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Performance analysis for time-delay systems: application to the control of an active mass damper

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Abstract: This paper proposes a method to analyze, beyond stability, the performances of linear time-delay systems. Using robust analysis techniques, a sufficient condition that analyzes the location of eigenvalues in the complex plane is presented. More precisely, a set of quadratic inequality constraints are designed to define an admissible region for the infinitely many eigenvalues of a time-delay system and the quadratic separation theorem is applied to assess that the eigenvalues are effectively belonging to that stability region. This method is then used for the control of an active mass damper. A standard state feedback control is replaced with a static output feedback plus a static delayed output feedback. This strategy avoids the full measurement of the state and shows that delays in the dynamic may be helpful for stabilization. The closed-loop system is then expressed as a time-delay system and the performance criterion is exploited to analyze the stability and the damping properties. Simulations and experimental tests support the approach.

Keywords: time-delay systems, robust analysis, quadratic separation, performance analysis, vibration control, static delayed output feedback.

1. INTRODUCTION

Analyzing the stability of a linear time delay system is a challenging task since this is an infinite dimensional system and it admits an infinite number of eigenvalues. Assessing the stability by inspecting the root locus is therefore complicated even if some studies have been carried out in this direction as Breda et al. (2015). In the literature many works have been dedicated to find numerically tractable conditions ensuring the stability of time-delay systems by using either Lyapunov approach in Fridman (2010), Fridman (2014) or Input-Output approach in Ariba et al. (2012), Kao and Rantzer (2007).
Regarding the performance analysis, results on the topic are scarce. In fact, only the exponential stability (also called α-stability) have been studied in, for example, Hien and Trinh (2016). Mainly two approaches can be reported: one relying on the classical transformation $z(t) = e^{\alpha t}x(t)$ (see Mondie and Kharitonov (2005)) and others on specific Lyapunov-Krasovskii functionals, incorporating the α exponential information into their structure. To the best of our knowledge, the input-output approach have never been proposed to tackle this problem. In this work, we aim at considering more general regions in which eigenvalues should lie. We make use of an input-output robust approach, namely, the quadratic separation method that has already been used in the context of the stability of delay systems in Ariba et al. (2018). Basically, starting from an existing stability condition expressed in the quadratic separation framework, we design a new set of quadratic inequality constraints to redefine the domain of definition of the Laplace variable. More specifically, this new set of inequality constraints enforces the uncertain transfer functions that models the dynamic system to be well defined in the prescribed regions. It thus implies that the eigenvalues do not belong to those regions. Therefore, instead of defining a region where eigenvalues should be located (which is impossible except for α-stability), we define a region where they should not be.

The second objective of the paper is to use this methodology to stabilize an active mass damper. In that case, following some recent papers Michiels and Niculescu (2007), we consider that the classical state feedback controller can be approximated by a delayed proportional feedback controller. The closed-loop system becomes thus a time delay system. The practical application [1] considered in this work is a bench-scale building like tall structure as depicted in Figure 1. The mechanical structure being flexible, a highly oscillatory behavior is observed when subjected to some disturbance forces. The principle of the active mass damper is to move an actuated cart on the top of the structure so as to counterbalance the oscillations. Then, the control problem is not only to stabilize the whole system (cart + structure), but also to dampen the vibrations.

[1] URL: https://www.quanser.com/products/active-mass-damper/

Figure 1. *Active mass damper experiment*

2. PROBLEM STATEMENT

We consider in this study a standard linear time-delay system of the form:

$$\dot{x}(t) = Ax(t) + A_d x(t-h) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, A and $A_d \in \mathbb{R}^{n\times n}$ are constant matrices, and h is a constant delay. The eigenvalues of the system (1) are solutions of the characteristic equation:

$$det(s1_n - A - A_d e^{-sh}) = 0.$$

It is well known (see for instance Fridman (2014)) that, this system admits an infinite number of solutions called characteristics roots or also eigenvalues, which satisfies the following properties:

(1) There exists a finite number of eigenvalues in any set of the form $|s| \le \alpha, \alpha > 0$.
(2) There exists a finite number of eigenvalues for which the real part is greater than a prescribed $\beta > 0$.
(3) The solutions of the characteristic equation satisfy

$$\lim_{|s|\to+\infty} Re(s) = -\infty.$$

It means also that the time-delay system is asymptotically stable if all the eigenvalues have their real part strictly negative. At this stage, analyzing the performances of such a system is rather complicated as stated in Hien and Trinh (2016). Indeed, for LTI systems, there is only a finite number of poles and their placement in the complex plane indicates the dynamic performances. Thus, the concept of $\mathcal{D}$-stability was developed (see Henrion et al. (2001) and references therein) and allowed the development of methods to ensure that the poles of a finite dimensional linear systems belong to some given regions. Convex regions of the complex plane can be easily expressed as an inequality of the form:

$$d_1 ss^* + d_2 s^* + d_2^* s + d_3 \le 0. \tag{2}$$

Depending on the choice of parameters $\{d_1, d_2, d_3\}$, various regions $\mathcal{D}$ can be defined Henrion et al. (2001). Typical regions are:

- half-plane, $\mathcal{D} = \{s = x + jy \in \mathbb{C} : ax + by + c < 0\}$, where a, b, c are real scalars, obtained with the set $\{d_1, d_2, d_3\} = \{0, a + jb, 2c\}$.
- disk of center $s_0 \in \mathbb{C}$ and radius $r > 0$, $\mathcal{D} = \{s \in \mathbb{C} : |s - s_0| < r\}$, obtained with the set $\{d_1, d_2, d_3\} = \{1, -s_0, s_0 s_0^* - r^2\}$.

In the case of time-delay systems, due to its infinite dimensional nature, it is impossible to impose the eigenvalues to belong to certain bounded set. At least, in the literature of time delay systems, one can find the concept of α-stability, meaning that the eigenvalues belong to the set for which $Re(s) \le \alpha$. If $\alpha < 0$, it means therefore that the system is exponentially stable with a convergence rate α (see Hien and Trinh (2016) and references therein). The idea in this paper is to define regions that are, in this case, forbidden for the eigenvalues. Knowing the specific properties of eigenvalues of time delay systems of the form (1), suitable regions can be chosen so as to state a certain level of performances.

3. MAIN RESULT

3.1 A first stability result

Two approaches are generally considered to study the stability of (1): the Lyapunov method and the robust analysis. This latter is employed in this paper with the quadratic separation method (see Kao and Rantzer (2007); Peaucelle et al. (2007) and references therein). It consists in modeling the time-delay system (1) as an uncertain feedback system as shown in Figure 2. The uncertainty ∇ embeds the delay dynamics in the feedback block (different types of uncertain models are presented in Peaucelle et al. (2007); Iwasaki and Hara (1998); Ariba et al. (2018)) and some linear relationships between system signals are specified in the feedforward block. The stability is then tested with the conditions stated in the following theorem.

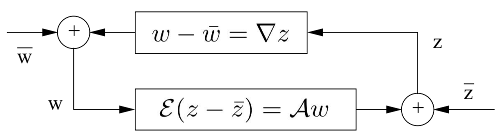

Figure 2. Feedback system.

Theorem 1. (Peaucelle et al. (2007)). Given the interconnection defined by Figure 2 where $\mathcal{E}$ and $\mathcal{A}$ are two real valued matrices and ∇ is a linear operator which represents the system uncertainties. This latter is assumed to belong to an uncertain set $\mathbb{W}$. For simplicity, we assume that $\mathcal{E}$ is full column rank. The uncertain feedback system of Figure 2 is well-posed and stable if and only if there exists a Hermitian matrix $\Theta = \Theta^*$ satisfying both conditions

$$\begin{bmatrix} 1 \\ \nabla \end{bmatrix}^* \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \le 0 \quad , \quad \forall \nabla \in \mathbb{W} \ , \tag{3}$$

$$[\, \mathcal{E} \ -\mathcal{A} \,]^{\perp *} \, \Theta \, [\, \mathcal{E} \ -\mathcal{A} \,]^{\perp} > 0. \tag{4}$$

Most of the work includes the modeling part (describing matrices $\mathcal{E}$, $\mathcal{A}$ and ∇) and finding a separator Θ (based on some inequality constraints w.r.t. the uncertainty ∇). This latter part is built such that the inequality (3) is satisfied for all possible uncertainties belonging to $\mathbb{W}$. Then, the second condition (4) provides the stability test, usually formulated as an LMI condition.

Let us consider, for instance, the stability criterion given below, extracted from (Ariba et al., 2018, Theorem3), and established with the quadratic separation approach. It is a pointwise delay stability condition [2] for systems of the form of (1). It makes use of the Bessel inequality and combines a set of transfer functions that describes the dynamical behaviour of (1). All these transfer functions are embedded in the ∇ matrix

$$\nabla = \left[s^{-1}1_{(N+1)n},\ e^{-hs}1_n,\ \tilde{\delta}_N(s)\right]. \tag{5}$$

Transfer functions s^{-1} and e^{-hs} are, respectively, the standard integrator and delay related transfer functions. The additional ones $\tilde{\delta}_N(s) = [\delta_0 1_n\ \delta_1 1_n\ \dots\ \delta_N 1_n]^T$, with $\delta_k(s) = \int_{-h}^{0} L_k(\theta)\ e^{s\theta}\ d\theta$ (L_k are Legendre polynomials defined over the interval $[-h, 0]$), are delay-related transfer functions introduced to reduce the conservatism of the approach. Matrix ∇ being considered as an uncertain feedback, each transfer function has to be bounded. The following quadratic inequalities have been proven in Peaucelle et al. (2007); Ariba et al. (2018) for all $s \in \mathbb{C}$ such that [3] $Re(s) > 0$.

$$\begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix}^* \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} 1_n \\ e^{-hs}1_n \end{bmatrix}^* \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 1_n \\ e^{-hs}1_n \end{bmatrix} \leq 0,$$

$$\sum_{k=0}^{N}(2k+1)\ \delta_k^* R \delta_k\ \leq\ h^2 R,$$

where P, Q and R are $n \times n$ positive definite matrices. From these inequalities, the separator Θ (9) is derived.

This particular choice of matrix ∇ connects internal signals $w(t)$ and $z(t)$ via $w(t) = \nabla z(t)$ with

$$w(t) = \begin{bmatrix} x(t) \\ \tilde{\delta}_{N-1}[x(t)] \\ x(t-h) \\ \tilde{\delta}_N[\dot{x}(t)] \end{bmatrix},\ z(t) = \begin{bmatrix} \dot{x}(t) \\ \tilde{\delta}_{N-1}[\dot{x}(t)] \\ x(t) \\ \dot{x}(t) \end{bmatrix}. \tag{6}$$

Then, the feedforward block is described by :

$$\underbrace{\left[\frac{1_{(N+3)n}}{0_{(N+1)n\times(N+3)n}}\right]}_{\mathcal{E}}\ z(t)\ =\ \mathcal{A}\ w(t) \tag{7}$$

with $\mathcal{A}$ defined in (8).

Applying Theorem 1 with this modeling strategy has led to the stability criterion below, which has shown very interesting results in terms of reduction of conservatism.

Theorem 2. (Ariba et al. (2018)). Assume that $A + A_d$ is a non singular matrix. For a given constant delay h and for a given $N \geq 1$, if there exist positive definite matrices $P \in \mathbb{R}^{(N+1)n\times(N+1)n}$, $Q, R \in \mathbb{R}^{n\times n}$ such that the following LMI is satisfied:

$$[\,\mathcal{E}\ -\mathcal{A}\,]^{\perp *}\,\Theta\,[\,\mathcal{E}\ -\mathcal{A}\,]^{\perp} > 0,$$

where matrices $\mathcal{E}$, $\mathcal{A}$ are defined in (7) and Θ by (9):

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^* & \Theta_3 \end{bmatrix} \tag{9}$$

$$\mathcal{A} = \left[\begin{array}{c|cccc|c|cccccc}
A & 0 & \dots & \dots & 0 & A_d & 0 & \dots & 0 & 0 & & \\ \hline
0 & 0 & \dots & \dots & 0 & 0 & 1 & & 0 & 0 & & \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & \vdots & & \\
0 & 0 & \dots & \dots & 0 & 0 & 0 & & 1 & 0 & & \\ \hline
1 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & & \\ \hline
A & 0 & \dots & \dots & 0 & A_d & 0 & \dots & 0 & 0 & & \\ \hline
1 & 0 & \dots & & 0 & -1 & -1 & 0 & 0 & \dots & 0 & \\
1 & -\frac{2}{h}1 & & & 0 & 1 & 0 & -1 & 0 & \dots & 0 & \\
0 & & & & & 0 & 1 & 0 & -1 & & 0 & \\
\vdots & & \ddots & & & \vdots & & \ddots & \ddots & \ddots & & \\
0 & 0 & & & -\frac{2(2N+1)}{h}1 & 0 & 0 & & 1 & 0 & -1 &
\end{array}\right]. \tag{8}$$

with

$$\Theta_1 = \mathrm{diag}\big(0_{(N+1)n}, -Q, -h^2 R\big),$$
$$\Theta_2 = \mathrm{diag}\big(-P, 0_n, 0_{n\times(N+1)n}\big),$$
$$\Theta_3 = \mathrm{diag}\big(0_{(N+1)n}, Q, R, 3R, \dots, (2N+1)R\big),$$

then system (1) is asymptotically stable for the given h.

Proof 1. See Ariba et al. (2018).

In this theorem, as in other results Iwasaki and Hara (1998); Peaucelle et al. (2007), the uncertain set $\mathbb{W}$ is chosen such that

$$\mathbb{W} = \{\nabla(s), Re(s) > 0\}.$$

We work on that set in the next paragraph to design a performance analysis criterion. The quadratic separation appears to be a convenient framework for this objective because all the structure of the uncertain feedback modeling (5)-(6)-(7) remains unchanged, only the separator matrix Θ needs to be recalculated if the domain of definition of s is modified.

3.2 Performance analysis

The performance of a time delay system is directly linked to the placement of its eigenvalues in the complex plane. Constraining the infinitely many eigenvalues to a desired region is impossible, except for vertical half planes. The principle of the proposed approach consists in specifying that the Laplace variable s, used in quadratic constraints for each transfer functions defined in ∇, belongs to the desired region $\mathcal{D}$. As mentioned earlier, starting from a given stability criterion, the key idea is to redefine condition (3) with a new separator Θ depending on $\mathcal{D}$, while the other matrices remain unchanged. The resulting condition of Theorem 1 will then imply that there is no eigenvalue in $\mathcal{D}$, what can be translated into some performance indexes (responsiveness, damping). This methodology requires additional constraints on the definition of the uncertain matrix ∇. The following lemmas show the new separator for each transfer functions in ∇.

The proofs of all lemmas and additional remarks are given in the long version Ariba and Gouaisbaut (2020) of this present paper.

Lemma 1. A quadratic constraint for s^{-1} is given by the following inequality for any positive definite matrix P in $\mathbb{R}^{n\times n}$,

$$\begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix}^* \begin{bmatrix} d_1 P & d_2 P \\ d_2^* P & d_3 P \end{bmatrix} \begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix} \leq 0,$$
$$\forall s \in \mathcal{D},\ s \neq 0.$$

[2] Note that any other sophisticated stability conditions (delay-range, uncertain delay, robust w.r.t system matrices...) could have been considered. The pointwise delay case having a less cumbersome LMI test is preferred for sake of simplicity.

[3] This condition ensures that transfer functions are defined for any s in the right half plane, and thus no eigenvalues lies in this zone.

Lemma 2. A quadratic constraint for e^{-hs} is given by the following inequality for any positive definite matrix Q in $\mathbb{R}^{n\times n}$,

$$\begin{bmatrix} 1_n \\ e^{-hs}1_n \end{bmatrix}^* \begin{bmatrix} -e^{2ah}Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 1_n \\ e^{-hs}1_n \end{bmatrix} \leq 0, \quad \forall s \in \mathcal{D},$$

where a is the real part of the leftmost point of the region $\mathcal{D}$ in the complex plan.

Lemma 3. A quadratic constraint for $\tilde{\delta}_N(s) = [\delta_0 \ \delta_1 \ \dots \ \delta_N]^T$ is given by the following inequality for any positive definite matrix $R \in \mathbb{R}^{n\times n}$,

$$\begin{bmatrix} 1_n \\ \delta_0 1_n \\ \delta_1 1_n \\ \vdots \\ \delta_N 1_n \end{bmatrix}^* \begin{bmatrix} \mu R & 0 & 0 & \dots & 0 \\ 0 & R & 0 & \dots & 0 \\ 0 & 0 & 3R & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & (2N+1)R \end{bmatrix} \begin{bmatrix} 1_n \\ \delta_0 1_n \\ \delta_1 1_n \\ \vdots \\ \delta_N 1_n \end{bmatrix} \leq 0,$$

$\forall s \in \mathcal{D}$ and $\mu = -\frac{h}{2a}\left(1 - e^{-2ah}\right)$.

We can now state a theorem for the performance analysis for time-delay systems (1).

Theorem 3. Assume that $A + A_d$ is a non singular matrix. For some given complex scalar parameters $\{d_1, d_2, d_3\}$ that define a region $\mathcal{D}$ in the complex plan (2). For a given constant delay h and for a given $N \geq 1$, if there exist positive definite matrices $P \in \mathbb{R}^{(N+1)n\times(N+1)n}$, Q, $R \in \mathbb{R}^{n\times n}$ such that the following LMI is satisfied:

$$[\,\mathcal{E} \ -\mathcal{A}\,]^{\perp *}\,\Theta\,[\,\mathcal{E} \ -\mathcal{A}\,]^{\perp} > 0,$$

where matrices $\mathcal{E}$, $\mathcal{A}$ are defined in (7) and Θ by:

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^* & \Theta_3 \end{bmatrix}$$

with

$$\Theta_1 = \mathrm{diag}\big(d_1 P, -e^{2ah}Q, -\frac{h}{2a}(1 - e^{-2ah})R\big),$$
$$\Theta_2 = \mathrm{diag}\big(d_2 P, 0_n, 0_{n\times(N+1)n}\big),$$
$$\Theta_3 = \mathrm{diag}\big(d_3 P, Q, R, 3R, \dots, (2N+1)R\big),$$

then system (1) has no eigenvalues in $\mathcal{D}$ for the given h.

The parameter N corresponds to the degree of the Legendre polynomial that is used to approximate the delay transfer function in the uncertain feedback modeling Ariba et al. (2018). It has been shown that increasing N reduces the conservatism of the criterion, at the expense of the numerical burden.

3.3 Numerical example

As an illustrative example, let us consider the following system with a delay $h = 1s$:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-1) \qquad (10)$$

Several tests have been run to illustrate the ability of Theorem 3 to assess the absence of any eigenvalue of (10) in some specific location. Those simulations are summarized in Table 1 and plotted on Figure 3. For each test, the forbidden region $\mathcal{D}$ for eigenvalues (orange zones) is compared to the actual location of (10). The eigenvalues location is determined with a MATLAB function that approximates the characteristic roots of linear delay differential equations (based on the work of Breda et al. (2015)). Obviously, when any region of $\mathcal{D}$ overlaps any pole spot, the LMI condition of Theorem 3 is unfeasible. It can be noticed that in some cases it is necessary to increase N so that Theorem 3 is able to detect the "$\mathcal{D}$-stability". Reducing the conservatism is especially required when a region gets closer to an eigenvalue.

test	regions	Thm. 3 result
1	half-plane: $\mathrm{Re}(s) > -0.3$ disc: $r = 2$, $s_0 = -3 + 20j$	LMI feasible for $N \geq 2$
2	half-plane: $\mathrm{Re}(s) > -0.3$ disc: $r = 3$, $s_0 = -8 + 6j$	LMI feasible for $N \geq 4$
3	half-plane: $\mathrm{Re}(s) > -0.3$ disc: $r = 1$, $s_0 = -4 + 11j$	LMI feasible for $N \geq 8$

Table 1. *Configuration parameters for the analysis of (10).*

4. APPLICATION TO AN ACTIVE MASS DAMPER SYSTEM

4.1 Description of the system

The practical application considered in this work is a bench-scale building like tall structure. The test stand is a Quanser experimental model and is depicted in Figure 1. The aim is to design a control system that dampens actively vibrations with an actuated cart on the top.

A schematic of the plant and notations are illustrated in Figure 4. For small floor deflection, the top of the structure is modeled as a linear spring-mass system. x_f is the floor horizontal displacement and x_c is the cart position. This latter is actuated with a DC motor that induces a linear force F_c. Applying the Lagrange's method and combining with the equation of the motor, a linear model of the plant is derived:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 278.9 & -18.6 & 0 \\ 0 & -336 & 5.9 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 2.99 \\ -0.96 \end{bmatrix} u(t) \qquad (11)$$

where the state space vector consists of $x = [x_c \ x_f \ \dot{x}_c \ \dot{x}_f]^T$. The control signal u is the voltage input of the DC motor driving the cart. A standard pole placement design provides the state feedback gain $K = [6.9 \ -103 \ -2.8 \ -26.9]$ to have the closed-loop poles $\{-8, -16, -6 \pm 2i\}$.

4.2 Delay based control

A state feedback control requires the full state x to be available. This requirement may be a major drawback in practice as engineers usually aim at limiting the use of sensors for several reasons: feasibility, reliability, cost, maintenance... It is proposed to approximate a state feedback control with a time-delay system approach. The key idea is to replace the static state feedback control with a static output feedback combined with a static delayed output feedback. This approach recently updated by Selivanov and Fridman (2018) assumes basically that,

$$u(t) = -Kx(t)$$

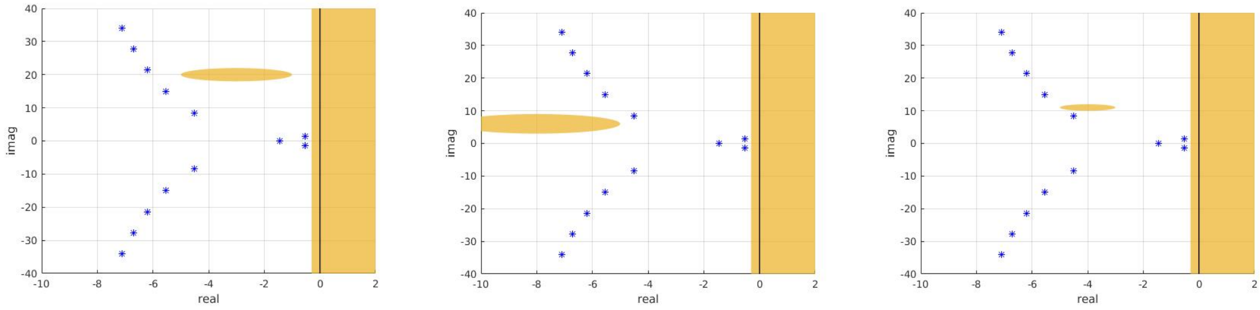

Figure 3. *Analysis of (10): test 1, test 2 and test 3 from left to right.*

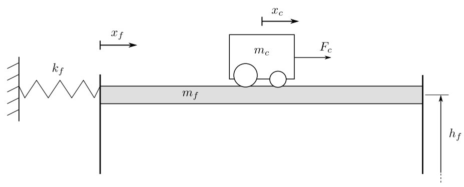

Figure 4. *Schematic of the plant and notations.*

can be approximated by

$$u(t) = -K_1 y(t) - K_2 y(t-h),$$

where $y = [x_c \; x_f]^T$ is the measured output, namely the positions. Note that the control law remains quite simple compared to a dynamical control system when a state observer is implemented. As often in mechanical system representation, the second part of the state x is the derivative of the first one. The method is based on the following approximation of the derivative non causal transfer function:

$$\dot{y}(t) \simeq \frac{y(t) - y(t-h)}{h}. \tag{12}$$

Applying a static state feedback control to system (11) with a state structured as $x = [y \; \dot{y}]^T$, the following closed-loop formulation is obtained

$$\dot{x}(t) = Ax(t) - BK_\alpha y(t) - BK_\beta \dot{y}(t)$$

where K_α and K_β are components of the state feedback gain $K = [K_\alpha \, K_\beta]$. Approximating the last term with (12), a delay based feedback formulation is obtained:

$$\begin{aligned}\dot{x}(t) &= Ax(t) - B\Big(K_\alpha + \frac{1}{h}K_\beta\Big)y(t) + \frac{1}{h}BK_\beta y(t-h),\\ &= Ax(t) - BK_1 y(t) + BK_2 y(t-h).\end{aligned} \tag{13}$$

The analysis of the properties of the above feedback system amounts to the analysis of a standard time-delay system:

$$\dot{x}(t) = A_1 x(t) + A_2 x(t-h), \tag{14}$$

with $A_1 = A - B\Big(K_\alpha + \frac{1}{h}K_\beta\Big)C$ and $A_2 = \frac{1}{h}BK_\beta C$. Let us exploit Theorem 3 to analyze the stability and performances of (14). In this application, besides stability, it is required to dampen the structure oscillations. This feature can be addressed with an appropriate setting of eigenvalues location. The objective is to prove that the properties of the theoretical closed-loop system (the linear system (11) with a state feedback) is preserved with the delay based feedback (static output feedback + static delayed output feedback).

4.3 Simulation

As mentioned above, a standard state feedback control with gain $K = [6.9 \; -103 \; -2.8 \; -26.9]$ for the linear system (11) leads to a stable and damped closed-loop system with poles $\{-8, -16, -6 \pm 2i\}$. The parameters for the delay based control are calculated according to (13). For instance, setting a delay $h = 60 \; ms$, the resulting static gains are

$$K_1 = [-40.5 \; -552.8] \quad \text{and} \quad K_2 = [-47.5 \; -449.5]\,.$$

The closed-loop system is then turned into a time-delay system as (14):

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 121 & 1936 & -18.6 & 0 \\ -39 & -866 & 6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -142 & 1347 & 0 & 0 \\ 46 & 431 & 0 & 0 \end{bmatrix} x(t - 0.06) \tag{15}$$

Theorem 3 can then be applied to perform a performance analysis and to confirm that the eigenvalues of the resulting system (15) are sufficiently damped (see Figure 5). We aim at ensuring that the oscillatory behavior of the response is reduced, similarly to the full state feedback control case. Figure 6 shows simulations of both control laws and an open-loop test. The initial condition for the floor position is $0.5m$. The open-loop response shows the highly oscillatory behavior of the flexible structure. The system response with the delay based control is fairly similar to the one with the state feedback control and a significant vibration reduction is observed. Several simulations have been run to find a delay h for the control (13) that minimizes the amplitude of oscillations. Results with $h = 60 \; ms$ are satisfactory as shown in Figure 6.

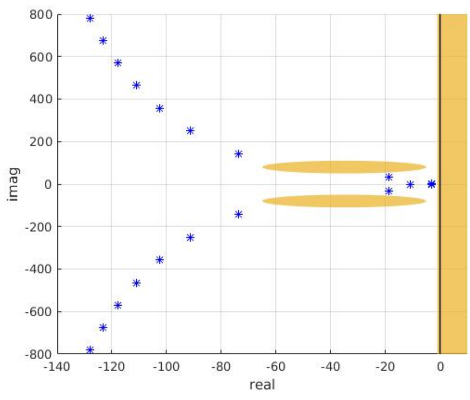

Figure 5. *Performance analysis for system (15).*

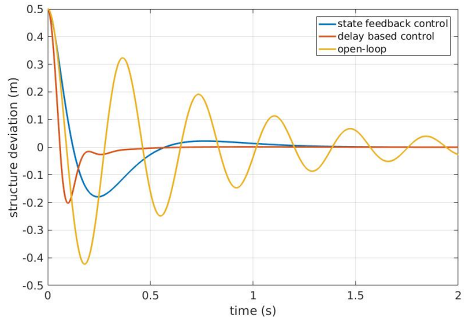

Figure 6. *Damping the floor deflection, simulation results.*

4.4 Experimentation

This paragraph presents the experimental test with the bench-scale building (see Figure 1). The delay based control law is now implemented on a data acquisition device, a *Q2-usb* board with real-time computing capability. The sampling period is 1 ms. The experimental results confirm the simulations of the previous paragraph. The experiment starts at rest (zero initial conditions), the system is stimulated by a brief push with the hand as an impulsive disturbance. The three configurations have been tested, and the corresponding measurements of the structure deviation x_f are plotted on Figure 7. Once again, the delay based control response is similar to the one with the state feedback control, and both are able to dampen oscillations. However, the former one requires only position information, that is two measures are used instead of four. The performance requirement in terms of damping was beforehand assessed with a performance analysis for the equivalent time-delay system.

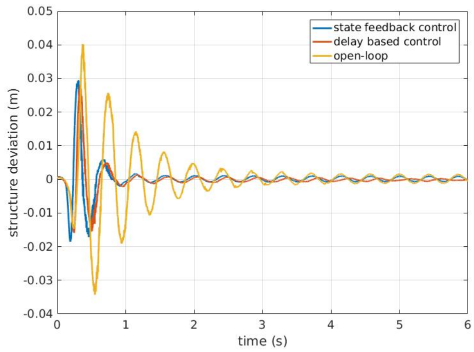

Figure 7. *Damping the floor deflection, experimental results.*

5. CONCLUSION

This paper studies the performance analysis for linear time-delay system. A sufficient condition ensuring that all eigenvalues are absent from a specific region $\mathcal{D}$ is proposed. The condition is built with the quadratic separation technique and is formulated as an LMI condition. Then, the criterion is used to analyze the performances of an active mass damper system controlled with a delayed output feedback. Indeed, the closed-loop system is expressed as a time-delay system, the delay h being also a design parameter. Analyzing the eigenvalues location in the complex plane, we are able to assess the damping property of the control law. The validity of the approach is demonstrated with simulations and experimental tests. Future works include the design of more complex delayed control laws and the conservatism reduction of performance analysis.

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