Boundary Control for Exponential Synchronization of Reaction-Diffusion Neural Networks Based on Coupled PDE-ODEs

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Abstract: This paper studies synchronization and exponential synchronization of reaction-diffusion neural networks based on semi-linear coupled partial differential equations-ordinary differential equations. Two kinds of boundary control methods are studied, one collocated boundary measurement based form and the other distributed measurement based form. Sufficient conditions for existence of the two proposed boundary controllers for synchronization and exponential synchronization are respectively obtained in terms of LMIs. Two examples are given to show the effectiveness of the proposed boundary control.

Keywords: Reaction-diffusion neural networks; Boundary control; Synchronization

1. INTRODUCTION

Synchronization of neural networks, an important dynamic behaviour, has attracted a great deal of attention during the last few decades. It has been widely applied to engineering fields such as secure communication (Song et al. (2018)), image encryption (Mani et al. (2019)), image protection (Hu et al. (2018)), and price prediction (Huang and Wang (2018)).

As been well known, the diffusion effects could not be ignored in fields as shown in Liang and Cao (2003) and Wang et al. (2016). Many important results for stability and synchronization of reaction-diffusion neural networks (RDNNs) were obtained. Vidhya et al. (2019) studied global asymptotic stability of stochastic Markovian jumping RDNNs with discrete and distributed delays; Yang et al. (2013) proposed pinning-impulsive control of RDNNs with Dirichlet boundary condition; Yang et al. (2018) proposed quantized control of markovian RDNNs with proportional delays; Rakkiyappan et al. (2015) using Jensenąŕs inequality and reciprocally convex technique discussed sampled-data controllers for synchronization of RDNNs with Dirichlet boundary conditions; Rakkiyappan and Dharani (2017) proposed sampled-data synchronization of randomly RDNNs with Markovian jumping and mixed delays; Dharani et al. (2017) studied a pinning sampled-data controller for synchronization of inertial RDNNs with time-varying delays; Wang et al. (2018a) proposed a finite-time controller of RDNNs with coupling delays; Stamova and Simeonov (2018) studied Mittag-Leffler-type criteria of fractional RDNNs with time-varying delays; Hou et al. (2019) studied pinning control of multi-weighted RDNNs; Xie et al. (2019) proposed pinning impulsive control of RDNNs with distributed delays and discrete delays. Actually, the mentioned literature assume that nodes can receive communication in all the spatial domain. When the nodes of the RDNNs can only receive communication in the spatial boundary domain, it is desired to design spatial boundary coupling. Therefore, it is important to research RDNN using spatial boundary coupling, which has not been solved.

Owing to the extensively exist of partial differential equations-ordinary differential equations (PDE-ODEs), this paper investigates exponential synchronization of stochastic reaction-diffusion neural networks (SRDNNs) based on semi-linear coupled PDE-ODEs. Two kinds of boundary control methods are studied, one collocated boundary measurement based form and the other distributed measurement based form. Using the Lyapunov direct method and Wirtinger's inequality, sufficient conditions for existence of the two proposed boundary controllers for synchronization and exponential synchronization are respectively obtained in terms of LMIs.

Notations: Some notations will be used below. I denotes identity matrix with proper order. $P < 0$ means negative definite. X^T denotes the transpose of X.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider a semi-linear parabolic PDE-ODEs based RDNN (PDE-ODERDNN) as

$$\begin{aligned}\dot{x}_i(t) &= f(x_i(t)) + A\int_0^L y_i(\xi,t)d\xi \\ &\quad + c_1\sum_{j=1}^N g_{ij}\Gamma_1 x_j(t) + u_i(t), \\ \frac{\partial y_i(\xi,t)}{\partial t} &= \Theta\frac{\partial^2 y_i(\xi,t)}{\partial \xi^2} + p(y_i(\xi,t)) + Bx_i(t) \\ &\quad + c_2\sum_{j=1}^N h_{ij}\Gamma_2 y_j(\xi,t),\end{aligned} \tag{1}$$

such that

$$\begin{aligned}&\left.\frac{\partial y_i(\xi,t)}{\partial \xi}\right|_{\xi=0} = 0,\ \left.\frac{\partial y_i(\xi,t)}{\partial \xi}\right|_{\xi=L} = U_i(t), \\ &x_i(0) = x_i^0, y_i(\xi,0) = y_i^0(\xi),\end{aligned} \tag{2}$$

where $(\xi,t) \in [0,L]\times[0,\infty)$ respectively stand for the spatial variable and the time variable; $i \in \{1,2,\cdots,N\}$; $x_i(t), y_i(\xi,t) \in \mathbb{R}^n$ are the states; $u_i(t), U_i(t) \in \mathbb{R}^n$ are control inputs; $x_i^0(t)$ and $y_i^0(\xi,t)$ are the initial states; $\alpha > 0$ is a known scalar; A and B are known $n\times n$ matrices; and $f(\cdot), p(\cdot) \in \mathbb{R}^n$ are sufficiently smooth nonlinear functions; and c_1, c_2 are the coupling strengths. Assume that the topological structure $G = (g_{ij})_{N\times N}$ is defined as: $g_{i,j} > 0 (i \neq j)$ if i is connected to j , otherwise $g_{i,j} = 0 (i \neq j)$; and $g_{ii} = -\sum_{j=1, j\neq i}^N g_{ij}$, $i,j \in \{1,2,\cdots,N\}$. And $H = (h_{ij})_{N\times N}$ is defined same to G.

The isolated node of the PDE-ODERDNN (1) is given as

$$\begin{aligned}\dot{s}(t) &= f(s(t)) + A\int_0^L \nu(\xi,t)d\xi, \\ \frac{\partial \nu(\xi,t)}{\partial t} &= \Theta\frac{\partial^2 \nu(\xi,t)}{\partial \xi^2} + p(\nu(\xi,t)) + Bs(t),\end{aligned} \tag{3}$$

such that the following boundary conditions and initial conditions:

$$\begin{aligned}&\left.\frac{\partial \nu(\xi,t)}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial \nu(\xi,t)}{\partial \xi}\right|_{\xi=L} = 0, \\ &s(0) = s^0, \nu(\xi,0) = \nu^0(\xi),\end{aligned} \tag{4}$$

where $s(\xi,t) = [s_1(\xi,t), s_2(\xi,t), \cdots, s_n(\xi,t)]^T$ is the state; $s^0(\xi)$ is the bounded and continuous initial function.

Let $e(t) \triangleq x_i(t) - s(t), \varepsilon_i(\xi,t) \triangleq y_i(\xi,t) - \nu(\xi,t)$. The synchronization error system can be obtained from (1) and (3) that

$$\begin{aligned}e_i(t) &= f(x_i(t)) - f(s(t)) + A\int_0^L \varepsilon_i(\xi,t)d\xi \\ &\quad + c_1\sum_{j=1}^N g_{ij}\Gamma_1 e_j(t) + u_i(t), \\ \frac{\partial \varepsilon_i(\xi,t)}{\partial t} &= \Theta\frac{\partial^2 \varepsilon_i(\xi,t)}{\partial \xi^2} + p(y_i(\xi,t)) - p(\nu(\xi,t)) \\ &\quad + Be_i(t) + c_2\sum_{j=1}^N h_{ij}\Gamma_2 \varepsilon_j(\xi,t),\end{aligned} \tag{5}$$

such that

$$\begin{aligned}&\left.\frac{\partial \varepsilon_i(\xi,t)}{\partial \xi}\right|_{\xi=0} = 0,\ \left.\frac{\partial \varepsilon_i(\xi,t)}{\partial \xi}\right|_{\xi=L} = U_i(t), \\ &e_i(\xi,0) = e_i^0, \varepsilon_i(\xi,0) = \varepsilon_i^0(\xi),\end{aligned} \tag{6}$$

where $e_i^0 \triangleq x_i^0 - s^0$ and $\varepsilon_i^0(\xi) \triangleq y_i^0(\xi) - \nu^0(\xi)$.

Definition 1. For the CSN (1) and the isolated node (3) with any initial conditions, if

$$lim_{t\to\infty}||y_i(\xi,t) - s(\xi,t)||_2 \to 0 \tag{7}$$

for any $i \in \{1,2,\cdots,N\}$, then the CSN (1) synchronizes the isolated node (3).

Lemma 2. (See Wang et al. (2018b).) $z \in W^{1,2}([0,1];\mathbb{R}^n)$ with $z(0) = 0$ or $z(1) = 0$, then

$$\int_0^1 z^T(s)z(s)ds \le \pi^{-2}\int_0^1 \dot{z}^T(s)\dot{z}(s)ds. \tag{8}$$

Assumption 1. Assume f and p satisfy the Lipschitz condition, i.e., for any sclars s_1 and s_2, there exist scalars $\gamma_1, \gamma_2 > 0$ such that

$$\begin{aligned}|f(s_1) - f(s_2)| &\le \gamma_1|s_1 - s_2|, \\ |p(s_1) - p(s_2)| &\le \gamma_2|s_1 - s_2|.\end{aligned} \tag{9}$$

3. SYNCHRONIZATION CONTROL VIA PROPORTIONAL CONTROL

To achieve synchronization of the PDE-ODERDNN (1), the boundary controller is first studied as:

$$\begin{aligned}u_i(t) &= -k_i e_i(t), \\ U_i(t) &= -d_i\varepsilon_i(L,t),\end{aligned} \tag{10}$$

where k_i and d_i are the feedback strengths to be determined.

Theorem 3. Under Assumption 1, using the boundary controller (10), the PDE-ODERDNN (1) synchronizes the isolated node (3) if there exist $K > 0$ and $D > 0$ such that

$$\Psi \triangleq \begin{bmatrix}\Psi_{11} & 0 & 0.5I_N\otimes(A+B^T) \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33}\end{bmatrix} < 0, \tag{11}$$

where

$$\begin{aligned}\Psi_{11} &\triangleq \gamma_1 L^{-1}I_{Nn} + 0.5c_1L^{-1}(G\otimes\Gamma_1 + G^T\otimes\Gamma_1^T) \\ &\quad - L^{-1}K\otimes I_n, \\ \Psi_{22} &\triangleq -0.25L^{-2}\pi^2 I_N\otimes\Theta - L^{-1}D\otimes\Theta, \\ \Psi_{23} &\triangleq 0.25L^{-2}\pi^2 I_N\otimes\Theta, \\ \Psi_{33} &\triangleq \gamma_2 I_{Nn} + 0.5c_2(H\otimes\Gamma_2 + H^T\otimes\Gamma_2^T) \\ &\quad - 0.25L^{-2}\pi^2 I_N\otimes\Theta, \\ K &\triangleq diag\{k_1, k_2, \ldots, k_N\}, \\ D &\triangleq diag\{d_1, d_2, \ldots, d_N\}.\end{aligned}$$

Proof. Consider the Lyapunov functional candidate as

$$V(t) = \frac{1}{2}\sum_{i=1}^N e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^N\int_0^L \varepsilon_i^T(\xi,t)\varepsilon_i(\xi,t)d\xi. \tag{12}$$

One has

$$
\begin{aligned}
\dot{V}(t) =& \sum_{i=1}^{N} e_i^T(t) \frac{de_i(t)}{dt} \\
&+ \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t) \frac{\partial \varepsilon_i(\xi,t)}{\partial t} d\xi \\
=& \sum_{i=1}^{N} e_i^T(t)(f(x_i(t)) - f(s(t))) \\
&+ \sum_{i=1}^{N} e_i^T(t) A \int_0^L \varepsilon_i(\xi,t) d\xi \\
&+ c_1 \sum_{i=1}^{N} e_i^T(t) \sum_{j=1}^{N} g_{ij}\Gamma_1 e_j(t) \\
&- k_i \sum_{i=1}^{N} e_i^T(t) e_i(t) \\
&+ \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t)\Theta \frac{\partial^2 \varepsilon_i(\xi,t)}{\partial \xi^2} d\xi \\
&+ \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t)(p(y_i(\xi,t)) - p(\nu(\xi,t))) d\xi \\
&+ \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t) B e_i(t) d\xi \\
&+ c_2 \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t) \sum_{j=1}^{N} h_{ij}\Gamma_2 \varepsilon_j(\xi,t) d\xi.
\end{aligned} \tag{13}
$$

Using Lemma 2, for $\Theta > 0$,

$$
\begin{aligned}
&\int_0^L \sum_{i=1}^{N} \varepsilon_i^T(\xi,t)\Theta\varepsilon_{i,\xi\xi}(\xi,t) d\xi \\
=& \sum_{i=1}^{N} \varepsilon_i^T(\xi,t)\Theta\varepsilon_{i,\xi}(\xi,t)\Big|_{x=0}^{x=L} \\
&- \int_0^L \sum_{i=1}^{N} \varepsilon_{i,\xi}^T(\xi,t)\alpha\varepsilon_{i,\xi}(\xi,t) d\xi \\
=& -\sum_{i=1}^{N} d_i \varepsilon_i^T(L,t)\Theta\varepsilon_i(L,t) \\
&- 0.25L^{-2}\pi^2 \sum_{i=1}^{N} \int_0^L (\varepsilon_i^T(\xi,t) - \varepsilon_i^T(L,t))\Theta \\
&\qquad (\varepsilon_i(\xi,t) - \varepsilon_i(L,t)) d\xi
\end{aligned} \tag{14}
$$

Using Assumption 1,

$$
\begin{aligned}
&\sum_{i=1}^{N} e_i^T(t)(f(x_i(t)) - f(s(t))) \\
\le& \gamma_1 \sum_{i=1}^{N} e_i^T(t) e_i(t),
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t)(p(y_i(\xi,t)) - p(\nu(\xi,t))) d\xi \\
\le& \gamma_2 \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t)\varepsilon_i(\xi,t) d\xi.
\end{aligned} \tag{16}
$$

With (18), substituting (7) and (8) into (8),

$$
\begin{aligned}
\dot{V}(t) \le& \sum_{i=1}^{N} e_i^T(t)(\gamma_1 - k_i)e_i(t) \\
&+ c_1 \sum_{i=1}^{N} e_i^T(t) \sum_{j=1}^{N} g_{ij}\Gamma_1 e_j(t) \\
&+ \gamma_2 \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t)\varepsilon_i(\xi,t) d\xi \\
&+ c_2 \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t) \sum_{j=1}^{N} h_{ij}\Gamma_2 \varepsilon_j(\xi,t) d\xi \\
&+ \sum_{i=1}^{N} \int_0^L e_i^T(t) A \varepsilon_i(\xi,t) d\xi \\
&+ \sum_{i=1}^{N} \int_0^L \varepsilon_i^T(\xi,t) B e_i(t) d\xi \\
&- d \sum_{i=1}^{N} \varepsilon_i^T(L,t)\Theta\varepsilon_i(L,t) \\
&- 0.25L^{-2}\pi^2 \sum_{i=1}^{N} \int_0^L (\varepsilon_i^T(\xi,t) - \varepsilon_i^T(L,t))\Theta \\
&\qquad (\varepsilon_i(\xi,t) - \varepsilon_i(L,t)) d\xi \\
=& \int_0^L \hat{\varepsilon}^T(\xi,t)\Psi\hat{\varepsilon}(\xi,t) d\xi < 0,
\end{aligned} \tag{17}
$$

where $\hat{\varepsilon}(\xi,t) = [e^T(t), \varepsilon^T(L,t), \varepsilon^T(\xi,t)]^T$, $e \triangleq [e_1^T, e_2^T, \cdots, e_N^T]^T$ and $\varepsilon \triangleq [\varepsilon_1^T, \varepsilon_2^T, \cdots, \varepsilon_N^T]^T$, and Ψ is defined in Eq.(11). This completes the proof. □

Theorem 4. Under Assumption 1, using the controller (10), the PDE-ODERDNN (1) exponentially synchronizes the isolated node (3) if

$$
\bar{\Psi} \triangleq \begin{bmatrix} \bar{\Psi}_{11} & 0 & 0.5I_N \otimes (A + B^T) \\ * & \bar{\Psi}_{22} & \bar{\Psi}_{23} \\ * & * & \bar{\Psi}_{33} \end{bmatrix} < 0, \tag{18}
$$

in which

$$
\begin{aligned}
\bar{\Psi}_{11} &\triangleq \Psi_{11} + \rho L^{-1} I_{Nn}, \\
\bar{\Psi}_{22} &\triangleq -0.25L^{-2}\pi^2 I_N \otimes \Theta - dL^{-1} I_N \otimes \Theta, \\
\bar{\Psi}_{23} &\triangleq 0.25L^{-2}\pi^2 I_N \otimes \Theta, \\
\bar{\Psi}_{33} &\triangleq \Psi_{33} + \rho I_{Nn}.
\end{aligned}
$$

Proof. Construct the Lyapunov functional candidate as

$$
\begin{aligned}
&\dot{V}(t) + \rho V(t) \\
\le& \int_0^L \hat{\varepsilon}^T(\xi,t)\bar{\Psi}\hat{\varepsilon}(\xi,t) d\xi \\
<& 0.
\end{aligned} \tag{19}
$$

This completes the proof. □

4. SYNCHRONIZATION VIA PROPORTIONAL CONTROL

To achieve synchronization of the PDE-ODERDNN (1), the proportional controller is first studied as:

$$\begin{aligned} u_i(t) &= -k_i e_i(t), \\ U_i(t) &= -d_i \int_0^L \varepsilon_i(\xi,t)d\xi, \end{aligned} \tag{20}$$

in which k_i and d_i are the feedback strengths to be determined.

With employing the adaptive controller (10) for the PDE-ODERDNN (1), the following result can be obtained.

Theorem 5. Under Assumption 1, using the boundary controller (20), the PDE-ODERDNN (1) synchronizes the isolated node (3) if

$$\Xi \triangleq \begin{bmatrix} \Psi_{11} & 0 & 0.5I_N \otimes (A+B^T) \\ * & \Xi_{22} & \Xi_{23} \\ * & * & \Xi_{33} \end{bmatrix} < 0, \tag{21}$$

in which

$$\begin{aligned} \Xi_{22} &\triangleq -0.25L^{-2}\pi^2 I_N \otimes \Theta \\ \Xi_{23} &\triangleq dI_N \otimes \Theta \\ \Xi_{33} &\triangleq \Psi_{33} - dI_N \otimes \Theta. \end{aligned}$$

Proof. Construct the Lyapunov functional candidate as

$$V(t) = \frac{1}{2}\sum_{i=1}^N e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^N \int_0^L \varepsilon_i^T(\xi,t)\varepsilon_i(\xi,t)d\xi. \tag{22}$$

Using Lemma 2, $\Theta > 0$,

$$\begin{aligned} &\int_0^L \sum_{i=1}^N \varepsilon_i^T(\xi,t)\Theta\varepsilon_{i,\xi\xi}(\xi,t)d\xi \\ =&\sum_{i=1}^N \varepsilon_i^T(\xi,t)\Theta\varepsilon_{i,\xi}(\xi,t)\Big|_{x=0}^{x=L} \\ &- \int_0^L \sum_{i=1}^N \varepsilon_{i,\xi}^T(\xi,t)\alpha\varepsilon_{i,\xi}(\xi,t)d\xi \\ =&-d\sum_{i=1}^N \varepsilon_i^T(L,t)\Theta\int_0^L \varepsilon_i(\xi,t)d\xi \\ &-0.25L^{-2}\pi^2\sum_{i=1}^N\int_0^L \tilde{\varepsilon}_i^T(\xi,t)\Theta\tilde{\varepsilon}_i(\xi,t)d\xi, \end{aligned} \tag{23}$$

where $\tilde{\varepsilon}_i(\xi,t) = \varepsilon_i(\xi,t) - \varepsilon_i(L,t)$. According to the Kronecker product for matrices and (18), substituting (7) and (8) into the time derivative of $V(t)$,

$$\begin{aligned} \dot{V}(t) \le& \sum_{i=1}^N e_i^T(t)(\gamma_1 - k_i)e_i(t) \\ &+ c_1\sum_{i=1}^N e_i^T(t)\sum_{j=1}^N g_{ij}\Gamma_1 e_j(t) \\ &+ \gamma_2\sum_{i=1}^N\int_0^L \varepsilon_i^T(\xi,t)\varepsilon_i(\xi,t)d\xi \\ &+ c_2\sum_{i=1}^N\int_0^L \varepsilon_i^T(\xi,t)\sum_{j=1}^N h_{ij}\Gamma_2\varepsilon_j(\xi,t)d\xi \\ &+ \sum_{i=1}^N\int_0^L e_i^T(t)A\varepsilon_i(\xi,t)d\xi \\ &+ \sum_{i=1}^N\int_0^L \varepsilon_i^T(\xi,t)Be_i(t)d\xi \\ &- d\sum_{i=1}^N \varepsilon_i^T(L,t)\Theta\int_0^L \varepsilon_i(\xi,t)d\xi \\ &- 0.25L^{-2}\pi^2\sum_{i=1}^N\int_0^L \tilde{\varepsilon}_i^T(\xi,t)\Theta\tilde{\varepsilon}_i(\xi,t)d\xi \\ =& \int_0^L \hat{\varepsilon}^T(\xi,t)\Xi\hat{\varepsilon}(\xi,t)d\xi, \end{aligned} \tag{24}$$

where $\check{\varepsilon}(\xi,t) = [e^T(t), \tilde{\varepsilon}^T(\xi,t), \varepsilon^T(\xi,t)]^T$ and Ξ is defined in Eq. (12).

Theorem 6. Under Assumption 1, using the boundary controller (10), the PDE-ODERDNN (1) exponentially synchronizes the isolated node (3) if

$$\bar{\Xi} \triangleq \begin{bmatrix} \bar{\Psi}_{11} & 0 & 0.5I_N \otimes (A+B^T) \\ * & \bar{\Xi}_{22} & \bar{\Xi}_{23} \\ * & * & \bar{\Xi}_{33} \end{bmatrix} < 0, \tag{25}$$

in which

$$\begin{aligned} \bar{\Xi}_{22} &\triangleq -0.25L^{-2}\pi^2 I_N \otimes \Theta, \\ \bar{\Xi}_{23} &\triangleq dI_N \otimes \Theta, \\ \bar{\Xi}_{33} &\triangleq \Xi_{33} + \rho I_{Nn}. \end{aligned}$$

Proof. Construct the Lyapunov functional candidate as

$$\begin{aligned} &\dot{V}(t) + \rho V(t) \\ &\le \int_0^L \hat{\varepsilon}^T(\xi,t)\bar{\Xi}\hat{\varepsilon}(\xi,t)d\xi \\ &<0. \end{aligned} \tag{26}$$

This completes the proof. □

Remark 7. Different from stabilization of PDE-ODE systems (Di Meglio et al. (2018); Wang and Wu (2019); Wang and Krstic (2019); Zhao et al. (2019)), this paper proposed synchronization control of PDE-ODE based RDNNs.

Remark 8. He (2018); Dai et al. (2019); Lan et al. (2019); Qi et al. (2019) studied synchronization control of many sorts of PDE based RDNNs and obtained many important results. Different from PDE based RDNNs, synchronization control for PDE-ODERDNNs is studied in this paper.

5. NUMERICAL SIMULATION

Example 1. Consider a semi-linear PDE-ODERDNN (1) composed of 4 nodes with coefficients listed as

$$\Theta = A = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, f(\cdot) = p(\cdot) = \tanh(\cdot), n = 2,$$
$$G = H = \begin{bmatrix} -5 & 2 & 1 & 2 \\ 1 & -6 & 1 & 4 \\ 2 & 4 & -7 & 1 \\ 2 & 0 & 1 & -3 \end{bmatrix}. \quad (27)$$

and with random initial conditions.

It is not difficult to verify that $f(\cdot) = p(\cdot)$ satisfies the Lipschitz condition with $\gamma_1 = \gamma_2 = 1$. According to Theorem 3, $k_1 = 7.6776, k_2 = 7.2776, k_3 = 6.8776, k_4 = 8.4776$ are obtained. Applying the controllers (8) with the feedback gains, $e_i(t)$ and $\varepsilon_i(\xi, t)$ are obtained as shown in Figs. 1 and 2. Obviously, the proposed controller (8) can guide the PDE-ODERDNN (1) to synchronize the isolated node (3).

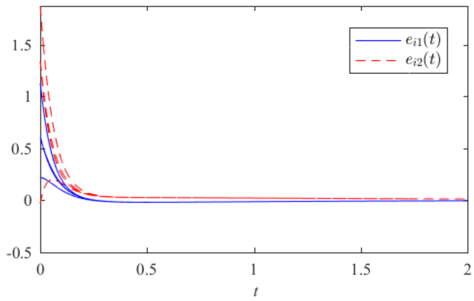

Fig. 1. $e_i(t)$

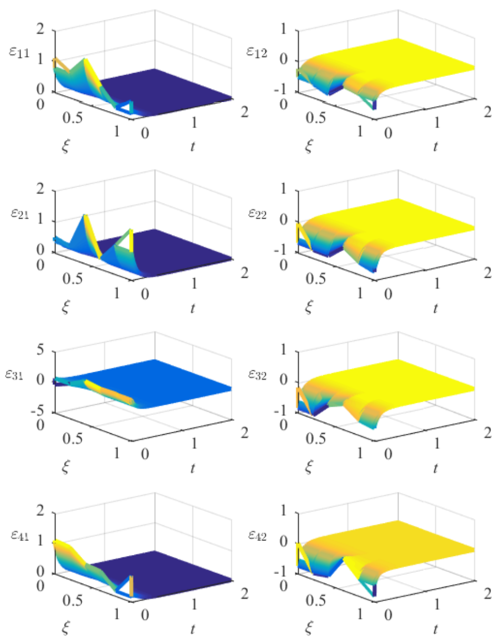

Fig. 2. $\varepsilon_i(\xi, t)$

Example 2. Consider a semi-linear PDE-ODERDNN (1) composed of 4 nodes with coefficients same to Example 1.

According to Theorem 5, $k_1 = 2.0451, k_2 = 2.0937, k_3 = 2.1174, k_4 = 1.9890$ are obtained. Applying the controllers (20) with the feedback gains, $\tilde{e}_i(t)$ and $\tilde{\varepsilon}_i(\xi, t)$ are obtained as shown in Figs. 3 and 4. Obviously, the proposed controller (20) can guide the PDE-ODERDNN (1) to synchronize the isolated node (3).

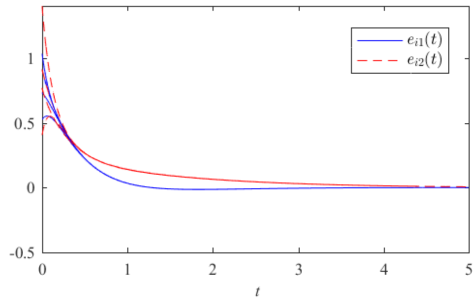

Fig. 3. $\tilde{e}_i(t)$

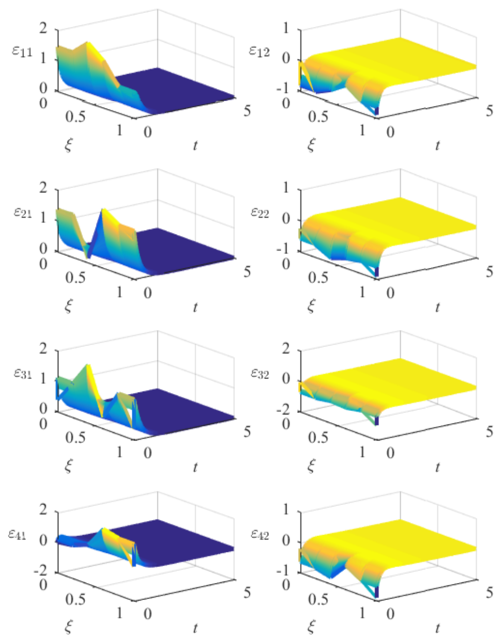

Fig. 4. $\tilde{\varepsilon}_i(\xi, t)$

6. CONCLUSIONS

Synchronization and exponential synchronization were respectively studied for RDNNs modelled by semi-linear parabolic PDE-ODEs. Two kinds of boundary control methods were studied, one collocated boundary measurement based form and the other distributed measurement based form. By using Lyapunov direct method and some inequalities, two sufficient synchronization criteria in terms of LMIs were obtained. Simulation results of numerical examples respectively verified the effectiveness of the proposed control respectively based on collocated boundary measurement and distributed measurement. One interesting topic in future is to further study pinning synchronization of PDE-ODERDNNs.

ACKNOWLEDGEMENTS

This work was jointly supported by the National Natural Science Foundation of China (Grant Nos. 61703193, 61773193, 61877033 and 61833005), the Natural Science Foundation of Shandong Province (Grant Nos. ZR2017MF022, ZR2019MF021 and ZR2019YQ28), and the Development Plan of Youth Innovation Team of University in Shandong Province (Grants No. 2019KJN007).

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