0 \\ -Y_{L_2L_1}(CDC^\top)^{-1} & I_n \end{pmatrix};$$
$$\tilde{D}^{-1} = \begin{pmatrix} D^{-1} & 0 \\ 0 & R^{-1} \end{pmatrix},$$

where we define

$$R := Y_{L_2L_2} + E_{L_2} - Y_{L_2L_1}(CDC^\top)^{-1}Y_{L_1L_2}.$$

Define $V'_L := -(\tilde{C}\tilde{D}\tilde{C}^\top)^{-1}\begin{pmatrix} Y_{L_1S_1} & Y_{L_1S_2} \\ Y_{L_2S_1} & Y_{L_2S_2} \end{pmatrix}\begin{pmatrix} V_{S_1} \\ V_{S_2} \end{pmatrix}$, which is a lower bound for the open-circuit voltages of the interconnected power grid.

Finally, let $(\tilde{C}^\top V_L')_{L_1}$ denote the entries of $\tilde{C}^\top V_L'$ corresponding to the load nodes of the original power grid, and similar for $(\tilde{C}^\top V_L')_{L_2}$ and the microgrid.

Theorem 13. (Plug-and-Play Solvability). Suppose that Assumptions 11 and 12 are satisfied. If the test $\mathcal{T}$, given by

$$\begin{aligned} &- R^{-1}Y_{L_2L_1}(CDC^\top)^{-1}[(\tilde{C}^\top V_L')_{L_1}]^{-1}P_{L_1} \\ &\qquad + R^{-1}[(\tilde{C}^\top V_L')_{L_2}]^{-1}P_{L_2} < \tfrac{1}{4}(\tilde{C}^\top V_L')_{L_2}, \end{aligned} \tag{7}$$

is satisfied, then the condition $\mathcal{C}$, given by

$$\tilde{D}^{-1}\tilde{C}^{-1}[\tilde{C}^\top V_L']^{-1}\begin{pmatrix} P_{L_1} \\ P_{L_2} \end{pmatrix} < \tfrac{1}{4}\tilde{C}^\top V_L' \tag{8}$$

holds for the virtual interconnected power grid. Therefore the interconnection of the power grid and the microgrid is feasible.

Proof. Due to the block triangular structure of $\tilde{C}$, the rows of (8) corresponding to the virtual power grid are given by

$$D^{-1}C^{-1}[(\tilde{C}^\top V_L')_{L_1}]^{-1}P_{L_1} < \tfrac{1}{4}(\tilde{C}^\top V_L')_{L_1}. \tag{9}$$

For the same reason we have

$$\begin{aligned} (\tilde{C}^\top V_L')_{L_1} &= -D^{-1}C^{-1}(Y_{L_1S_1}V_{S_1} + Y_{L_1S_2}V_{S_2}) \\ &\geq -D^{-1}C^{-1}Y_{L_1S_1}V_{S_1} = C^\top V_{L_1}', \end{aligned}$$

which implies $[(\tilde{C}^\top V_L')_{L_1}]^{-1} \leq [C^\top V_{L_1}']^{-1}$. Assumption 11 implies that

$$\begin{aligned} D^{-1}C^{-1}[(\tilde{C}^\top V_L')_{L_1}]^{-1}P_{L_1} &\leq D^{-1}C^{-1}[C^\top V_{L_1}']^{-1}P_{L_1} \\ &< \tfrac{1}{4}C^\top V_{L_1}' \leq \tfrac{1}{4}(\tilde{C}^\top V_L')_{L_1}. \end{aligned}$$

Therefore (9) is satisfied.

Note that (7) is equivalent to the rows of (8) corresponding to the virtual microgrid. Hence, if (7) holds, then so does (8).

If (8) is satisfied, then by Theorem 8 the virtual interconnection

$$\left(\begin{pmatrix} Y_{L_1L_1}+E_{L_1} & Y_{L_1L_2} \\ Y_{L_2L_1} & Y_{L_2L_2}+E_{L_2} \end{pmatrix}, \begin{pmatrix} Y_{L_1S_1} & Y_{L_1S_2} \\ Y_{L_2S_1} & Y_{L_2S_2} \end{pmatrix}\begin{pmatrix} V_{S_1} \\ V_{S_2} \end{pmatrix}, \begin{pmatrix} P_{L_1} \\ P_{L_2} \end{pmatrix}\right)$$

is feasible. Since $\widehat{E}_{L_1} \leq E_{L_1}$ and $\widehat{E}_{L_2} \leq E_{L_2}$, we may use Lemma 10 to conclude that the physical interconnection

$$\left(\begin{pmatrix} Y_{L_1L_1}+\widehat{E}_{L_1} & Y_{L_1L_2} \\ Y_{L_2L_1} & Y_{L_2L_2}+\widehat{E}_{L_2} \end{pmatrix}, \begin{pmatrix} Y_{L_1S_1} & Y_{L_1S_2} \\ Y_{L_2S_1} & Y_{L_2S_2} \end{pmatrix}\begin{pmatrix} V_{S_1} \\ V_{S_2} \end{pmatrix}, \begin{pmatrix} P_{L_1} \\ P_{L_2} \end{pmatrix}\right)$$

is feasible as well. □

Note that Assumption 11 and (8) are of the same form. Hence, microgrids can be attached to a power grid in an iterative fashion by using Theorem 13 to guarantee feasibility. This provides a method to determine if a power grid remains feasible after connecting a specific microgrid.

A numerical example of this process was omitted but is available in the preprint of this paper Jeeninga et al. [2019].

6. CONCLUSION

In this paper we studied the interconnection of purely resistive DC microgrids with constant-power loads. We have presented a sufficient condition $\mathcal{C}$ for the feasibility of a DC power grid. In addition, we have presented a test $\mathcal{T}$ for the interconnection of a microgrid such that condition $\mathcal{C}$ also holds for their interconnection. This establishes a method to determine if a power grid remains feasible after connecting a specific microgrid.

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