

# Stochastic Model Predictive Control with Adaptive Chance Constraints based on Empirical Cumulative Distributions <sup>★</sup>

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**Abstract:** Individual chance constraints can be used to systematically seek trade-offs between control performance and constraint violation for a given disturbance description. This paper presents a Stochastic Model Predictive Control (SMPC) approach with adaptive individual chance constraints that relies on online adaptation of the disturbance description using the empirical cumulative distribution (ECDF). Individual chance constraints are ensured by using a suitable worst-case confidence interval derived from the ECDF. The confidence interval may, however, be excessively conservative due to the empirical nature of the ECDF. To reduce this conservatism, the proposed approach accounts for the updated disturbance information, which is sampled online from the one-step ahead prediction error. Hence, the initial ECDF can be obtained from a reduced number of samples since the conservative handling of the chance constraint is continuously mitigated. This will also allow for using simpler models of the stochastic system disturbances. Convergence and recursive feasibility of the proposed adaptive approach are established. A DC-DC converter benchmark problem is used to illustrate the usefulness of the proposed approach.

*Keywords:* Stochastic MPC; Tube-based MPC; Offset-free tracking; Chance Constraints; Empirical Distributions

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## 1. INTRODUCTION

Model predictive control (MPC) is a popular control strategy due to its ability to deal simultaneously with control and state constraints, multiple performance objectives and multivariable systems (Rawlings and Mayne, 2009). Disturbance handling is a crucial aspect in MPC, which affects the control performance, feasibility, robust stability and constraint satisfaction (Kouvaritakis and Cannon, 2015; Mesbah, 2016). In this context, Stochastic MPC (SMPC) with chance constraints has been proposed to systematically tolerate constraint violation in a probabilistic sense in order to improve control performance and to increase the domain of attraction (Cannon et al., 2011; Mesbah, 2016; Lorenzen et al., 2017; Heirung et al., 2018; Paulson et al., 2019; D' Jorge et al., 2020).

Chance constraints are converted into deterministic constraints (Kouvaritakis et al., 2010; Kouvaritakis and Cannon, 2015) by using some knowledge of the disturbance distribution in tube-based SMPC. However, in most of the SMPC approaches, the disturbance distribution is assumed to be perfectly known. This assumption does not hold in practice since the estimation of the statistical moments of the disturbance distribution is subject to un-

certainty (Van Parys et al., 2015). Under standard assumption, useful results from non-parametric statistical theory can be applied to bound uncertainty of an empirical cumulative distribution function (ECDF) with a desired prescribed confidence level such as Wilson, Clopper–Pearson, Agresti–Coull, Wald and Jeffreys intervals (Wilson, 1927; Brown et al., 2001; Wasserman, 2006). In these confidence interval approaches, the upper and lower bounds of the ECDF depend on the number of samples. Hence, an exceedingly large number of samples is often required to reduce the conservatism of the confidence interval. Moreover, the uncertainty in the disturbance distribution may partly mitigate or even fully eliminate the benefits of using chance constraints when the constraint violation levels are tight relative to the confidence interval uncertainty.

For linear constrained systems with hard input bounds, this paper presents an offset-free tracking SMPC approach with online improvement of individual chance constraints. The individual chance constraints are adapted at each time step based on an updated ECDF. This strategy is an extension of Santos et al. (2018) and Paulson et al. (2019) to enable the use of online additive disturbance information to reduce the uncertainty of the ECDF. In contrast to previous works, individual chance constraints are continuously adapted during closed-loop operation based on the updated ECDF. Hence, a new offset-free

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tracking SMPC is proposed to ensure recursive feasibility despite the individual chance constraints adaptation.

**Notation.**  $[A]_j$  denotes the  $j$ -th row of matrix  $A$ . For a matrix  $M$  or scalar  $\lambda$  and a given set  $\mathbb{X}$ ,  $M\mathbb{X} = \{Mx : x \in \mathbb{X}\}$  and  $\lambda\mathbb{X} = \{\lambda x : x \in \mathbb{X}\}$ .  $\mathbb{N}$  represents the set of non-negative integers and  $\mathbb{N}^*$  denotes the set of positive integers. A positive definite (semi-definite) matrix  $T$  is denoted by  $T > 0$  ( $T \geq 0$ ). The matrix  $I$  denotes the identity matrix with appropriate dimension. The Minkowski sum of the set  $\mathbb{S}$  with the set  $\mathbb{T}$  is denoted by  $\mathbb{S} \oplus \mathbb{T} = \{v = s+t : s \in \mathbb{S}, t \in \mathbb{T}\}$ , and the Pontryagin set difference is denoted by  $\mathbb{S} \ominus \mathbb{T} = \{s \in \mathbb{S} : s+t \in \mathbb{S}, \forall t \in \mathbb{T}\}$ . The vector  $x_k$  describes the measured state at  $k$ , while  $x_{l|k}$  represents the  $l$  step ahead prediction of the state based on the information available at  $k$ .  $\mathcal{E}[X]$  and  $\mathcal{V}[X]$  denote, respectively, the expected value and the variance of a random variable  $X$ .  $\mathbb{P}[s \in \mathbb{S}]$  is the probability of the event  $s \in \mathbb{S}$ . The notation  $\mathbb{P}_k[s \in \mathbb{S}]$ , also described by  $\mathbb{P}[s \in \mathbb{S} | x_k]$ , represents the conditional probability of the event  $s \in \mathbb{S}$  given the realization  $x_k$ . For a sequence of random variables  $X_k$ ,  $X_k \rightarrow X$  denotes  $L_1$ -convergence, i.e.,  $\lim_{k \rightarrow \infty} \mathcal{E}[\|X_k - X\|_1] = 0$ , to random variable  $X$ .

## 2. PROBLEM STATEMENT AND PRELIMINARIES

Relevant previous results are briefly revisited in this section to formulate the proposed SMPC problem based on ECDF estimation for chance constraint adaptation.

### 2.1 System Description

The following linear time-invariant (LTI) system subject to additive stochastic disturbances is considered

$$x_{k+1} = Ax_k + Bu_k + B_w w_k, \quad (1a)$$

$$y_k = Cx_k + Du_k, \quad (1b)$$

where  $x_k \in \mathbb{R}^n$ ,  $u_k \in \mathbb{R}^m$ , and  $y_k \in \mathbb{R}^p$  represent the system states, control inputs, and outputs, respectively. Without loss of generality, it is assumed that  $w_k$  is an independent and identically distributed (i.i.d.) random variable with zero mean.<sup>1</sup> Moreover,  $w_k$  has finite support such that  $w_k \in \mathbb{W}$ , where  $\mathbb{W}$  is assumed to be a known closed set.

Hard input constraints and state chance constraints are defined using the following polyhedral sets

$$\mathbb{U} = \{u \mid Gu \leq g, u \in \mathbb{R}^m\},$$

$$\mathbb{X} = \{x \mid Hx \leq h, x \in \mathbb{R}^n\},$$

where  $G \in \mathbb{R}^{n_u \times m}$ ,  $g \in \mathbb{R}^{n_u \times 1}$ ,  $H \in \mathbb{R}^{n_c \times n}$ , and  $h \in \mathbb{R}^{n_c \times 1}$ . Hence, chance constraints are defined as

$$\mathbb{P}[[H]_j x_k \leq [f]_j] \geq 1 - \epsilon_j, \quad j = 1, \dots, n_c, k \in \mathbb{N}^*, \quad (2)$$

where  $\epsilon_j \in [0, 1)$  defines an acceptable level of probabilistic constraint violation.

### 2.2 Probabilistic tubes

The notion of probabilistic tubes can be directly used to achieve recursive feasibility in the presence of individual chance constraints (Kouvaritakis and Cannon, 2015,

<sup>1</sup> The i.i.d. assumption is useful for bounding the ECDF uncertainty, but ECDF can be a consistent estimator for non-i.i.d. random variables under mild assumption (Azriel and Schwartzman, 2015). Random variables with non-zero mean can be handled as discussed in (Paulson et al., 2019).

Chapter 3). Let  $u_k = Kx_k$  be an unconstrained stabilizing control law, where  $\Phi = A + BK$  is a Schur matrix. For prediction, consider  $x_k = x_{0|k}$ ,  $e_{0|k} = 0$ , and the nominal constrained control law is given by  $v_{i|k} = Kz_{i|k} + c_{i|k}$  where,  $z_{i|k}$  is the nominal prediction for  $x_{k+i}$  at  $k$ . Then, nominal predictions can be described by

$$x_{i|k} = z_{i|k} + e_{i|k}, \quad (3)$$

$$v_{i|k} = Kz_{i|k} + c_{i|k}, \quad (4)$$

$$z_{i+1|k} = \Phi z_{i|k} + Bc_{i|k}, \quad (5)$$

$$e_{i+1|k} = \Phi e_{i|k} + B_w w_{i|k}, \quad (6)$$

where  $c_{i|k}$  is a free decision variable used for constraint satisfaction (Mayne et al., 2000).

The notion of probabilistic tubes can be directly applied by rewriting the chance constraints in terms of a polyhedral constraint based on the cumulative distribution function. The constraint for  $x_{k|1}$  can be defined as

$$\mathbb{Z}_1 = \{z \mid Hz \leq h - \Gamma, z \in \mathbb{R}^n\} \quad (7)$$

with  $\Gamma = [\gamma_1^* \ \gamma_2^* \ \dots \ \gamma_{n_c}^*]^\top$ , where every  $\gamma_j^*$  is obtained offline by solving the auxiliary problem (Kouvaritakis and Cannon, 2015)

$$\begin{aligned} \gamma_j^* &= \min \gamma_j \\ \text{s.t. } &\mathbb{P}[[H]_j B_w w_k \leq \gamma_j] \geq 1 - \epsilon_j. \end{aligned} \quad (8)$$

Due to the hard input constraints  $\mathbb{U}$ , the initial input constraint is defined by  $\mathbb{V}_0 = \mathbb{U}$ . Then, based on  $\mathbb{Z}_1$  and  $\mathbb{V}_0$ , the tighter constraints for recursive feasibility purposes can be recursively defined by

$$\mathbb{Z}_{j+1} = \mathbb{Z}_j \ominus \Phi^j B_w \mathbb{W}, \quad j \in \mathbb{N}^*, \quad (9)$$

$$\mathbb{V}_{j+1} = \mathbb{V}_j \ominus K\Phi^j B_w \mathbb{W}, \quad j \in \mathbb{N}. \quad (10)$$

The main advantage of this recursive definition arises from the fact that no Minkowski sum is required. When polytopes are considered, the Pontryagin difference can be easily computed by solving a simple linear programming problem (Borrelli et al., 2017, Chapter 4).

### 2.3 Stochastic MPC for offset free tracking

Offset-free tracking with individual chance constraints and enlarged domain of attraction can be achieved by using an artificial reference with an augmented terminal set as proposed in (Santos et al., 2018). In this offset-free tracking approach, the unconstrained control law is defined by  $u_k = K(x_k - \bar{x}_t) + \bar{u}_t$ , where  $(A - I)\bar{x}_t + B\bar{u}_t = 0$  and  $[\bar{x}_t^\top \ \bar{u}_t^\top]^\top$  represents the steady-state target. Alternatively, the tracking control law can be parametrized by

$$u_k = Kx_k + L\theta_k, \quad (11)$$

where  $L = [-K \ I]M_\theta$ ,  $[\bar{x}_t^\top \ \bar{u}_t^\top]^\top = M_\theta \theta_k$ ,  $M_\theta = [M_x^\top \ M_u^\top]^\top$  is obtained from linear independent vectors that define a base for the nullspace of  $[(A - I) \ B]$ , and  $\theta_k$  is the free parameter which determines the steady-state vector (Limon et al., 2008). Note that  $\bar{y}_t = M_y \theta_k$ , where  $M_y = [C \ D]M_\theta$ .

**Problem 1. (SMPC for offset-free tracking)** Consider the measured state  $x_k$  and a desired output target  $y_t$ , the SMPC for tracking is obtained from the solution of the following optimal control problem

$$\min_{\mathbf{c}_k, \theta_k} J_N(\mathbf{c}_k, \theta_k; x_k, y_t) \quad (12a)$$

$$\text{s.t. } z_{0|k} = x_k, \quad (12b)$$

$$\begin{bmatrix} \bar{x}_t^\top & \bar{u}_t^\top \end{bmatrix}^\top = M_\theta \theta_k, \quad (12c)$$

$$z_{i+1|k} = \Phi z_{i|k} + BL\theta_k + Bc_{i|k}, \quad i \in \mathbb{N}_{[0, N-1]}, \quad (12d)$$

$$v_{i|k} = Kz_{i|k} + c_{i|k} + L\theta_k, \quad i \in \mathbb{N}_{[0, N-1]}, \quad (12e)$$

$$z_{i|k} \in \mathbb{Z}_i, \quad i \in \mathbb{N}_{[1, N-1]}, \quad (12f)$$

$$v_{i|k} \in \mathbb{V}_i, \quad i \in \mathbb{N}_{[0, N-1]}, \quad (12g)$$

$$\begin{bmatrix} z_{N|k}^\top & \theta_k^\top \end{bmatrix}^\top \in \mathbb{Z}_f^a, \quad (12h)$$

where

$$J_N(\mathbf{c}_k; x_k) = \sum_{i=0}^{N-1} \left( \|z_{i|k} - \bar{x}_t\|_Q^2 + \|v_{i|k} - \bar{u}_t\|_R^2 \right) + \|z_{N|k} - \bar{x}_t\|_P^2 + V_o(y_t - \bar{y}_t).$$

$V_o(\cdot)$  is a convex offset cost that penalizes deviations between the desired target  $y_t$  and the artificial target  $\bar{y}_t$ ,  $\theta_k$  is a decision variable, and  $\mathbb{Z}_f^a$  is an augmented admissible robust invariant set for tracking.

The robust admissible invariant set  $\mathbb{Z}_f^a$  should be computed for the following constrained autonomous system with disturbances

$$\begin{bmatrix} z_{N|k+1} \\ \theta_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi & BL \\ 0 & I \end{bmatrix} \begin{bmatrix} z_{N|k} \\ \theta_k \end{bmatrix} + \begin{bmatrix} \Phi^N B_w \\ 0 \end{bmatrix} w_k,$$

where the augmented state is subject to

$$\mathbb{Z}_N^a = \left\{ \begin{bmatrix} z \\ \theta \end{bmatrix} \in \mathbb{R}^{n+m} \mid \begin{array}{l} z \in \mathbb{Z}_N \\ Kz + L\theta \in \mathbb{V}_N \\ M_x \theta \in \lambda \mathbb{Z}_N \\ M_u \theta \in \lambda \mathbb{V}_N \end{array} \right\} \quad (13)$$

and  $\lambda \in (0, 1)$  ensures that the invariant set is finitely determined (Limon et al., 2008). The robust polyhedral invariant sets can be computed as discussed in (Kolmanovskiy and Gilbert, 1998).

The actual control is derived in a receding-horizon fashion at every sampling instant  $k$  given by

$$u_k = Kx_k + c_{0|k}^*(x_k, y_t) + L\theta_k^*(x_k, y_t), \quad (14)$$

where  $*$  denotes the optimal value of the decision variables obtained from (12).

#### 2.4 Empirical Cumulative Distribution Function

In most of the related works (Kouvaritakis et al., 2010; Santos et al., 2018; Paulson et al., 2019; D' Jorge et al., 2020), it is assumed that the disturbance distribution is exactly known. However, this assumption is an important challenge since the disturbance distribution is also subject to uncertainties.

The ECDF is a simple tool that can be directly used to obtain the required information for chance constraint handling (Lorenzen et al., 2017; Kouvaritakis and Cannon, 2015; Santos et al., 2019). Let  $\mathcal{R} = \{\xi_1, \xi_2, \dots, \xi_{N_s}\}$  be an ascending sequence of samples of a stochastic process  $\chi$ . The ECDF of  $\chi$  is given by

$$\hat{F}_\chi(q) = \frac{1}{N_s} \sum_{i=1}^{N_s} \mathbf{1}(\xi_i \leq q),$$

where  $\mathbf{1}(\xi_i \leq q) = 1$  if  $\xi_i \leq q$  and  $\mathbf{1}(\xi_i \leq q) = 0$  if  $\xi_i > q$ . When  $\chi$  is identically independent distributed (IID),

the random variable  $N_s \hat{F}_\chi(q)$  has a binomial distribution (Owen, 2001). In this case, the expected value and variance of  $\hat{F}_\chi(q)$  are given by

$$\begin{aligned} \mathcal{E}[\hat{F}_\chi(q)] &= F_\chi(q), \\ \mathcal{V}[\hat{F}_\chi(q)] &= \frac{(1 - F_\chi(q))F_\chi(q)}{N_s}, \end{aligned}$$

where  $F_\chi(q)$  denotes the true cumulative distribution function. As expected, the variance of the ECDF is reduced by increasing the number of samples. Moreover, the backoff parameter  $\hat{\gamma}_j^*$  can be estimated from a sequence  $\mathcal{H}_j = \{h_j^\top B_w w_1, h_j^\top B_w w_2, \dots, h_j^\top B_w w_{N_s}\}$ , where  $Bw_{k-1} = x_k - Ax_{k-1} - Bu_{k-1}$ . This leads to

$$\hat{\gamma}_j^* = \hat{F}_{h_j^\top B_w W}^{-1}(1 - \epsilon_j).$$

Two classes of ECDF bounding approaches can be considered in practical applications. Confidence intervals are used when a point-wise value of the cumulative distribution functions (CDF) should respect the confidence limits for a given probability level. Confidence bands are used when the entire CDF should lie inside the bands for a prescribed probability level. A confidence interval is used in this work because the desired information is specified at a given value of the ECDF inverse. In the context of confidence intervals, several approaches can be considered (Brown et al., 2001). Here, we use the Wilson method (Wilson, 1927) since it is widely accepted for applications with either small ( $\approx 40$ ) or large number of samples (Brown et al., 2001). The Wilson interval can be defined by

$$F(\chi) \in [F_l(\chi), F_u(\chi)], \quad (15)$$

$$F_l(\chi) = \frac{\hat{F}(\chi) + \Psi_{\alpha/2}^2/(2N_s) - \rho}{1 + \Psi_{\alpha/2}^2/N_s}, \quad (16)$$

$$F_u(\chi) = \frac{\hat{F}(\chi) + \Psi_{\alpha/2}^2/(2N_s) + \rho}{1 + \Psi_{\alpha/2}^2/N_s}, \quad (17)$$

where  $\rho = \Psi_{\alpha/2} \sqrt{\frac{\hat{F}(\chi)(1 - \hat{F}(\chi)) + \Psi_{\alpha/2}^2/(4N_s)}{N_s}}$  and  $\Psi_{\alpha/2}$  denotes the upper  $\alpha/2$ -quantile of the normal distribution.

The value  $\alpha/2$ -quantile is a free specification parameter which defines the desired confidence level. Moreover, an increased confidence level enlarges the confidence intervals ( $F_l(\chi)$  and  $F_u(\chi)$ ). In summary, for a fixed number of sample, confidence intervals are enlarged if  $\alpha/2$ -quantile is increased. Since  $\hat{\gamma}_j^*$  is obtained from the inverse of the ECDF,  $F_u^{-1}(1 - \epsilon_j)$  should be used for a given  $1 - \epsilon_j$  because  $F_u^{-1}(1 - \epsilon_j) > F^{-1}(1 - \epsilon_j) > F_l^{-1}(1 - \epsilon_j)$ . The worst-case backoff, provided by  $F_u^{-1}(1 - \epsilon_j)$ , ensures that constraint violation probability is smaller than the acceptable bound since a tighter deterministic constraint is provided.

Also note that a low-pass filter can be used to attenuate undesired spikes. For example, the filter  $\hat{\Gamma}_k = \beta \hat{\Gamma}_{k-1} + (1 - \beta) \hat{\Gamma}_k^l$  can be used, where  $\hat{\Gamma}_k^l$  represents an unfiltered backoff obtained from the lower Wilson-based ECDF.

### 3. ADAPTIVE SMPC BASED ON ECDF UPDATE

The main contribution of this work arises from the fact that  $\hat{\Gamma}_k$  is adapted online. To ensure recursive feasibility, the augmented nominal vector is defined as  $\xi_{i|k} =$

$[z_{i|k}^\top \hat{\Gamma}_k^\top]^\top$ , where  $\hat{\Gamma}_k$  is assumed to be constant at a given prediction horizon and  $\xi_{k+i|k} \in \mathbb{R}^{n+n_c}$ .<sup>2</sup>

Accordingly, the new probabilistic tubes are defined using the initial set

$$\mathbb{X}_1^\xi = \{\xi \in \mathbb{R}^{n+n_c} \mid [H \ I]\xi \leq h\}. \quad (18)$$

Based on  $\mathbb{X}_1^\xi$ , the probabilistic tubes for recursive feasibility are

$$\mathbb{X}_{j+1}^\xi = \mathbb{X}_j^\xi \ominus (\Phi^j B_w \mathbb{W} \times \{0\}), \quad j \in \mathbb{N}^*. \quad (19)$$

Note that  $[z^\top \Gamma^\top]^\top \in \mathbb{X}_1^\xi$  is equivalent to  $z \in \mathbb{Z}_1$ . Then, a new invariant set should be defined due to the time-varying nature of  $\hat{\Gamma}_k$ , which depends on  $\Gamma_{\max} = [\bar{\gamma}_1 \dots \bar{\gamma}_{n_c}]^\top$  and  $\delta \hat{\Gamma}_k = \hat{\Gamma}_k - \Gamma_{\max}/2$ , where

$$\begin{aligned} \bar{\gamma}_j &= \max [H]_j B_w w \\ \text{s.t. } w &\in \mathbb{W}. \end{aligned} \quad (20)$$

Now,  $\delta \hat{\Gamma}_k = \hat{\Gamma}_k - \Gamma_{\max}/2$  is defined as a translated version of  $\hat{\Gamma}_k$  to ensure that the origin is inside the interior of the augmented invariant set. To ensure recursive feasibility, the augmented dynamics are defined by

$$\begin{bmatrix} z_{N|k+1} \\ \theta_{k+1} \\ \delta \hat{\Gamma}_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi & BL & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_{N|k} \\ \theta_k \\ \delta \hat{\Gamma}_k \end{bmatrix} + \begin{bmatrix} \Phi^N B_w \\ 0 \\ 0 \end{bmatrix} w_k.$$

The constraints that define the maximal admissible robust invariant sets (Kolmanovsky and Gilbert, 1998) are

$$\mathbb{X}_N^{\xi,a} = \left\{ \begin{bmatrix} z \\ \theta \\ \delta \hat{\Gamma} \end{bmatrix} \in \mathbb{R}^{n_a} \mid \begin{array}{l} [z^\top \ \delta \hat{\Gamma}^\top] \in \mathbb{X}_N^\xi, \\ Kz + L\theta \in \mathbb{V}_N, \\ [(M_x \theta)^\top \ (\frac{\Gamma_{\max}}{2})^\top] \in \lambda \mathbb{X}_N^\xi, \\ M_u \theta \in \lambda \mathbb{V}_N, \\ -\frac{\Gamma_{\max}}{2} < \delta \hat{\Gamma} < \frac{\Gamma_{\max}}{2}, \end{array} \right\}, \quad (21)$$

where  $n_a = n + m + n_c$ .

**Assumption 1.** Recursive feasibility and convergence to a minimal robust positively invariant set (mRPI) around a desired admissible target are based on the following assumptions:

- (1) The pair  $(A, B)$  is controllable;
- (2) The pair  $(A, Q^{1/2})$  is observable;
- (3)  $Q \geq 0$  and  $R > 0$ ;
- (4)  $K$  is defined such that  $\Phi = A + BK$  has all its eigenvalues strictly inside the unitary circle;
- (5)  $P > 0$  is given by  $\Phi^\top P \Phi - P = -(Q + K^\top R K)$ ;
- (6)  $V_o : \mathbb{R}^p \rightarrow \mathbb{R}$  is a convex, positive-definite and subdifferentiable function with  $V_o(0) = 0$ ;
- (6)  $V_\gamma : \mathbb{R}_c^n \rightarrow \mathbb{R}$  is a convex, positive-definite and subdifferentiable function with  $V_\gamma(0) = 0$ ;
- (7)  $\mathbb{X}_f^{\xi,a}$  is a non-empty admissible robust invariant set such that  $\mathbb{X}_f^{\xi,a} \subseteq \mathbb{X}_N^{\xi,a}$  and  $[x^\top \ \theta^\top \ \Gamma^\top]^\top \in \mathbb{X}_f^{\xi,a} \Rightarrow [(\Phi x + \Phi^N B_w w + BL\theta)^\top \ \theta^\top \ \Gamma^\top]^\top \in \mathbb{X}_f^{\xi,a}, \forall w \in \mathbb{W}$ .

The main modification of Assumption 1 with respect to Santos et al. (2018) comes from the robust invariant set (7) that is defined in order to consider the backoff adaptation.

**Problem 2. (SMPC for offset-free tracking with adaptive backoff)** Consider the measured state  $x_k$  and

<sup>2</sup> In the case of symmetric constraints,  $\hat{\Gamma}_k$  consists of the backoff of non-symmetric half-spaces and  $\xi_{k+i|k} \in \mathbb{R}^{n+n_c/2}$ .

a desired output target  $y_t$ , the SMPC for tracking is obtained from the solution of the following optimal control problem

$$\min_{\mathbf{c}_k, \theta_k, \bar{\Gamma}_k} J_N(\mathbf{c}_k, \theta_k, \bar{\Gamma}_k; x_k, \hat{\Gamma}_k, y_t) \quad (22a)$$

$$\text{s.t. } z_{0|k} = x_k, \quad (22b)$$

$$\bar{\Gamma}_k \geq \min(\bar{\Gamma}_{k-1}^*, \hat{\Gamma}_k), \quad (22c)$$

$$z_{i+1|k} = \Phi z_{i|k} + BL\theta_k + Bc_{i|k}, \quad i \in \mathbb{N}_{[0, N-1]}, \quad (22d)$$

$$v_{i|k} = Kz_{i|k} + c_{i|k} + L\theta_k, \quad i \in \mathbb{N}_{[0, N-1]}, \quad (22e)$$

$$\xi_{i|k} = [z_{i|k}^\top \ \bar{\Gamma}_k^\top]^\top, \quad i \in \mathbb{N}_{[0, N-1]}, \quad (22f)$$

$$\xi_{i|k} \in \mathbb{X}_i^\xi, \quad i \in \mathbb{N}_{[1, N-1]}, \quad (22g)$$

$$v_{i|k} \in \mathbb{V}_i, \quad i \in \mathbb{N}_{[0, N-1]}, \quad (22h)$$

$$[\bar{x}_t^\top \ \bar{u}_t^\top]^\top = M_\theta \theta_k, \quad (22i)$$

$$\xi_N^a = [z_{N|k}^\top \ \theta_k^\top \ (\bar{\Gamma}_k - \frac{\Gamma_{\max}}{2})^\top]^\top, \quad (22j)$$

$$\xi_N^a \in \mathbb{X}_f^{\xi,a}, \quad (22k)$$

where the objective function takes the form of  $J_N(\mathbf{c}_k, \theta_k, \bar{\Gamma}_k; x_k, \hat{\Gamma}_k, y_t) = J_{N,k}(\mathbf{c}_k, \theta_k, \bar{\Gamma}_k)$  for notation simplicity such that

$$\begin{aligned} J_{N,k}(\mathbf{c}_k, \theta_k, \bar{\Gamma}_k) &= \sum_{i=0}^{N-1} (\|z_{i|k} - \bar{x}_t\|_Q^2 + \|v_{i|k} - \bar{u}_t\|_R^2) \\ &\quad + \|z_{N|k} - \bar{x}_t\|_P^2 + V_o(y_t - \bar{y}_t) \\ &\quad + V_\gamma(\bar{\Gamma}_k - \hat{\Gamma}_k). \end{aligned}$$

$V_o(\cdot)$  is a convex cost that penalizes deviations between the desired  $y_t$  and the artificial  $\bar{y}_t$  targets,  $V_\gamma(\cdot)$  is a convex costs that penalizes deviations between the estimated  $\hat{\Gamma}_k$  and the artificial  $\bar{\Gamma}_k$  backoffs,  $\theta_k$  is a decision variable, and  $\mathbb{X}_f^{\xi,a}$  is an augmented admissible robust invariant set for tracking with adaptive backoff. For initialization purposes,  $\bar{\Gamma}_{-1}^* = \hat{\Gamma}_0$ . The proposed SMPC control law is given by

$$u_k = Kx_k + c_{0|k}^*(x_k, \hat{\Gamma}_k, y_t) + L\theta_k^*(x_k, \hat{\Gamma}_k, y_t). \quad (23)$$

Problem 2 is a standard quadratic programming problem because  $\bar{\Gamma}_{k-1}^*$ ,  $\hat{\Gamma}_k$ , and  $\min(\bar{\Gamma}_{k-1}^*, \hat{\Gamma}_k)$  are known vectors at instant  $k$ . In the proposed SMPC strategy, if  $\min(\bar{\Gamma}_{k-1}^*, \hat{\Gamma}_k) = \hat{\Gamma}_k$ , then  $\bar{\Gamma}_k^* = \hat{\Gamma}_k$  once a less conservative backoff vector provides relaxed constraints that enable reducing the cost function. On the other hand, if  $\min(\bar{\Gamma}_{k-1}^*, \hat{\Gamma}_k) \neq \hat{\Gamma}_k$ , feasibility is ensured while the distance between  $\bar{\Gamma}_k^*$  and  $\hat{\Gamma}_k$  can be progressively reduced in an optimal sense.

**Proposition 1.** Consider that Assumption 1 holds;  $\hat{\Gamma}_k \geq \Gamma$  (Wilson method);  $y_t$  asymptotically converges to a given steady-state value; and  $w_k \in W$ . Then, the closed-loop system  $x_{k+1} = Ax_k + Bu_k + B_w w_k$  with  $u_k$  defined by (23) satisfies the following conditions:

- (i) For all feasible initial conditions  $x_0$  and every target  $y_t$ , the evolution of the system is robustly feasible, i.e.,  $u_k \in U$  and  $\mathbb{P}([H]_j x_k \leq [f]_j) \geq 1 - \epsilon_j, j = 1, \dots, n_c$  for all  $k \in \mathbb{N}^*$
- (ii) The output converges to a mRPI around a desired admissible target, i.e.,  $y_k \in \mathcal{E}[y_k] \oplus (C + DK)\mathcal{R}_\infty$ , where  $\mathcal{R}_\infty = \bigoplus_{j=0}^\infty \Phi^j B_w \mathbb{W}$  is the mRPI.
- (iii) The expected value of the output converges to the target  $\lim_{k \rightarrow \infty} \mathcal{E}[y_k] = y_t$  whenever  $y_t$  is reachable.

If  $y_t$  is not reachable, then the mean converges to a value  $\lim_{k \rightarrow \infty} \mathcal{E}[y_k] = \tilde{y}_t$  that minimizes the offset cost, i.e.,

$$\tilde{y}_t = \min_{\bar{y}_t \in \mathcal{Y}_s} V_o(\bar{y}_t - y_t),$$

where the set of reachable targets is  $\mathcal{Y}_s = \{y = M_y \theta \in \mathbb{R}^p \mid [M_x \theta^\top \ \theta^\top \ \frac{\hat{\Gamma}_{\max}}{2}^\top]^\top \in \mathbb{Z}_f^{\xi, a}\}$ .

(iv) The artificial backoff converges to the estimated one in steady-state, i.e.,

$$\lim_{k \rightarrow \infty} \hat{\Gamma}_k - \bar{\Gamma}_k^* = 0.$$

*Proof.* Proofs mostly follow the same arguments of Santos et al. (2018). The difference arises from the fact that  $\lim_{k \rightarrow \infty} \hat{\Gamma}_k = \Gamma$  due to the binomial nature of the ECDF and the Wilson method property since  $\lim_{N_s \rightarrow \infty} F_l(\chi) = \lim_{N_s \rightarrow \infty} F_u(\chi) = F(\chi)$ . Hence,  $\hat{\Gamma}_k$  converges to the true backoff in steady-state.<sup>3</sup>

The proof of Property (i) proof is given in Appendix A.1 while Property (ii) is demonstrated in Appendix A.2. Dominated Convergence Theorem is used to show Property (iii) as follows. From Appendix A.2,  $\lim_{k \rightarrow \infty} \theta_k^* = \bar{\theta}^*$ . Then, the nominal target is defined by  $\bar{x}_t^* = (A + BK)\bar{x}_t^* + BL\bar{\theta}^*$  or, alternatively,  $\bar{x}_t^* = [I - (A + BK)]^{-1}BL\bar{\theta}^*$ . The one step ahead expected value is given by

$$\lim_{k \rightarrow \infty} \mathcal{E}[x_{k+1}] = \lim_{k \rightarrow \infty} \{\mathcal{E}[(A + BK)x_k] + \mathcal{E}[BL\theta_k^*]\}. \quad (24)$$

Now, from the finite convergence of  $\mathcal{E}[x_k]$  and  $\mathcal{E}[\theta_k^*]$ , (24) can be expressed by

$$\lim_{k \rightarrow \infty} \mathcal{E}[x_{k+1}] = (A + BK) \lim_{k \rightarrow \infty} \mathcal{E}[x_k] + BL \lim_{k \rightarrow \infty} \mathcal{E}[\theta_k^*]$$

with  $\lim_{k \rightarrow \infty} \mathcal{E}[x_{k+1}] = \lim_{k \rightarrow \infty} \mathcal{E}[x_k]$ . Since  $\theta_k^*$  has finite support due to the terminal set constraints, an integrable dominating random variable can then be defined such that  $\lim_{k \rightarrow \infty} \mathcal{E}[B\theta_k^*] \rightarrow B\bar{\theta}^*$  by virtue of the Dominated Convergence Theorem (Whittle, 2000). Thus,  $\lim_{k \rightarrow \infty} \mathcal{E}[x_k] = [I - (A + BK)]^{-1}BL\bar{\theta}^* \rightarrow [I - (A + BK)]^{-1}BL\bar{\theta}^* = \bar{x}_t^*$ . Likewise,  $\lim_{k \rightarrow \infty} \mathcal{E}[u_k] = \lim_{k \rightarrow \infty} \mathcal{E}[Kx_k + L\theta_k^*] \rightarrow K\bar{x}_t^* + L\bar{\theta}^* = \bar{u}_t^*$ . Hence,  $\lim_{k \rightarrow \infty} \mathcal{E}[y_k] \rightarrow C\bar{x}_t^* + D\bar{u}_t^* = \bar{y}_t^*$ .

Moreover,  $\theta_k^*$  minimizes  $V_o(M_y \theta_k^* - y_t)$  subject to  $[(M_x \theta_k^*)^\top \ \theta_k^{*\top} \ \Gamma^\top]^\top \in \mathbb{X}_f^{\xi, a}$ , otherwise  $\lim_{k \rightarrow \infty} c_{0|k}^* = 0$  would be contradicted due to optimality principle and convexity of  $V_o(\cdot)$ . Since the cost is composed of a sum of convex functions, the optimality achieved by  $\lim_{k \rightarrow \infty} c_{0|k}^* = 0$  cannot sacrifice the optimality of  $V_o(M_y \theta_k^* - y_t)$ . Details can be found in Ferramosca et al. (2012).

Property (iv) is established from the definition of the terminal constraint for  $\theta_k$  given by  $[(M_x \theta_k)^\top \ (\frac{\Gamma_{\max}}{2})^\top] \in \lambda \mathbb{X}_N^\xi$ . From this set definition, admissible steady-state parameter defined by  $\theta_k$  ( $\bar{y}_t$ ,  $\bar{x}_t$ , and  $\bar{u}_t$  as a consequence) is limited by the worst-case backoff  $\delta \hat{\Gamma} = \Gamma_{\max}/2$ . Hence, Property (iv) is directly established because:  $\lim_{k \rightarrow \infty} x_k$  is inside the robust admissible invariant set, control and state constraints are respected by using a linear feedback law, and  $\hat{\Gamma}_k$  has no effect in terms of cost function, except from  $V_\gamma(\bar{\Gamma}_k - \hat{\Gamma}_k)$  once  $\theta_k$  is not sensitive to  $\hat{\Gamma}_k$ . Then,

<sup>3</sup> The notation  $\lim_{N_s \rightarrow \infty} F_l(\chi) = \lim_{N_s \rightarrow \infty} F_u(\chi) = F(\chi)$  means that the ECDF upper and lower bounds converge to the true CDF as the number of samples of the random variable tend to infinity.

$\lim_{k \rightarrow \infty} \bar{\Gamma}_k = \lim_{k \rightarrow \infty} \hat{\Gamma}_k$  is the best solution with respect to the cost function.  $\square$

Proposition 1 indicates that the SMPC strategy for offset-free tracking can be interpreted as a generalized version of the results presented in Santos et al. (2018) with online constraint adaptation.

#### 4. NUMERICAL EXAMPLE

The SMPC approach with adaptive backoff is illustrated on the DC-DC converter benchmark problem (Lorenzen et al., 2017; Santos et al., 2018; Paulson et al., 2019; D' Jorge et al., 2020). The linearized converter model given by

$$A = \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix}, B = \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix}, C = [0 \ 1], B_w = I.$$

Stochastic disturbances are assumed to be bounded by  $\|w_k\|_\infty \leq 0.2$ , where  $w_k$  is defined from a truncated normal distribution  $\mathcal{N}(0, 0.06^2 I)$ . The hard input constraints are given by  $\|u\|_\infty \leq 0.4$ , while the state constraint sets are defined as

$$\begin{aligned} \mathbb{P}[[x_{k+1}]_1 \leq 2] &\geq 0.9, & \mathbb{P}[[x_{k+1}]_1 \geq -2] &\geq 0.9, \\ \mathbb{P}[[x_{k+1}]_2 \leq 3] &\geq 0.9, & \mathbb{P}[[x_{k+1}]_2 \geq -3] &\geq 0.9. \end{aligned}$$

The SMPC parameters are chosen to be  $Q = \text{diag}(1, 10)$ ,  $R = 1$ ,  $N = 5$ ,  $V_o(y_t - \bar{y}_t) = \|y_t - \bar{y}_t\|_{1000}^2$ , and  $V_\gamma(\hat{\Gamma}_k - \bar{\Gamma}_k) = \|\hat{\Gamma}_k - \bar{\Gamma}_k\|_{1000I}^2$ ,  $K$  is the LQR solution. A constant value of  $\lambda = 0.99$  is used for computing the robust invariant set.

The initial backoff vector is obtained from 150 samples ( $\bar{\Gamma}_{-1}^* = \hat{\Gamma}_0$ ). Symmetric constraint property is used to reduce the number of decision variables. The confidence interval of the ECDF is derived from the Wilson method with  $\alpha = 10^{-6} \rightarrow \Psi_{\alpha/2} = 3.554$ . The Low-pass filter parameter was defined as  $\beta = 0.9$ . The setpoint is defined as  $y_t = 2$  up until the sampling instant 1400, after which it changed to  $y_t = -2$ .

Fig. 1 shows the evolution of the closed-loop state for the cases of standard SMPC and proposed SMPC approaches. As expected, a similar closed-loop response is observed when adaptive SMPC has a small number of samples for computing the ECDF since less than 20 steps are necessary to achieve the mRPI around  $y_t = 2$ . Indeed, almost no constraint violation occurs when the system is evolved from the initial condition to this mRPI in both cases. However, once the target is changed to  $y_t = -2$  at the sampling instant 1400, the adaptive SMPC approach shows a significant increase in constraint violation. The ECDF of  $[x_k]_1$  with  $k \in [1405, 1410]$  is shown in Fig. 2, indicating that the proposed SMPC approach with adaptive backoff results in 9% constraint violation (almost equal to the chosen value of 10%) while the standard SMPC approach leads to less than 3% constraint violation. In the latter case, the conservatism of the confidence interval approach largely eliminates the benefits of the chance constraint.

The main benefit of the proposed adaptive approach to chance constraint handling arises from the fact that the initial ECDFs can be derived using fewer samples since the ECDF is continuously improved online. As can be observed

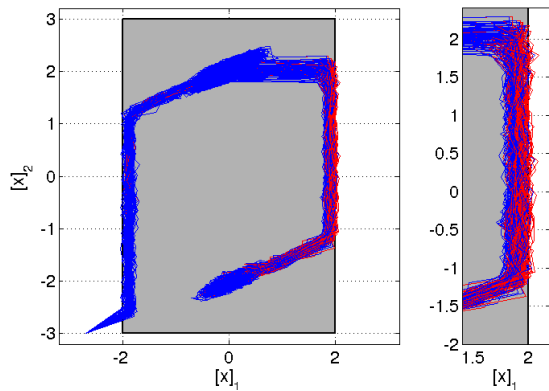


Fig. 1. State evolution from  $x(0) = [-2.7 \ -3]^T$  under 100 Monte Carlo disturbance realizations for standard SMPC (blue) and adaptive SMPC (red). Shaded region represents the original state constraints  $\mathbb{X}$ . The target is  $y_t = 2$ , which is changed to  $y_t = -2$  at the sampling instant 1400.

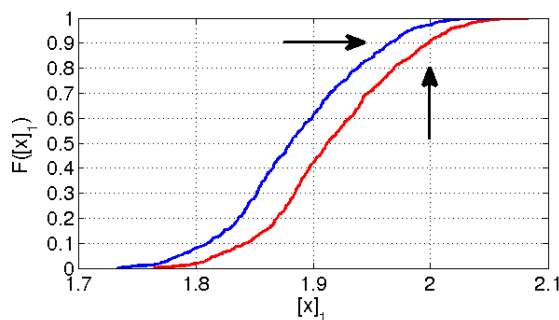


Fig. 2. Empirical cumulative distribution functions for  $\mathbb{S} = \{[x_{1405}]_1, [x_{1406}]_1 \dots [x_{14010}]_1\}$  under 100 disturbance realizations for standard SMPC (blue) and Adaptive SMPC (red). Arrows indicate the desired theoretical pair ( $[x]_1 = 2, F([x]_1) = 0.9$ ).

from Fig. 1, similar performance is achieved once the state converges to the mRPI around the admissible targets as the control law is implicitly derived from the solution of the unconstrained problem. However, desired chance constraint violation is almost achieved during the second set-point change due to online chance constraint update. We note that plant-model mismatch effect can be handled as an additional source of ECDF uncertainty. On the other hand, the ECDF uncertainty can be significantly reduced by using high fidelity modeling of system uncertainties (Paulson et al., 2019).

## 5. CONCLUSIONS

This paper presented a tracking SMPC approach with online adaptation of chance constraints using the notion of ECDFs. The Wilson confidence interval method is applied to ensure satisfaction of the individual chance constraints despite uncertainty in disturbance distributions. The proposed SMPC approach with backoff improvement ensures recursive feasibility and convergence of the expected value of the output to an admissible desired target. The extension of the proposed approach to the output feedback problem is an interesting topic for future work.

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## Appendix A. RECURSIVE FEASIBILITY AND CONVERGENCE

Initially consider that a feasible solution obtained at  $k$  is given by  $\mathbf{c}^*(k) = [c_{0|k}^{*T} \ c_{1|k}^{*T} \ \dots \ c_{N-1|k}^{*T}]^T$ ,  $\theta_k^*$  and  $\bar{\Gamma}_k^*$ . The candidate solution for the next step is defined by  $\tilde{\mathbf{c}}(k+1) = [c_{1|k+1}^{*T} \ \dots \ c_{N-1|k+1}^{*T} \ 0^T]^T$ ,  $\tilde{\theta}_{k+1} = \theta_k^*$  and  $\tilde{\bar{\Gamma}}_{k+1} = \bar{\Gamma}_k^*$ . Recursive feasibility and convergence are shown by following the standard steps from these solutions.

### A.1 Recursive Feasibility

The nominal prediction at  $k$  are defined by  $z_{0|k} = x_k$ ,  $z_{0|k+1} = x_{k+1}$ ,  $u_k = v_{0|k} = Kx_k + c_{0|k}^* + L\theta_k^*$ , and  $z_{1|k} = \Phi z_{0|k} + B(c_{0|k}^* + L\theta_k^*)$ . Based on the feasible candidate,  $z_{0|k+1}$  and  $z_{1|k}$  can be related by:

$$z_{0|k+1} = Ax_k + Bu_k + B_w w_k \quad (\text{A.1})$$

$$= (A + BK)x_k + B(c_{0|k}^* + L\theta_k^*) + B_w w_k \quad (\text{A.2})$$

$$= z_{1|k} + B_w w_k. \quad (\text{A.3})$$

Then, Eq. (A.3) can be rewritten by multiplying both sides by  $\Phi$  with the following addition  $B(c_{1|k}^* + L\theta_k^*)$ :

$$\begin{aligned} \Phi z_{0|k+1} + B(c_{1|k}^* + L\theta_k^*) &= \Phi[z_{1|k} + B_w w_k] \\ &\quad + B(c_{1|k}^* + L\theta_k^*) \end{aligned} \quad (\text{A.4})$$

or alternatively

$$z_{1|k+1} = z_{2|k} + \Phi B_w w_k. \quad (\text{A.5})$$

By induction, optimal predicted solution and feasible candidate are related by:

$$z_{j-1|k+1} = z_{j|k} + \Phi^{j-1} B_w w_k, \quad j \in [1, N]. \quad (\text{A.6})$$

Once the control candidate is defined by:

$$\tilde{v}_{j-1|k+1} = Kz_{j-1|k+1} + c_{j|k}^* + L\theta_k^*, \quad (\text{A.7})$$

$$v_{j|k}^* = Kz_{j|k} + c_{j|k}^* + L\theta_k^*, \quad (\text{A.8})$$

then

$$\tilde{v}_{j-1|k+1} = v_{j|k}^* + K\Phi^{j-1} B_w w_k, \quad j \in [1, N-1]. \quad (\text{A.9})$$

The recursive feasibility is verified from Eqs. (A.6), (A.9) combined with the definition of (9),(10):

$$[z_{i|k}^T \ \hat{\Gamma}_k^T]^T \in \mathbb{X}_i^\xi \Rightarrow [z_{i-1|k+1}^T \ \hat{\Gamma}_k^T]^T \in \mathbb{X}_{i-1}^\xi, \quad i \in [2, N], \quad (\text{A.10})$$

$$v_{j|k}^* \in \mathbb{V}_j \Rightarrow \tilde{v}_{j-1|k+1} \in \mathbb{V}_{j-1}, \quad j \in [1, N-1]. \quad (\text{A.11})$$

Moreover, the feasibility of  $z_{N|k+1}$  is shown from  $\tilde{v}_{N-1|k+1} = Kz_{N-1|k+1} + L\theta_k^*$  or simply  $\tilde{v}_{N-1|k+1} = K(z_{N|k} + \Phi^{N-1} B_w w_k) + L\theta_k^*$ . Once  $[z_{N|k}^T \ \theta_k^{*T} \ \hat{\Gamma}_k^{*T}]^T \in \mathbb{X}_f^\xi$ , define  $v_{N|k} = Kz_{N|k} + L\theta_k^* \in \mathbb{V}_N$ . As  $\tilde{v}_{N-1|k+1} = v_{N|k} + K\Phi^{N-1} B_w w_k$ , then  $v_{N|k} \in \mathbb{V}_N \Rightarrow \tilde{v}_{N-1|k+1} \in \mathbb{V}_{N-1}$  because  $\mathbb{V}_N = \mathbb{V}_{N-1} \ominus K\Phi^{N-1} B_w \mathbb{W}$  by definition.

Also observe from  $z_{N-1|k+1} = z_{N|k} + \Phi^{k-1} w_k$  and  $\tilde{v}_{N-1|k+1} = K(z_{N|k} + \Phi^{N-1} B_w w_k) + L\theta_k^*$  that the terminal prediction candidate is given by:

$$z_{N|k+1} = Az_{N-1|k+1} + B\tilde{v}_{N-1|k+1} \quad (\text{A.12})$$

$$\begin{aligned} &= A(z_{N|k} + \Phi^{k-1} B_w w_k) \\ &\quad + BK(z_{N|k} + \Phi^{N-1} B_w w_k) + BL\theta_k^* \\ &= \Phi z_{N|k} + BL\theta_k^* + \Phi^N B_w w_k. \end{aligned} \quad (\text{A.13})$$

Finally,  $[z_{N|k}^T \ \theta_k^{*T} \ \hat{\Gamma}_k^{*T}]^T \in \mathbb{X}_f^\xi$  implies in  $[z_{N|k+1}^T \ \theta_k^{*T} \ \hat{\Gamma}_k^{*T}]^T \in \mathbb{X}_f^\xi$  by definition presented in (21), which completes the proof.

### A.2 Convergence

Once the feasible candidate is defined with  $\bar{\Gamma}_{k+1} = \bar{\Gamma}_k^*$ , the decreasing property of the cost function is exactly the same of Ferramosca et al. (2012). A detailed input-to-state stability (ISS) proof is provided in Ferramosca et al. (2012). However, if  $K$  is given by the Linear Quadratic Regulator (LQR) solution, a simplified sketch of proof can be compactly presented.

In this case, the cost function can be rewritten by

$$J_{N,k}(\tilde{\mathbf{c}}(k), \theta_k, \hat{\Gamma}_k) = \sum_{j=0}^{N-1} \|c_{j|k}\|_{\Upsilon}^2 + V_o(\bar{y}_t - y_t) + V_\gamma(\bar{\Gamma}_k - \hat{\Gamma}_k)$$

where  $\Upsilon = R + B'PB$ . The cost function variation based on the feasible candidate is

$$J_{N,k+1}(\tilde{\mathbf{c}}(k+1), \theta_k^*, \hat{\Gamma}_k^*) - J_{N,k}(\mathbf{c}^*(k), \theta_k^*, \hat{\Gamma}_k^*) = -\|c_{0|k}\|_{\Upsilon}^2 \quad (\text{A.14})$$

where  $\Upsilon \geq 0$ . Hence, the optimal solution is such that

$$J_{N,k+1}(\mathbf{c}^*(k+1), \theta_{k+1}^*, \hat{\Gamma}_{k+1}^*) \leq J_{N,k+1}(\tilde{\mathbf{c}}(k+1), \theta_k^*, \hat{\Gamma}_k^*). \quad (\text{A.15})$$

Then,  $\lim_{k \rightarrow \infty} c_{0|k}^* = 0$ ,  $\lim_{k \rightarrow \infty} u^*(k) = Kx_k + L\theta_k^*$ , and  $\lim_{k \rightarrow \infty} J_{N,k}(\mathbf{c}^*(k), \theta_k^*, \hat{\Gamma}_k^*) = J_\infty$  for a given  $y_t$  with  $\lim_{k \rightarrow \infty} \hat{\Gamma}_k = \Gamma$ . Moreover, these facts ensure that  $\lim_{k \rightarrow \infty} x_k$  is inside the maximal robust admissible invariant set for tracking. Thus,  $\lim_{k \rightarrow \infty} \mathbf{c}^*(k) = [0^T \ 0^T \ \dots \ 0^T]^T$  is a feasible candidate.

The terminal constraint definition is such that  $\bar{y}_t^*$  does not depend on  $\hat{\Gamma}_k$ . Due to the continuity with respect to the decision variables, then  $\lim_{k \rightarrow \infty} J_{N,k}(\mathbf{c}^*(k), \theta_k^*, \bar{\Gamma}_k^*) = J_\infty \leq J_{N,k}(\mathbf{c}^*(k), \theta_k^*, \Gamma) = V_o(M_y \theta_k^* - y_t)$ . However,  $J_{N,k}(\mathbf{c}^*(k), \theta_k^*, \bar{\Gamma}_k^*) \geq V_o(M_y \theta_k^* - y_t)$ ,  $\forall k$  which shows the convergence of  $\theta_k^*$ , i.e.,  $\lim_{k \rightarrow \infty} \theta_k^* = \theta_\infty^*$ .

Finally,  $\lim_{k \rightarrow \infty} u_k = Kx(k) + \theta_\infty^*$  so that

$$\lim_{k \rightarrow \infty} x_k \in \bar{x}_t^* \oplus \mathcal{R}_\infty, \quad (\text{A.16})$$

$$\lim_{k \rightarrow \infty} u_k \in \bar{u}_t^* \oplus K\mathcal{R}_\infty, \quad (\text{A.17})$$

where  $\mathcal{R}_\infty = \bigoplus_{j=0}^{\infty} \Phi^j B_w \mathbb{W}$  which completes the proof. Details can be found in Ferramosca et al. (2012).