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# Dissipative PI control for a class of semilinear heat equations with actuator disturbance

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**Abstract:** The problem of dissipative proportional-integral (PI) output-feedback control for a class of semilinear heat equations with actuator disturbance is addressed. Sufficient conditions for strict dissipativity in the state and thus exponential stability of the closed-loop system are derived which are stated in terms of the dissipavity properties of the nonlinearity, the controller gains and the actuator and sensor shape functions (i.e., their form and localization). Based on these conditions a dissipation maximization procedure is proposed to appropriately choose the degrees of freedom using optimization algorithms. Numerical simulation results illustrate the performance of the proposed controller.

*Keywords:* Semilinear partial differential equations, PI control, dissipativity, actuator disturbance.

## 1. INTRODUCTION

Control of partial differential equations with nonlinearities has received increasing attention during the last decades. Most of the common approaches like the modal (or spectral) decomposition (Curtain and Zwart, 1995) or backstepping based control (Krstic and Smyshlyaev, 2008) apply only to linear or linearized models, with few exceptions (Vazquez and Krstic, 2008a,b). Lyapunov-based techniques (Bastin and Coron, 2016; Hagen and Mezic, 2003; Hagen, 2006; Hasan, 2015; Hu et al., 2015; Hasan, 2016) offer a way to extend these approaches to explicitely account for the destabilization potential of the nonlinear terms. Dissipativity-based approaches (Hagen and Mezic, 2003; Schaum and Meurer, 2019b,a) allow a characterization to include both stabilizing as well as destabilizing nonlinear components, e.g. by means of sector conditions (Lur'e and Postnikov, 1944; Popov, 1959; Khalil, 1996; Sepulchre et al., 1997).

The dissipative approach presents a general framework closely connected to Lyapunov's direct method and basically exploits the particular dissipativity properties of each component of the system and its particular interaction with the other parts. By specific characterization of dissipation properties it is thus possible to determine (sufficient) conditions under which a multi-system interconnection has some desired stability properties. In the case that according stabilizability features are ensured the required dissipation properties can be introduced or improved through feedback control.

The dissipativity-based approach has been sucessfully implemented e.g. in (Schaum and Meurer, 2019b) for the exponential stabilization of a class of heat equations with a nonlinear (destabilizing) globally (in space) acting distributed output-feedback term by means of a local in-domain stabilizing linear output-feedback injection. The key design degrees of freedom herein are the localization of the collocated sensor-actuator pair and the control gain. In a similar fashion in (Schaum and Meurer, 2019a) a dissipative boundary backstepping controller is proposed for the global stabilization of a semi-linear first-order partial integro-differential equation. In both scenarios the nonlinear terms are explicitely accounted for in the design by means of sector conditions, which are equivalent to quadratic dissipation inequalities.

The above results did not consider the effects of actuator disturbances, measurement noise or other kind of model and parameter uncertainties. Thus some open question with respect to the stabilization capabilities and associated sufficient conditions for exponential stability or input-to-state stability have still to be answered. Motivated by this fact, in the present work a first extension of the result in (Schaum and Meurer, 2019b) is developed to explicitely account for actuator disturbances in the form of a constant offset. From finite-dimensional linear control theory it is quite well known that these kind of perturbations can in principle be compensated by a proportional-integral (PI) controller. Nevertheless, even though intuitively appealing, an explicit proof as well as sufficient conditions for the choice of controller gains to ensure the exponential stability of a distributed parameter nonlinear system model with actuator offset is still an open task. This gap is filled in the present study by establishing sufficient conditions for exponential stability using dissipative PI control.

The results extend the ones presented in (Schaum and Meurer, 2019b) in the sense that (i) actuator offset is included, (ii) integral action is combined with proportional one, (iii) the functioning assessment includes conditions on the collocated actuator-sensor position and shape, and (iv) the restriction to symmetric (sector) dissipativity conditions is revealed.

The paper is organized as follows. In Section 2 the problem set-up is formulated. Some basic notions, concepts and results from dissipativity theory are recalled in Section 3. In Section 4 the main results on the exponential stabilization using dissipative PI control are presented and related to the optimal sensor and actuator location problem. Section 5 contains numerical simulation results and discussions for the application of the presented theory to a representative case example. Conclusions are drawn in Section 6.

*Notation*

The space of square integrable functions over the domain $[0,1]$ is given by $L^2(0,1)$. For a given space $\mathbb{S}$ the inner product is denoted by $\langle\cdot,\cdot\rangle_{\mathbb{S}}$ and the induced norm by $\|v\|_{\mathbb{S}} = \sqrt{\langle v,v\rangle}$. In particular for $\boldsymbol{x},\boldsymbol{y}\in\mathbb{S}=\mathbb{R}^n$ one has $\langle\boldsymbol{x},\boldsymbol{y}\rangle_{\mathbb{R}^n} = \boldsymbol{x}^\mathsf{T}\boldsymbol{y}$ and for functions $u,v\in\mathbb{S}=L^2(0,1)$ one has $\langle u,v\rangle_{L^2(0,1)} = \int_0^1 uv\,\mathrm{d}z$. Correspondingly, for the product space $\mathbb{S}=L^2\times\mathbb{R}$ it holds that for any $[u,x]^\mathsf{T},[v,y]^\mathsf{T}\in L^2(0,1)\times\mathbb{R}$ one has $\langle[u,x]^\mathsf{T},[v,y]^\mathsf{T}\rangle_{\mathbb{S}} = \langle u,v\rangle_{L^2(0,1)} + \langle x,y\rangle_{\mathbb{R}}$ and the induced norm is given by $\|[u,x]^\mathsf{T}\|_{\mathbb{S}} = \sqrt{\|u\|^2_{L^2(0,1)}+|x|^2}$ with $|\cdot|$ denoting the absolute value of a real number. The arguments of functions are provided only when necessary to improve the comprehensibility.

## 2. PROBLEM STATEMENT

Consider a heat equation with globally (in space) distributed nonlinear output injection $\varphi\in\mathcal{C}^1$ and locally acting control disturbed by a constant offset $w$, i.e.,

$$\partial_t x(z,t) = D\partial_z^2 x(z,t) + g\varphi(\sigma(t)) + b(z)(u(t)+w) \quad (1a)$$
$$x(0,t) = x(1,t) = 0 \quad (1b)$$
$$\sigma(t) = Hx(z,t) = \langle h,x\rangle_{L^2(0,1)} \quad (1c)$$
$$x(z,0) = x_0(z) \quad (1d)$$
$$y(t) = \sigma(t) \quad (1e)$$

with time $t\in[0,\infty)$, the state $x\in L^2(0,1)$, space $z\in[0,1]$, the diffusion coefficient $D>0$, a distributed constant gain $g\in\mathbb{R}$, the actuator shape function $b\in L^2(0,1)$, the control input $u:[0,\infty)\to\mathbb{R}$ and the output $y=\sigma$ with $\sigma$ being a weighted integral of the state $x$ with the weight given by the function $h\in L^2(0,1)$.

In the following only the collocated set-up is analyzed, i.e., the case that sensor and actuator shape functions $h$ and $b$ are identical so that

$$\int_0^1 bx\,\mathrm{d}z = \int_0^1 hx\,\mathrm{d}z = Hx = \sigma. \quad (1f)$$

The control design task for (1) amounts to achieve exponential stability of the zero profile $x=0$. For this purpose the following PI control scheme is proposed

$$u(t) = -ky(t) - l\int_0^t y(\tau)\,\mathrm{d}\tau. \quad (2)$$

The problem of determining $k$ and $l$ is solved by means of ensuring a desired dissipativity property of the closed-loop system. To further address this point, first some notions and results from dissipativity theory are briefly recalled.

## 3. DISSIPATIVITY CONCEPTS

The focus on dissipativity theory used in the sequel follows the general ideas presented in the early considerations for finite-dimensional systems (Willems, 1972a,b; Hill and Moylan, 1976, 1980), and extensions to the infinite-dimensional setup (Pandolfi, 1998; Brogliato et al., 2007).

In the following denote by $\Sigma(\mathcal{A},\mathcal{G},\mathcal{H})$ a system described by the abstract differential equation

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \mathcal{A}\boldsymbol{x}(t) + \mathcal{G}\boldsymbol{\nu}(t) \quad (3a)$$
$$\boldsymbol{\sigma}(t) = \mathcal{H}\boldsymbol{x}(t). \quad (3b)$$

with the state $\boldsymbol{x}\in\mathbb{X}$ with inner product $\langle\cdot,\cdot\rangle_{\mathbb{X}}$, the operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\to\mathbb{X}$, the matrix $\mathcal{G}\in\mathbb{R}^{n\times p}$, input $\boldsymbol{\nu}\in\mathbb{U}$, and output $\boldsymbol{\sigma}\in\mathbb{S}$ associated to the output operator $\mathcal{H}:\mathbb{X}\to\mathbb{S}$.

*Definition 1.* $\Sigma(\mathcal{A},\mathcal{G},\mathcal{H})$ is called *strictly state dissipative with dissipation rate* $\kappa$ if there exist a storage functional $\mathcal{S}:\mathbb{X}\to\mathbb{R}$ with $\mathcal{S}\succ 0$ and a supply rate $\omega:\mathbb{S}\times\mathbb{U}\to\mathbb{R}$, so that

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} \preceq \omega(\boldsymbol{\sigma},\boldsymbol{\nu}) - \kappa\|\boldsymbol{x}\|^2. \quad (4)$$

For the case of a quadratic supply rate

$$\omega(\boldsymbol{\sigma},\boldsymbol{\nu}) = \langle\boldsymbol{\sigma},Q\boldsymbol{\sigma}\rangle_{\mathbb{S}} + 2\langle\boldsymbol{\sigma},S\boldsymbol{\nu}\rangle_{\mathbb{S}} + \langle\boldsymbol{\nu},R\boldsymbol{\nu}\rangle_{\mathbb{U}}. \quad (5)$$

with operators $Q:\mathbb{S}\to\mathbb{S}$, $S:\mathbb{U}\to\mathbb{S}$ and $R:\mathbb{U}\to\mathbb{U}$ the following notions are given:

(a) The system $\Sigma(\mathcal{A},\mathcal{G},\mathcal{H})$ is called $(Q,S,R)$*-strictly state dissipative with dissipation rate* $\kappa$ if (4) holds with $\omega$ given in (5).
(b) The static map $\boldsymbol{\varphi}(\boldsymbol{\sigma})$ is $(Q,S,R)$-dissipative if it holds that $\omega(\boldsymbol{\varphi},\boldsymbol{\sigma})\succeq 0$.

◇

In many situations it is possible to show that the particular nonlinearity at hand is contained in a linear sector, i.e. that there exist constants $k_1,k_2$ so that

$$(k_2\sigma-\varphi(\sigma))(\varphi(\sigma)-k_1\sigma)\geq 0 \quad (6)$$

holds true for all $\sigma\in\mathbb{R}$ (see Figure 1 for an illustration). In this case the nonlinearity is said to belong to the sector $[k_1,k_2]$ (see, e.g., (Khalil, 1996; Sepulchre et al., 1997)), denoted by $\varphi(\sigma)\in[k_1,k_2]$.

The sector condition (6) can be rewritten as

$$\begin{aligned} &-\varphi^2(\sigma)+(k_1+k_2)\varphi(\sigma)\sigma-k_1k_2\sigma^2\\ &=\langle\varphi,-\varphi\rangle_{\mathbb{R}}+\langle\sigma,(k_1+k_2)\varphi\rangle_{\mathbb{R}}+\langle\sigma,-k_1k_2\sigma\rangle_{\mathbb{R}}\geq 0\end{aligned}$$

implying that any $\varphi\in[k_1,k_2]$ is $(Q,S,R)$-dissipative with

$$Q=-1,\quad R=-k_1k_2,\quad S=\frac{1}{2}(k_1+k_2).$$

For a symmetric sector (i.e., for $k_1=-k_2$) it holds that $S=0$ and $R=k_1^2$. The converse is only valid under some conditions on $Q$ (see e.g. Schaum and Meurer (2019a)).

For the case of the quadratic storage functional

$$\mathcal{S}(\boldsymbol{x}) = \langle\boldsymbol{x},P\boldsymbol{x}\rangle_{\mathbb{X}} \quad (7)$$

with a symmetric positive definite operator $P:\mathbb{X}\to\mathbb{X}$ satisfying

$$P(z)=P^*(z)\succ 0\,\forall\,z\in[0,1]$$

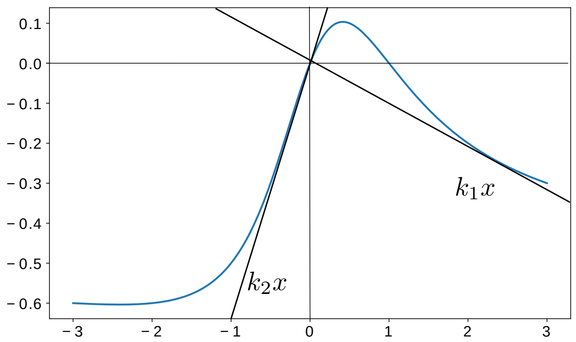


Fig. 1. Sector condition for $\varphi(\sigma) = k_0\sigma(1-\sigma)/(1+\sigma^2)$.

the following lemma is a straight-forward consequence of Definition 1 and has been stated in the literature muliple times (with slight modifications) (Hill and Moylan, 1976), (Hill and Moylan, 1980), (Pandolfi, 1998) and (Schaum and Meurer, 2019b).

*Lemma 1.* $\Sigma(\mathcal{A},\mathcal{G},\mathcal{H})$ is $(Q,S,R)$-*strictly state dissipative with dissipation rate* $\kappa$ if there exists $P = P^* > 0$ satisfying the inequality

$$\begin{aligned}&\langle \mathcal{A}\boldsymbol{x}, P\boldsymbol{x}\rangle_{\mathbb{X}} + \langle \boldsymbol{x}, P\mathcal{A}\boldsymbol{x}\rangle_{\mathbb{X}} + 2\langle \boldsymbol{x}, P\mathcal{G}\boldsymbol{\nu}\rangle_{\mathbb{X}}\\ &\preceq -\kappa\|\boldsymbol{x}\|^2 + \langle \boldsymbol{\sigma}, Q\boldsymbol{\sigma}\rangle_{\mathbb{S}} + 2\langle \boldsymbol{\sigma}, S\boldsymbol{\nu}\rangle_{\mathbb{S}} + \langle \boldsymbol{\nu}, R\boldsymbol{\nu}\rangle_{\mathbb{U}}\,.\end{aligned}$$

◇

Note that in order to simplify the calculations in application examples the operators $P,Q,R,S$ are typically chosen as constant matrices or even scalars, e.g., for one-dimensional sector conditions.

## 4. DISTURBANCE REJECTION BY DISSIPATIVE PI CONTROL

With the PI controller (2) the closed-loop dynamics are given by

$$\begin{aligned}\partial_t x &= D\partial_z^2 x + g\varphi(\sigma) - b(k\sigma + \varpi - w) && (8a)\\ x(0,t) &= x(1,t) = 0 && (8b)\\ \dot{\varpi} &= l\sigma && (8c)\\ \sigma &= Hx && (8d)\\ x(z,0) &= x_0(z), \quad \varpi(0) = \varpi_0. && (8e)\end{aligned}$$

Introducing the disturbance compensation error

$$\tilde{w} = \varpi - w \tag{9}$$

the associated error dynamics are given by

$$\begin{aligned}\partial_t x &= D\partial_z^2 x + g\varphi(\sigma) - bk\sigma - b\tilde{w} && (10a)\\ x(0,t) &= x(1,t) = 0 && (10b)\\ \dot{\tilde{w}} &= l\sigma && (10c)\\ \sigma &= Hx && (10d)\\ x(z,0) &= x_0(z), \quad \tilde{w}(0) = \tilde{w}_0. && (10e)\end{aligned}$$

In order to put the stability analysis of the preceding error dynamics into the dissipativity framework (10) is rewritten in the form of the Lur'e type two-subsystem interconnection

$$\begin{aligned}\partial_t x &= D\partial_z^2 x - bk\sigma - b\tilde{w} + g\nu && (11a)\\ x(0,t) &= x(1,t) = 0 && (11b)\\ \dot{\tilde{w}} &= l\sigma && (11c)\\ \sigma &= Hx && (11d)\\ x(z,0) &= x_0(z), \quad \tilde{w}(0) = \tilde{w}_0 && (11e)\\ \nu &= \varphi(\sigma). && (11f)\end{aligned}$$

To show the exponential stability along the lines of the dissipativity-based approach outlined in Section 3 the dissipation properties of the linear subsystem (11a)-(11e) have to be determined. Note that this subsystem can be written in the abstract form (3) with $\boldsymbol{x} = [x,\tilde{w}]^\intercal \in L^2(0,1)\times\mathbb{R}$ and

$$\mathcal{A} = \begin{bmatrix} D\partial_z^2 - bkH & -b\\ lH & 0\end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} g\\ 0\end{bmatrix}, \quad \mathcal{H} = [H\ \ 0]$$

$$\mathcal{D}(\mathcal{A}) = \{[v,w]^\intercal \in L^2(0,1)\times\mathbb{R}\,|\, v(0) = v(1) = 0\}.$$

According to Lemma 1, assuming that the nonlinearity is $(q,s,r)$-dissipative for some $q<0, s,r\in\mathbb{R}$ the exponential stability of the closed-loop system is ensured if the linear system $\Sigma(\mathcal{A},\mathcal{G},\mathcal{H})$ is $(-r,-s,-q)$-*strictly state dissipative with dissipation rate* $\kappa$ . To establish sufficient conditions for this property consider the particular storage functional

$$\mathcal{S}(x,\tilde{w}) = \int_0^1 \left(\frac{1}{2}x^2 + \alpha x\tilde{w}\right)\mathrm{d}z + \frac{1}{2}\tilde{w}^2 \tag{12}$$

with $\alpha \in \mathcal{C}^2(0,1)$ being a degree of freedom satisfying $\alpha(z) \geq 0$ for all $z \in [0,1]$ and

$$\alpha^2(z) \leq \alpha^+ < 1 \quad \forall z\in[0,1] \tag{13}$$

so that

$$\mathcal{S}(x,\tilde{w}) \succ 0$$

holds true. From the definition of $\mathcal{S}$ in (12) it follows that

$$\begin{aligned}&\frac{1}{2}\|x\|^2 + \frac{1}{2}|\tilde{w}|^2 - \left|\int_0^1 \alpha x\tilde{w}\,\mathrm{d}z\right|\\ &\leq \mathcal{S}(x,\tilde{w})\\ &\leq \frac{1}{2}\|x\|^2 + \frac{1}{2}|\tilde{w}|^2 + \left|\int_0^1 \alpha x\tilde{w}\,\mathrm{d}z\right|.\end{aligned}$$

The integral term on the other hand satisfies

$$\begin{aligned}\left|\int_0^1 \alpha x\tilde{w}\,\mathrm{d}z\right| &\leq \alpha^+\left|\int_0^1 x\tilde{w}\,\mathrm{d}z\right|\\ &\leq \alpha^+\int_0^1 |x|\,|\tilde{w}|\,\mathrm{d}z\\ &\leq \alpha^+\int_0^1 \frac{1}{2}\left(|x|^2 + |\tilde{w}|^2\right)\mathrm{d}z\\ &= \frac{\alpha^+}{2}\left(\|x\|^2 + |\tilde{w}|^2\right).\end{aligned}$$

In consequence it follows that

$$\beta_1\left(\|x\|^2 + |\tilde{w}|^2\right) \leq \mathcal{S}(x,\tilde{w}) \leq \beta_2\left(\|x\|^2 + |\tilde{w}|^2\right), \tag{14a}$$

with

$$\beta_1 = \frac{1-\alpha^+}{2}, \quad \beta_2 = \frac{1+\alpha^+}{2} \tag{14b}$$

showing that $\mathcal{S}$ is quadratically bounded with respect to the norm of the product space $L^2(0,1)\times\mathbb{R}$.

Taking the time derivative of $\mathcal{S}$, substituting the system dynamics (11), integrating by parts and substituting (1f) it follows that

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} &= \int_0^1 \left(x\partial_t x + \alpha\partial_t x\tilde{w} + \alpha x\dot{\tilde{w}}\right)\,\mathrm{d}z + \tilde{w}\dot{\tilde{w}} \\
&= \int_0^1 (x+\alpha\tilde{w})\left(D\partial_z^2 x - bk\sigma - b\tilde{w} + g\nu\right)\,\mathrm{d}z \\
&\qquad + \int_0^1 \alpha x l\sigma\,\mathrm{d}z + \tilde{w}l\sigma \\
&= \int_0^1 (x+\alpha\tilde{w})D\partial_z^2 x\,\mathrm{d}z - \int_0^1 xbk\sigma\,\mathrm{d}z \\
&\quad - k\left(\int_0^1 \alpha b\,\mathrm{d}z\right)\tilde{w}\sigma + \int_0^1 (x+\alpha\tilde{w})\left(-b\tilde{w}+g\nu\right)\,\mathrm{d}z \\
&\qquad + \int_0^1 \alpha x l\sigma\,\mathrm{d}z + \tilde{w}l\sigma \\
&= (x+\alpha\tilde{w})D\partial_z x\Big|_0^1 - \int_0^1 (\partial_z x + \alpha'\tilde{w})D\partial_z x\,\mathrm{d}z - k\sigma^2 \\
&\quad - k\left(\int_0^1 \alpha b\,\mathrm{d}z\right)\tilde{w}\sigma + \int_0^1 (x+\alpha\tilde{w})\left(-b\tilde{w}+g\nu\right)\,\mathrm{d}z \\
&\qquad + \int_0^1 \alpha x l\sigma\,\mathrm{d}z + \tilde{w}l\sigma.
\end{aligned}
$$

Substituting the homogeneous boundary conditions (11b) and integrating by parts once more one obtains

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} &= \left(\alpha\tilde{w}D\partial_z x - \alpha'\tilde{w}Dx\right)\Big|_0^1 - \int_0^1 D\left(\partial_z x\right)^2\,\mathrm{d}z \\
&\quad + \int_0^1 D\alpha''\tilde{w}x\,\mathrm{d}z - k\sigma^2 - \left(k\int_0^1 \alpha b\,\mathrm{d}z + 1 - l\right)\tilde{w}\sigma \\
&\quad - \int_0^1 \alpha b\,\mathrm{d}z\tilde{w}^2 + \int_0^1 (x+\alpha\tilde{w})g\nu\,\mathrm{d}z + \int_0^1 \alpha x l\sigma\,\mathrm{d}z
\end{aligned}
$$

Taking into account the homogeneous Dirichlet boundary conditions again, choosing $0 \leq \alpha < 1$ so that

$$\alpha(0) = \alpha(1) = 0, \tag{15}$$

and applying Wirtinger's inequality (Hardy et al., 1952) (given the homogeneous Dirichlet boundary conditions)

$$-\int_0^1 D\left(\partial_z x\right)^2\,\mathrm{d}z \leq -D\pi^2\int_0^1 x^2\,\mathrm{d}z \tag{16}$$

it follows that

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} &\leq -D\pi^2\|x\|^2 + D\tilde{w}\int_0^1 \alpha'' x\,\mathrm{d}z - \left(\int_0^1 \alpha b\,\mathrm{d}z\right)\tilde{w}^2 \\
&\quad - \left(k\int_0^1 \alpha b\,\mathrm{d}z + 1 - l\right)\tilde{w}\sigma - k\sigma^2 + l\sigma\int_0^1 \alpha x\,\mathrm{d}z \\
&\quad + \int_0^1 xg\nu\,\mathrm{d}z + \tilde{w}\int_0^1 \alpha g\nu\,\mathrm{d}z.
\end{aligned}
$$

Accordingly, the $(-r,-s,-q)$ strict state dissipativity with dissipation rate $\kappa > 0$ is ensured if

$$
\begin{aligned}
&- D\pi^2\|x\|^2 + D\tilde{w}\int_0^1 \alpha'' x\,\mathrm{d}z - \left(\int_0^1 \alpha b\,\mathrm{d}z\right)\tilde{w}^2 \\
&- \left(k\int_0^1 \alpha b\,\mathrm{d}z + 1 - l\right)\tilde{w}\sigma - k\sigma^2 + l\sigma\int_0^1 \alpha x\,\mathrm{d}z \\
&+ \int_0^1 xg\nu\,\mathrm{d}z + \tilde{w}\int_0^1 \alpha g\nu\,\mathrm{d}z \\
&\leq -\kappa\left(\|x\|^2 + \tilde{w}^2\right) - r\sigma^2 - 2s\int_0^1 \nu\,\mathrm{d}z\sigma - q\int_0^1 \nu^2\,\mathrm{d}z
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&- (D\pi^2 - \kappa)\|x\|^2 + D\tilde{w}\int_0^1 \alpha'' x\,\mathrm{d}z \\
&- \left(\int_0^1 \alpha b\,\mathrm{d}z - \kappa\right)\tilde{w}^2 - \left(k\int_0^1 \alpha b\,\mathrm{d}z + 1 - l\right)\tilde{w}\sigma \\
&- (k-r)\sigma^2 + l\sigma\int_0^1 \alpha x\,\mathrm{d}z + \int_0^1 xg\nu\,\mathrm{d}z + \tilde{w}\int_0^1 \alpha g\nu\,\mathrm{d}z \\
&+ 2s\sigma\int_0^1 \nu\,\mathrm{d}z + q\int_0^1 \nu^2\,\mathrm{d}z \leq 0
\end{aligned}
$$

Given that $b$ is zero outside a compact sub-domain in $[0,1]$ the above inequality cannot be satisfied pointwise for all $x \in [0,1]$ but only in an integral manner over the complete domain $[0,1]$. Thus note that in virtue of the Cauchy-Schwarz inequality a sufficient condition is given by

$$
\begin{aligned}
&- (D\pi^2 - \kappa)\|x\|^2 + D|\tilde{w}|\|\alpha''\|\,\|x\| \\
&- \left(\int_0^1 \alpha b\,\mathrm{d}z - \kappa\right)|\tilde{w}|^2 + \left(k\left|\int_0^1 \alpha b\,\mathrm{d}z\right| + |1-l|\right)|\tilde{w}|\,|\sigma| \\
&- (k-r)|\sigma|^2 + |l|\,|\sigma|\|\alpha\|\,\|x\| + |g|\,\|x\|\,\|\nu\| \\
&+ |\tilde{w}|\,|g|\,\|\alpha\|\,\|\nu\| + 2|s|\,|\sigma|\|\nu\| + q\|\nu\|^2 \leq 0.
\end{aligned}
$$

Introducing the vector $\boldsymbol{\xi} = [\|x\|\ |\tilde{w}|\ |\sigma|\ \|\nu\|]^\mathsf{T}$ and the matrix

$$
M = \begin{bmatrix}
-D\pi^2 + \kappa & \frac{D\|\alpha''\|}{2} & \frac{\|\alpha\||l|}{2} & \frac{|g|}{2} \\
\frac{D\|\alpha''\|}{2} & -\int_0^1 \alpha b\,\mathrm{d}z + \kappa & \frac{|1-l|+k\left|\int_0^1 \alpha b\,\mathrm{d}z\right|}{2} & \frac{\|\alpha\||g|}{2} \\
\frac{\|\alpha\||l|}{2} & \frac{|1-l|+k\left|\int_0^1 \alpha b\,\mathrm{d}z\right|}{2} & -k+r & |s| \\
\frac{|g|}{2} & \frac{\|\alpha\||g|}{2} & |s| & q
\end{bmatrix}
$$

the preceding inequality is compactly written as

$$\boldsymbol{\xi}^\mathsf{T} M\boldsymbol{\xi} \preceq 0$$

and holds true if

$$M \preceq 0. \tag{17}$$

The matrix $M$ can be written in compact form as

$$
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},\ M_{11} = \begin{bmatrix} -D\pi^2 + \kappa & \frac{D\|\alpha''\|}{2} \\ \frac{D\|\alpha''\|}{2} & -\int_0^1 \alpha b\,\mathrm{d}z + \kappa \end{bmatrix},
$$

$$
M_{12} = \begin{bmatrix} \frac{\|\alpha\||l|}{2} & \frac{|g|}{2} \\ \frac{|1-l|+k\left|\int_0^1 \alpha b\,\mathrm{d}z\right|}{2} & \frac{\|\alpha\||g|}{2} \end{bmatrix}, \quad M_{21} = M_{12}^\mathsf{T},
$$

$$
M_{22} = \begin{bmatrix} -k+r & |s| \\ |s| & q \end{bmatrix}.
$$

For $\mathrm{tr}\,(M_{22}) < 0, \det\,(M_{22}) > 0$ it holds that $M_{22} \prec 0$. This, in turn, is ensured if

$$q < 0, \quad k > r + \frac{|s|^2}{|q|}. \tag{18a}$$

Provided this holds true, it follows from standard arguments using the Schur complement (Dym, 2007) that

$$M \preceq 0 \quad \Leftarrow \quad \Delta_{21} = M_{11} - M_{12}M_{22}^{-1}M_{21} \preceq 0. \tag{18b}$$

Note that the remaining degrees of freedom to satisfy this condition are $\kappa, l, \alpha$ and $b = h$. Thus, the condition can be interpreted in terms of dissipation optimization, i.e. maximization of $\kappa$ in terms of the sensor (and actuator) placement and shape. In accordance with the conditions on $M_{22}$, the condition (18b) on $\Delta_{21}$ holds true if

$$\mathrm{tr}\,(\Delta_{21}) < 0 \tag{18c}$$

$$\det\,(\Delta_{21}) > 0. \tag{18d}$$

This is summarized in the following theorem.

*Theorem 1.* Consider the semilinear heat equation (1) with constant actuator offset. Let the nonlinearity $\varphi$ be $(q, s, r)$-dissipative with $q < 0$. If there exists a function $\alpha \in \mathcal{C}^2([0, 1], [0, 1))$ with $\alpha(0) = \alpha(1) = 0$, a collocated sensor-actuator shape function $b$ (or $h$) and controller gains $k, l > 0$ so that (18a),(18c) and (18d) hold true for a positive $\kappa$, then the linear subsystem $\Sigma(A - bkH, g, H)$ is $(-r, -s, -q)$-*strictly state dissipative with dissipation rate* $\kappa$ and the closed-loop semilinear system (11) is exponentially stable. $\diamond$

**Remark:** Note that for a given triplet $(q, s, r)$ the search for appropriate functions $\alpha, b$ (or $h$) and the determination of the controller gains $(k, l)$ to ensure the conditions of Theorem 1 can be carried e.g. using an optimization of the dissipation rate $\kappa$. Considering e.g. the particular (typical) functions

$$\alpha(z) = z(1 - z) \tag{19}$$

$$b(z) = \begin{cases} \frac{1}{2\epsilon}, & z \in [\zeta - \epsilon, \zeta + \epsilon] \\ 0, & \text{else} \end{cases} \tag{20}$$

this can be implemented in the form of the constrained static optimization problem

$$\min_{k,l,\zeta,\epsilon} \frac{1}{\kappa}$$

$$s.t. \quad \kappa > 0, \quad k > r + \frac{|s|^2}{|q|},$$

$$\operatorname{tr}(\Delta_{21}) < 0, \quad \det(\Delta_{21}) > 0,$$

which can be solved using standard algorithms. Note that this interpretation enables a numerical approach to maximizing the dissipation rate $\kappa$ by choosing the sensor and actuator position $\zeta$, length $2\epsilon$ and the control gains $k, l$.

## 5. CASE STUDY

Consider the semilinear heat equation (1) with

$$\varphi(\sigma) = \frac{k_0\sigma(1 - \sigma)}{1 + \sigma^2}, \quad k_0 = 2.0. \tag{21a}$$

The graph of this function is shown in Figure 1 and contained in the sector $[k_1, k_2] = [-0.42, 2.5]$. The initial condition is chosen as $x_0(z) = 0.05 \sin^2(\pi z)$ and the following parameters and constant actuator offset are set

$$D = 0.1, \quad g = 1.5, \quad w = 0.5. \tag{21b}$$

The control gains $(k, l)$ and the sensor-actuator size $2\epsilon$ and position $\zeta$, as well as the predicted dissipation rate $\kappa$ have been determined following the optimization procedure described in Remark 1 set with $\alpha$ and $b$ given in (19) and using the sequential least squares programming algorithm `SLSQP` that is implemented as a standard solver in `scipy.minimize`. The optimized values are given by

$$\zeta = 0.5, \ \epsilon = 0.01, \ k = 2.132, \ l = 3.433, \ \kappa = 0.31. \tag{22}$$

To illustrate the controller performance numerical simulations have been carried out based on a finite difference approximation of the system dynamics with $N = 400$ discretization points and solving the resulting ode system using the standard algorithm `dopri5` implemented in the package `scipy.integrate`.

In Figure 2 the open-loop behavior is shown. It can be seen that in spite of the small initial deviation the profile converges to the non-zero stationary profile in about 2 time units, illustrating that in open-loop the solution $x = 0$ is repulsive.

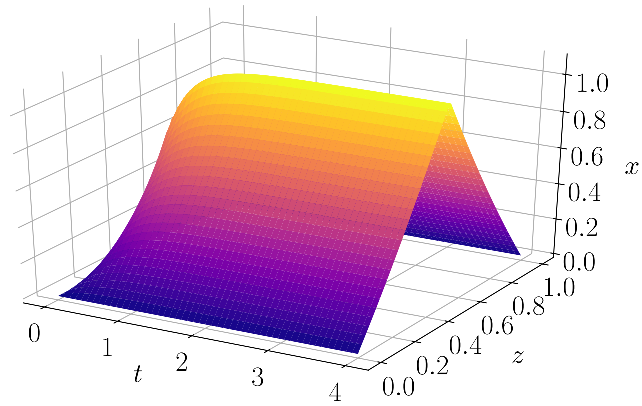


Fig. 2. Open-loop profile evolution for the semilinear heat equation (1) with (21).

The behavior of the closed-loop system with the dissipative PI control (2) using the optimized parameters (22) is shown in Figure 3. It can be seen that the actuator disturbance is completely attentuated and the exponential stability of the solution $x = 0$ is achieved with convergence in about 2 time units.

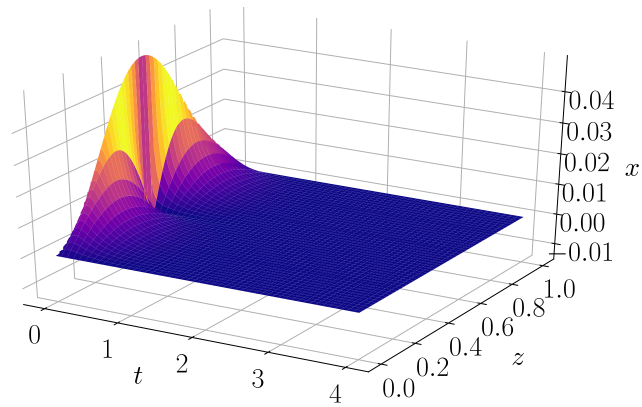


Fig. 3. Closed-loop profile evolution for the semilinear heat equation (1) with (21) and dissipative PI control (2).

The associated control input is shown in Figure 4 where it can be seen that the constant actuator offset of $w = 0.5$ is completely compensated while the control action stays within considerable small amplitudes.

## 6. CONCLUSIONS

The problem of stabilization of a semilinear heat equation with actuator disturbance has been resolved using a dissipative PI output feedback control. Sufficient conditions for strict dissipativity in the state and thus exponential stability of the closed loop system have been derived, depending on the dissipativity characteristics of the nonlinearity (e.g., in form of a sector condition), the controller gains and the actuator and sensor shape functions and localization. The sensor and actuator positioning problem has been

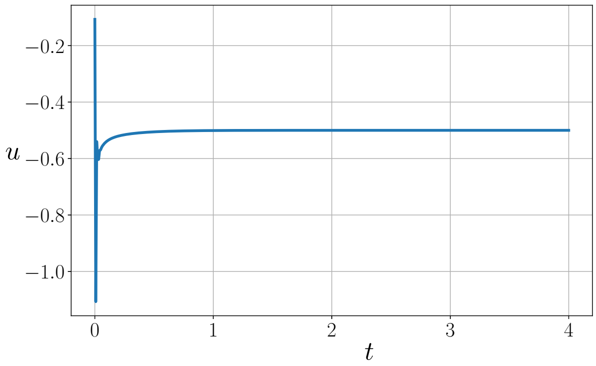


Fig. 4. Closed-loop control input signal for the parameters given in (22).

reformulated as a dissipation maximization problem that can be solved using standard tools for static constrained optimization. Numerical simulations illustrate the performance of the proposed approach.

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